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Superconformal String Theory

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Abstract

Earlier, bosonic string theory was found to lack fermionic states. To solve for this unphysical property, the theory is extended through a new symmetry to what is called Superstring Theory. This theory does turn out to include fermions. The aim of this report is to analyze the effects produced on the system with the introduction of fermions in $\mathcal{N} = 1$ superstring theory. The discussion will cover the NS-R model of superstring theory, where a worldsheet supersymmetry gives the right tools to construct a consistent quantum string theory in critical spacetime dimension $D = 10$.

Contents

1 Introduction

This report is based on the presentation of *Superconformal String Theory*, given in the Spring Semester of 2013 at ETH Zürich for the Proseminar in Conformal Field Theory & String Theory. The topic of the talk has been deeper analyzed and supplemented with certain computations.

Superstring theory arises as an attempt to solve certain shortcomings of the bosonic string theory, such as the absence of fermions and the presence of tachyons in the mass spectrum. In this report fermionic strings will be introduced through the NS-R model in $\mathcal{N} = 1$ superstring theory. The system is described by a two-dimensional action invariant under global worldsheet supersymmetry, whose properties and conserved currents are explored in sections 2.2 and 2.3. As an aside, in sec. 2.4 supersymmetry is formulated in superspace. Superspace plays the same role for supersymmetry as Minkowski space plays for Lorentz symmetry.

The variation of the superconformal action leads to the Euler-Lagrange equations of motion for the bosonic and fermionic coordinates. Since the former has been already discussed in previous lectures, section 2.5 deals with the solutions for the fermionic part only. The vanishing of the surface term is achieved through the implementation of periodic (Ramond) or antiperiodic (Neveu-Schwarz) boundary conditions, whose realization in spacetime will give rise to both bosonic and fermionic states. In section 2.6 the Fourier modes of the conserved currents are analyzed. These turn out to satisfy the super-Virasoro algebra, whose central extension will be crucial in the process of quantization.

The quantization procedure is explored through two different approaches. The old-covariant approach, given in section 3.1, provides a first insight on how the parameters of the theory are constrained under the requirement of unitarity and decouple of unphysical states. Secondly, the light-cone gauge quantization, in section 3.2, exploits the residual gauge freedom of the system to completely fix the gauge and restrict the Fock space to the physical space. The effect of the non-covariant gauge choice is the appearance of an anomaly term in the Lorentz algebra after the process of quantization. The Lorentz symmetry will be restored constraining the values of the mass shift constant and the spacetime dimension to particular values.

Eventually, the global worldsheet supersymmetry, section 4, will be generalized via the Noether method to a locally supersymmetric action, the supergravity action.

2 Classical Theory

2.1 Superconformal action

The Polyakov action, describing the bosonic string theory, represents a two-dimensional free field theory of scalar fields, X^μ. In order to obtain fermionic strings, worldsheet spinors are introduced. Nevertheless, the generalization of the previous theory is made through the gauge-fixed Polyakov action, whose simpler expression offers a more intuitive generalization, while the fundamental gauge-invariant action of the system[1] will be examined later. The gauge-fixed superstring action or superconformal action is

$$\mathcal{S} = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma \Big\{\partial_\alpha X^\mu(\sigma)\partial^\alpha X_\mu(\sigma) - i\bar{\psi}^\mu(\sigma)\rho^\alpha\partial_\alpha\psi_\mu(\sigma)\Big\}, \tag{1}$$

where the string tension is $T = 1/\pi$. The spacetime coordinates of the string, X^μ, are supplemented with a D-plet of two-component spinors, ψ^μ. Although ψ^μ_A behaves as a spinor under worldsheet transformations, A denotes the spinor index, it transforms under the vector representation of the Lorentz group $SO(D-1,1)$. The unusual characteristic, for fermionic coordinates, of transforming as vectors in spacetime does not contradict the spin-statistics theorem. The theorem, in fact, does not give any constraint on how a fermionic field should transform under internal symmetries and, since we are dealing with a classical field theory on the worldsheet, spacetime transformations are merely internal symmetries from the worldsheet point of view.

In the formula above, ρ^α are two-dimensional Dirac matrices satisfying the Clifford Algebra $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$. In the following they will be chosen to be

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

This choice allows to consider the fermionic coordinates as two-component Majorana spinors. To be more precise, a Majorana spinor is a Dirac spinor on which the reality condition has been imposed

$$\psi = \psi_C,$$

where ψ_C denotes the charge conjugate field. This condition corresponds to the request $\psi^\dagger = \psi^T$, i.e. the Majorana fermion is a real spinor. From the form of the superconformal action the equation of motion for ψ^μ_A is simply the Dirac equation, which is a real first order differential equation, thanks to the choice of the Dirac matrices, and hence may have a real solution. Furthermore, the reality condition is Lorentz invariant, being the Dirac matrices purely imaginary. It seems reasonable, therefore, to deal with Majorana spinors instead of Weyl or Dirac spinors.

In the classical theory the fermionic coordinates, just introduced, are described by anticommuting variables. This is due to the fact that a consistent quantization of fields with integer spin requires the use of commutation relations ($[\phi, \dot{\phi}] \sim \hbar$), whereas for fields with half-integer spin requires anti-commutation relations ($\{\psi, \psi^\dagger\} \sim \hbar$). Looking at the classical field theory as a limit of the quantum system ($\hbar \to 0$), the bosonic coordinates, X^μ, are represented classically by commuting variables while

[1]The gauge-invariant action of superstring theory will be argument of the last section, 4.

the fermionic coordinates, ψ^μ_A, are anticommuting variables. The generalization of commuting numbers are the so-called Grassmann numbers[2]. They represent, in fact, both commuting variables, even Grassmann numbers, and anticommuting variables, odd Grassmann numbers.

2.2 Supersymmetry

The gauge-fixed action (1) preserves the conformal invariance[3] present in the bosonic string theory, but in addition, results invariant under global supersymmetry transformations. The supersymmetry transformations are

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \qquad\qquad \delta\psi^\mu = -i\rho^\alpha\partial_\alpha X^\mu\epsilon. \tag{2}$$

ϵ is an anticommuting infinitesimal Majorana spinor indipendent of the worldsheet coordinates. As pure computation, it appears that the commutator of two supersymmetry transformations gives a spatial translation on the worldsheet.

$$\begin{aligned} [\delta_1,\delta_2]X^\mu &= a^\alpha\partial_\alpha X^\mu \\ [\delta_1,\delta_2]\psi^\mu &= a^\alpha\partial_\alpha\psi^\mu \end{aligned} \tag{3}$$

The infinitesimal supersymmetry transformation is a fermionic object that multiplied by another fermionic transformation gives a vector on the worldsheet $a^\alpha = 2i\bar{\epsilon}_1\rho^\alpha\epsilon_2$. This can be easily check computing first the commutation on the bosonic coordinates

$$\begin{aligned} [\delta_1,\delta_2]X^\mu &= \delta_1(\bar{\epsilon}_2\psi^\mu) - \delta_2(\bar{\epsilon}_1\psi^\mu) \\ &= -i\bar{\epsilon}_2\rho^\alpha\partial_\alpha X^\mu\epsilon_1 + i\bar{\epsilon}_1\rho^\alpha\partial_\alpha X^\mu\epsilon_2 \\ &= 2i\bar{\epsilon}_1\rho^\alpha\epsilon_2\partial_\alpha X^\mu \\ &= a^\alpha\partial_\alpha X^\mu. \end{aligned} \tag{4}$$

Where the third step is achieved using the anticommutative relation of odd Grassmann numbers and the definition of adjoint ($i\bar{\epsilon}_1\rho^\alpha\epsilon_2 = i\epsilon_1^\dagger\rho^0\rho^\alpha\epsilon_2 = -i\bar{\epsilon}_2\rho^\alpha\epsilon_1$).

In the case of fermionic coordinates, the computation is slightly more involved and it is better to proceed writing down explicitly the spinor indices.

$$\begin{aligned} [\delta_1,\delta_2]\psi^\mu_A &= \delta_1(-i(\rho^\alpha\epsilon_2)_A\partial_\alpha X^\mu) - \delta_2(-i(\rho^\alpha\epsilon_1)_A\partial_\alpha X^\mu) \\ &= -i(\rho^\alpha\epsilon_2)_A\partial_\alpha(\bar{\epsilon}_{1B}\psi^\mu_B) + i(\rho^\alpha\epsilon_1)_A\partial_\alpha(\bar{\epsilon}_{2B}\psi^\mu_B) \\ &= i\bar{\epsilon}_{1A}(\rho^\alpha\epsilon_2)_B\partial_\alpha\psi^\mu_B + i\bar{\epsilon}_{1B}(\rho^\alpha\epsilon_2)_B\partial_\alpha\psi^\mu_A + (1\leftrightarrow 2) && \text{(5a)} \\ &= i\bar{\epsilon}_{1B}(\rho^\alpha\epsilon_2)_B\partial_\alpha\psi^\mu_A - i\bar{\epsilon}_{2B}(\rho^\alpha\epsilon_1)_B\partial_\alpha\psi^\mu_A && \text{(5b)} \\ &= a^\alpha\partial_\alpha\psi^\mu_A \end{aligned}$$

In (5a) we have used the Fierz identity: $\chi_A(\xi_B\eta_B) = -\xi_A(\chi_B\eta_B) - (\xi_B\chi_B)\eta_A$, together with the independence of ϵ to the worldsheet coordinates. While (5b) can be found

[2]Grassmann numbers form a non-commutative ring with $\mathbb{Z}_2$ grading. Being $\mathbb{Z}_2$ graded, they can be classified into even, $|\chi| = 0$, or odd, $|\chi| = 1$. The product between two Grassmann numbers is $\chi\psi = (-1)^{|\chi||\psi|}\psi\chi$.

[3]In the bosonic string theory, the invariance under reparametrization and Weyl scaling gives the possibility to fix the gauge setting the world sheet metric equal to the flat Minkowsky metric. This gauge-fixing condition leaves the theory invariant under conformal transformations, coordinate tranformation that leaves the metric unchanged up to a scale factor.

requiring ψ^μ on-shell. The reason why the commutator between two supersymmetry is equal to a translation on the worldsheet comes from the algebra of the generators of Poincaré and supersymmetry transformations. Later we will come back on this point, for the moment it is important to notice that the closure of the algebra generated by superconformal transformations is achieved only if ψ^μ satisfies the Dirac equation, $\rho^\alpha\partial_\alpha\psi^\mu = 0$, i.e. ψ^μ is an on-shell spinor.

2.3 Conserved currents

The invariance of the action, $\mathcal{S}$, under continuous transformations implies, through Noether's theorem, the existence of conserved charges. The conserved charges corresponding to infinitesimal conformal transformation are $j_\alpha = T_{\alpha\beta}\epsilon^\beta$ [4], where $T_{\alpha\beta}$ is the stress-energy tensor. $T_{\alpha\beta}$ corresponds to the charge density related to the invariance of the system under translations on the worldsheet. In superstring theory, the bosonic part of the stress-energy tensor remains equal to the expression found in the bosonic string theory, however, the presence of the spinor fields brings about some new terms,

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu + \tfrac{i}{4}\bar{\psi}^\mu\rho_\alpha\partial_\beta\psi_\mu + \tfrac{i}{4}\bar{\psi}^\mu\rho_\beta\partial_\alpha\psi_\mu - (trace). \tag{6}$$

The conserved current of supersymmetry transformations is determined through the Noether method. Acting with a local supersymmetry transformation on the globally invariant action, the variation of the action turns out to be proportional to the current multiplied by the derivative of the parameter ϵ,

$$\delta\mathcal{S} = \frac{2}{\pi}\int d^2\sigma \partial_\alpha\bar{\epsilon}J^\alpha. \tag{7}$$

From this expression the supercurrent is derived.

$$\begin{aligned}
\delta\mathcal{S} &= -\frac{1}{\pi}\int \mathrm{d}^2\sigma\Big\{\partial_\alpha(\bar{\epsilon}\psi^\mu)\partial^\alpha X_\mu - \tfrac{i}{2}\big(i\bar{\epsilon}\rho^\beta\partial_\beta X^\mu\rho^\alpha\partial_\alpha\psi_\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha(\rho^\beta\partial_\beta X^\mu\epsilon)\big)\Big\} \\
&\overset{P.I.}{=} -\frac{1}{\pi}\int \mathrm{d}^2\sigma\Big\{\partial_\alpha(\bar{\epsilon}\psi^\mu)\partial^\alpha X_\mu + \tfrac{1}{2}\bar{\epsilon}\rho^\beta\partial_\beta X^\mu\rho^\alpha\partial_\alpha\psi_\mu + \tfrac{1}{2}\partial_\alpha\bar{\psi}^\mu\rho^\alpha\rho^\beta\partial_\beta X^\mu\epsilon\Big\} \\
&= -\frac{1}{\pi}\int \mathrm{d}^2\sigma\Big\{\partial_\alpha(\bar{\epsilon}\psi^\mu)\partial^\alpha X_\mu + \bar{\epsilon}\rho^\beta\partial_\beta X^\mu\rho^\alpha\partial_\alpha\psi_\mu\Big\} \\
&= -\frac{1}{\pi}\int \mathrm{d}^2\sigma\, \partial_\beta X_\mu\Big(\partial^\beta\bar{\epsilon}\psi^\mu + \bar{\epsilon}\partial^\beta\psi^\mu + \bar{\epsilon}\rho^\beta\rho^\alpha\partial_\alpha\psi^\mu\Big) \\
&= -\frac{1}{\pi}\int \mathrm{d}^2\sigma\, \partial_\beta X_\mu\Big(\partial^\beta\bar{\epsilon}\psi^\mu - \bar{\epsilon}\rho^{\alpha\beta}\partial_\alpha\psi^\mu\Big) \\
&\overset{P.I.}{=} -\frac{1}{\pi}\int \mathrm{d}^2\sigma\, \partial_\beta X_\mu\Big(\partial^\beta\bar{\epsilon}\psi^\mu + \partial_\alpha\bar{\epsilon}\rho^{\alpha\beta}\psi^\mu\Big) \\
&= -\frac{1}{\pi}\int \mathrm{d}^2\sigma\, \partial_\beta X_\mu\Big(-\partial_\alpha\bar{\epsilon}\rho^\beta\rho^\alpha\psi^\mu\Big) \\
&= \frac{2}{\pi}\int \mathrm{d}^2\sigma\, \Big(\tfrac{1}{2}\partial_\alpha\bar{\epsilon}\rho^\beta\rho^\alpha\psi^\mu\partial_\beta X_\mu\Big).
\end{aligned}$$

[4] The parameter ϵ^β is the infinitesimal variation of the worldsheet coordinates ($\sigma^\alpha \to \sigma^\alpha + \epsilon^\alpha$). For further information the reader could refer to [3, p. 5].

In the previous calculation $P.I.$ means that we have performed an integration by part. The variation of the action gives for the bosonic part two equal terms, whereas for the fermionic part includes the variation of both the spinor and the adjoint spinor: $\delta\bar{\psi}^\mu = \delta\psi^{\dagger\mu}\rho^0 = i\epsilon^\dagger\rho^0\rho^\alpha\rho^0\partial_\alpha X^\mu\rho^0 = i\bar{\epsilon}\rho^\alpha\partial_\alpha X^\mu$. Other useful identities are

$$\bar{\chi}\rho^\alpha\rho^\beta\psi = \bar{\psi}\rho^\beta\rho^\alpha\chi,$$

$$\rho^\alpha\rho^\beta = \tfrac{1}{2}\{\rho^\alpha, \rho^\beta\} + \tfrac{1}{2}[\rho^\alpha, \rho^\beta] = -\eta_{\alpha\beta} + \rho^{\alpha\beta}.$$

From the above calculation, the conserved current corresponding to the supersymmetry invariance of the action is a fermionic object whose explicit form is

$$J_\alpha = \tfrac{1}{2}\rho^\beta\rho_\alpha\psi^\mu\partial_\beta X_\mu \tag{9}$$

As stated by Noether's theorem, the currents satisfy conservation law, i.e. $\partial_\alpha T^{\alpha\beta} = 0$ and $\partial_\alpha J^\alpha = 0$. Moreover, due to the invariance of the theory under conformal transformation, the currents must also be traceless[5]

$$T^\alpha_\alpha = 0 \qquad\qquad \rho^\alpha J_\alpha = 0,$$

where the vanishing of the latter expression is due to the fact that $\rho^\alpha\rho^\beta\rho_\alpha = 0$.

2.4 Superspace

The global worldsheet supersymmetry becomes manifest when formulated in the superspace. Instead of having the usual two-dimensional worldsheet parametrized by the coordinates $\sigma^\alpha = (\sigma, \tau)$, two other coordinates θ^A, anitcommuting variables forming a two-component Majorana spinor, are introduced. The space parametrized by these coordinates is called superspace. The usefulness of the superspace will be soon clear.

A general function in superspace, called superfield, can be expanded in power series of θ, noticing that θ^A is an odd Grassmann number and hence $\theta^A\theta^A = 0$.

$$Y^\mu(\sigma, \theta) = X^\mu(\sigma) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma). \tag{10}$$

The superfield depends not only on the bosonic and fermionic coordinates of the string, but also on a new field B^μ, the so-called auxiliary field. In superspace the generator Q of supersymmetry takes the following form

$$Q_A = \frac{\partial}{\partial\bar{\theta}^A} + i(\rho^\alpha\theta)_A\partial_\alpha. \tag{11}$$

The infinitesimal supersymmetry transformation on the coordinates σ^α and θ^A can be found from the commutator between the coordinates and the supercharge,

$$\begin{aligned}\delta\theta^A &= [\bar{\epsilon}Q, \theta^A] = \epsilon^A,\\ \delta\sigma^\alpha &= [\bar{\epsilon}Q, \sigma^\alpha] = i\bar{\epsilon}\rho^\alpha\theta,\end{aligned} \tag{12}$$

[5] The requirement that the current associated to scale transformation is conserved, implies the traceless condition, [6, p. 65].

for convenience the generator of the transformaion is multiplied by an arbitrary infinitesimal anticommuting parameter ϵ_A. Thus, in superspace a supersymmetry transformation is a geometrical transformation, in particular a translation on superspace. As an example, let us compute the variation of the odd Grassmann coordinate, θ^A.

$$\begin{aligned}
\delta\theta_A f &= [\bar{\epsilon}Q, \theta_A] f \\
&= \bar{\epsilon}Q(\theta_A f) - \theta_A \bar{\epsilon}Q(f) \\
&= \bar{\epsilon_B}\Big(\frac{\partial}{\partial\bar{\theta^B}} + i(\rho^\alpha\theta)_B\partial_\alpha\Big)(\theta_A f) - \theta_A\bar{\epsilon_B}\Big(\frac{\partial}{\partial\bar{\theta^B}} + i(\rho^\alpha\theta)_B\partial_\alpha\Big)(f) \\
&= \epsilon_A f + \theta_A\bar{\epsilon_B}\Big(\frac{\partial}{\partial\bar{\theta^B}} + i(\rho^\alpha\theta)_B\partial_\alpha\Big)(f) - \theta_A\bar{\epsilon_B}\Big(\frac{\partial}{\partial\bar{\theta^B}} + i(\rho^\alpha\theta)_B\partial_\alpha\Big)(f). \\
&= \epsilon_A f
\end{aligned} \tag{13a}$$

The supercharge can be used to define the transformation property of the superfields

$$\delta Y^\mu = [\bar{\epsilon}Q, Y^\mu] = \bar{\epsilon}QY^\mu, \tag{14}$$

whose expansion in powers of θ yealds the supersymmetry transfromations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu$$

$$\begin{aligned}
\delta\psi^\mu &= -i\rho^\alpha\partial_\alpha X^\mu\epsilon + B^\mu\epsilon \\
\delta B^\mu &= -i\bar{\epsilon}\rho^\alpha\partial_\alpha\psi^\mu.
\end{aligned}$$

Moreover, the commutator between two supersymmetry transformations is

$$[\delta_1, \delta_2]Y^\mu = -a^\alpha\partial_\alpha Y^\mu, \tag{15}$$

since the commmutator of two supercharges is $[\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] = -2i\bar{\epsilon}_1\rho^\alpha\epsilon_2\partial_\alpha = -a^\alpha\partial_\alpha$. This equation corresponds to equation (3), previously derived. Nonetheless, the presence of the auxiliary field B^μ allows the closure of the algebra of the generators of supersymmetry and Poincaré transformations without the on-shell condition for the fermionic field. In fact, as it can be seen above, the vanishing of the variation of the auxiliary field corresponds to the Dirac equation.

The action expanded in components reads

$$\mathcal{S} = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma \Big\{\partial_\alpha X^\mu\partial^\alpha X_\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu - B^\mu B_\mu\Big\}\ . \tag{16}$$

From the variation of $\mathcal{S}$, the Euler-Lagrange equation for B^μ turns out to be $B^\mu = 0$. Therefore, B^μ can be set equal to zero on-shell, which corresponds to the absence of the auxiliary field as in section 2.2. The necessity of the field B^μ to achieve the closure of the algebra is due to a mismatch between the bosonic and the fermionic degrees of freedom off-shell[6].

[6] For further information the reader could refer to [5, p. 99].

2.5 Dirac equation & boundary conditions

The analysis of the equations of motion of the spinor field is conveniently performed in light-cone coordinates, $\sigma^{\pm} = \tau \pm \sigma$ and $\partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$, where the upper and lower components of the spinor $\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$ completely decouple. Since the analysis of the bosonic part of the action is identical to the discussion in the bosonic string theory, this section will focus on the fermionic part, which is

$$\mathcal{S}_f = \frac{i}{\pi} \int \mathrm{d}^2\sigma \{\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+\}.$$

Varying the fermionic action, the equations of motion for the field ψ appear together with surface terms,

$$\begin{aligned}
\delta \mathcal{S}_f &= \frac{i}{\pi} \int \mathrm{d}^2\sigma \Big(\delta\psi_- \partial_+ \psi_- + \psi_- \partial_+ \delta\psi_- + \delta\psi_+ \partial_- \psi_+ + \psi_+ \partial_- \delta\psi_+ \Big) && \text{(17a)} \\
&\overset{P.I.}{=} \frac{i}{\pi} \int \mathrm{d}^2\sigma \Big(\delta\psi_- \partial_+ \psi_- + \partial_+ \big(\psi_- \delta\psi_-\big) - \partial_+ \psi_- \delta\psi_- + (+ \leftrightarrow -) \Big) \\
&= \frac{i}{\pi} \int \mathrm{d}^2\sigma \Big(\delta\psi_- \partial_+ \psi_- + \partial_+ \big(\psi_- \delta\psi_-\big) + \delta\psi_- \partial_+ \psi_- + (+ \leftrightarrow -) \Big) && \text{(17b)} \\
&= \frac{2i}{\pi} \int \mathrm{d}^2\sigma \Big(\delta\psi_- \partial_+ \psi_- + \delta\psi_+ \partial_- \psi_+ \Big) + \\
&+ \frac{i}{2\pi} \int \mathrm{d}^2\sigma \Big((\partial_\tau + \partial_\sigma)(\psi_- \delta\psi_-) + (\partial_\tau - \partial_\sigma)(\psi_+ \delta\psi_+) \Big) && \text{(17c)}
\end{aligned}$$

In the last expression, the first term leads to the equations of motion

$$\begin{aligned} \partial_+ \psi_-^\mu &= 0 \\ \partial_- \psi_+^\mu &= 0, \end{aligned} \tag{18}$$

which implies that ψ_-^μ is a right-moving spinor while ψ_+^μ is a left-moving spinor. The second term is the surface term which, discarding the total derivative with respect to τ, becomes

$$\frac{i}{2\pi} \int \mathrm{d}\tau \, \{\psi_- \delta\psi_- - \psi_+ \delta\psi_+\}\big|_{\sigma=0}^{\sigma=\pi}. \tag{19}$$

The vanishing of this surface term can be implemented in different ways, depending on the type of strings and sector considered.

2.5.1 Closed strings

The cancellation of the surface term for closed strings is ensured either by periodicity condition, Ramond boundary condition

$$\psi_A^\mu(\sigma, \tau) = \psi_A^\mu(\sigma + \pi, \tau)$$

or antiperiodicity condition, Neveu-Schwarz boundary condition

$$\psi_A^\mu(\sigma, \tau) = -\psi_A^\mu(\sigma + \pi, \tau)$$

Figure 1: Closed string

for each component of the spinor. The possibillity of having an antiperiodicity condition, absent in the bosonic string theory, is due to the fact that we are dealing with spinors on the worldsheet. A spinor after a complete rotation around the string can transform to plus or minus itself. A Lorentz transformation on a spinor is affected by a sign ambiguity, being the square root of the vector representation of the Lorentz group[7]. Therefore, for each component of the spinor periodicity or antiperiodicity has to be imposed to guarantee the vanishing of the surface term. Periodicity condition is, consequently, associated to integer modes, while antiperiodicity condition to half-integer modes.

The general solutions of the Dirac equation in Fourier modes is

$$\psi_-^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \quad (R)$$

or

$$\psi_-^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-2ir(\tau-\sigma)} \quad (NS)$$

for the right-moving component ($\partial_+\psi_-^\mu = 0$), and

$$\psi_+^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} \quad (R)$$

or

$$\psi_+^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)} \quad (NS)$$

for the left-moving component ($\partial_-\psi_+^\mu = 0$).

For closed strings the oscillating modes of left- and right-moving spinor are indipendent apart from the level matching condition[8]. Four closed-string sectors are obtained, corresponding to the different pairings (ψ_-^μ, ψ_+^μ): $(NS-NS), (NS-R), (R-NS), (R-R)$[9].

[7]In other words the spin bundle is the square root of the tangent bundle.

[8]The level matching condition for the R-sector is $\alpha' M^2 = N^\alpha + N^d = \sum_{n=1}^\infty \alpha_{-n}\cdot\alpha_n + \sum_{n=1}^\infty d_{-n}\cdot d_n = \sum_{n=1}^\infty \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \sum_{n=1}^\infty \tilde{d}_{-n}\cdot\tilde{d}_n$, for the NS-sector instead of the integer oscillators there are half-integer oscillators.

[9]The significance of these four sectors in spacetime will be discussed in section 3.1.

2.5.2 Open strings

In the case of open strings, the vanishing of the surface terms requires

$$\psi_+\delta\psi_+ = \psi_-\delta\psi_- \qquad \sigma = 0, \pi.$$

As a consequence, at each end of the string $\psi_+ = -\psi_-$. The overall sign between ψ_+ and ψ_- is just a matter of convention, what acquires significance is the sign at one end of the string when at the other end the relative sign has already been fixed. For this reason, the relative sign at $\sigma = 0$ can be chosen to be

$$\psi_+^\mu(0, \tau) = \psi_-^\mu(0, \tau),$$

without loss of generality. The relative sign in $\sigma = \pi$ defines two sectors:

- The Ramond sector
$$\psi_+^\mu(\pi, \tau) = \psi_-^\mu(\pi, \tau) \qquad (R)$$

- The Neveu-Schwarz sector
$$\psi_+^\mu(\pi, \tau) = -\psi_-^\mu(\pi, \tau) \qquad (NS)$$

The general solutions of the Dirac equation in Fourier modes is

$$\psi_-^\mu(\sigma, \tau) = \tfrac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)}$$

$$\psi_+^\mu(\sigma, \tau) = \tfrac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)},$$

with Ramond boundary condition,

$$\psi_-^\mu(\sigma, \tau) = \tfrac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau-\sigma)}$$

$$\psi_+^\mu(\sigma, \tau) = \tfrac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau+\sigma)},$$

with Neveu-Schwarz boundary condition. As before, the Ramond boundary condition is associated to integer modes, while the Neveu-Schwarz to half-integer.

2.6 Super-Virasoro modes

In view of the decoupling of right- and left-moving components in light-cone coordinates, the conserved currents assume a more compact form. The supercurrents, J_- and J_+, as linear combination of lower and upper component of the fermionic current could in principle be two-component spinors, instead a short calculation shows that only the right-moving component of the former and the left-moving component of the latter are not vanishing

$$J_- = \psi_-^\mu \partial_- X_\mu.$$

$$J_+ = \psi_+^\mu \partial_+ X_\mu$$

The conserved current under translation or stress-energy tensor becomes

$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \tfrac{i}{2}\psi_-^\mu \partial_- \psi_{-\mu},$$

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu + \tfrac{i}{2}\psi_+^\mu \partial_+ \psi_{+\mu}$$

where T_{+-} and T_{-+} are zero by the tracelessness condition[10].

The $\mathcal{N} = 1$[11] superconformal algebra in two dimension satisfied by these currents corresponds to the closed algebra previously introduced with equation(3), where the commutation relation between supercurrents is

$$\begin{aligned}
&\{J_-(\sigma), J_-(\sigma')\} = \pi\delta(\sigma - \sigma')T_{--}(\sigma) \\
&\{J_+(\sigma), J_+(\sigma')\} = \pi\delta(\sigma - \sigma')T_{++}(\sigma) \\
&\{J_+(\sigma), J_-(\sigma')\} = 0.
\end{aligned}$$

By analogy to the bosonic string theory, where the equation of motion for the gauge field $h_{\alpha\beta}$ leads to the Virasoro constraints $T_{++} = T_{--} = 0$ after having fixed the gauge, it seems reasonable to generalize the Virasoro constraints to super-Virasoro constraints

$$T_{++} = T_{--} = J_+ = J_- = 0. \tag{21}$$

In superstring theory the constraint equations would arise from the gauge-invariant action, also called supergravity action, whose invariance under local supersymmetry gives rise to constraint equations as equations of motion of a new gauge field[12].

The Fourier modes of the conserved currents $T_{\alpha\beta}$ and J_α, as in the bosonic string theory, correspond to the super-Virasoro modes[13] L_m, and F_m for the R-sector, G_r for the NS-sector. For open strings they are

$$\begin{aligned}
L_m &= \frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma \{e^{im\sigma}T_{++} + e^{-im\sigma}T_{--}\} = \frac{1}{\pi}\int_{-\pi}^\pi \mathrm{d}\sigma e^{im\sigma}T_{++} \\
F_m &= \frac{\sqrt{2}}{\pi}\int_0^\pi \mathrm{d}\sigma \{e^{im\sigma}J_+ + e^{-im\sigma}J_-\} = \frac{\sqrt{2}}{\pi}\int_{-\pi}^\pi \mathrm{d}\sigma e^{im\sigma}J_+ \\
G_r &= \frac{\sqrt{2}}{\pi}\int_0^\pi \mathrm{d}\sigma \{e^{ir\sigma}J_+ + e^{-ir\sigma}J_-\} = \frac{\sqrt{2}}{\pi}\int_{-\pi}^\pi \mathrm{d}\sigma e^{ir\sigma}J_-.
\end{aligned}$$

The utility in writing down the last term in the above expressions is due to the orthogonality of the exponential functions in the interval from $-\pi$ to π. For closed string there are two sets of super-Viraroso generators, one given by the mode expansions of T_{++} and J_+ that correspond to L_m, F_m, G_r and the other given by the mode expansions of T_{--} and J_- that correspond to the set $\tilde{L}_m, \tilde{F}_m, \tilde{G}_r$. The expansion

[10] T_{+-} and T_{-+} are linear combinations of T_{00} and T_{11}, where the zero-component corresponds to the worldsheet coordinate τ while the first-component to σ.

[11] The condition $\mathcal{N} = 1$ refers to the fact that a supersymmetry transformation is performed via one Majorana spinor.

[12] The fundamental action of superstring theory, invariant under local supersymmetry transformation, is described in section 4. For a more detailed explanation of the subject we refer to [2, p. 129].

[13] The super-Virasoro modes are the generators of the super-Virasoro algebra, whose central extension is written explicitly in section 3.1.1.

of the super-Virasoro modes in term of oscillators is derived by inserting the mode expansions for the bosonic and fermionic coordinates. Below we present an example of computation.

$$\begin{aligned} L_m &= \frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma\Big\{e^{im\sigma}T_{++}+e^{-im\sigma}T_{--}\Big\} \\ &= \frac{1}{\pi}\int_{-\pi}^\pi \mathrm{d}\sigma\Big\{e^{im\sigma}T_{++}\Big\} \\ &\overset{(6)}{=} \frac{1}{\pi}\int_{-\pi}^\pi \mathrm{d}\sigma\Big\{\partial_+X^\mu\partial_+X_\mu+\tfrac{i}{2}\psi_+^\mu\partial_+\psi_{+\mu}\Big\} \\ &= \frac{1}{4\pi}\int_{-\pi}^\pi \mathrm{d}\sigma\Big\{e^{im\sigma}\Big(\sum_{n,p}\alpha_n\cdot\alpha_p e^{-i(n+p)\sigma}+\sum_{n,p}d_n\cdot d_p p e^{-i(n+p)\sigma}\Big)\Big\} \\ &= \frac{1}{2}\Big(\sum_n\alpha_n\cdot\alpha_{m-n}+\sum_n d_n\cdot d_{m-n}(m-n)\Big) \\ &= \frac{1}{2}\Big(\sum_n\alpha_{-n}\cdot\alpha_{m+n}+\sum_n(n+\tfrac{1}{2}m)d_{-n}\cdot d_{m+n}\Big). \end{aligned}$$

The mode expansion is evaluated at $\tau=0$ and considering for the bosonic coordinates $\partial_+X^\mu=\frac{1}{2}(\dot{X}^\mu+X'^\mu)=\frac{1}{2}\sum_n\alpha_n^\mu e^{-in(\tau+\sigma)}$, whereas for the fermionic coordinates in the R-sector $\psi_+^\mu=\frac{1}{\sqrt{2}}\sum_n d_n^\mu e^{-in(\tau+\sigma)}$. In the last step we have used the identity $\sum_n(m-n)d_n\cdot d_{m-n}=-\sum_n nd_n\cdot d_{m-n}$ and the possibility of changing the sign of the index on which we are summing over.

Proceeding in the same way for the other super-Virasoro modes, explicit expansions in terms of oscillators are found. In the case of R-sector

$$L_m=L_m^\alpha+L_m^d \tag{23}$$

$$L_m^\alpha=\tfrac{1}{2}\sum_{n\in\mathbb{Z}}\alpha_{-n}\alpha_{m+n}$$

$$L_m^d=\tfrac{1}{2}\sum_{n\in\mathbb{Z}}(n+\tfrac{1}{2}m)d_{-n}d_{m+n}$$

$$F_m=\sum_{n\in\mathbb{Z}}\alpha_{-n}d_{m+n}.$$

While in the NS-sector

$$L_m=L_m^\alpha+L_m^b \tag{24}$$

$$L_m^\alpha=\tfrac{1}{2}\sum_{n\in\mathbb{Z}}\alpha_{-n}\alpha_{m+n}$$

$$L_m^b=\tfrac{1}{2}\sum_{r\in\mathbb{Z}+1/2}(r+\tfrac{1}{2}m)b_{-r}b_{m+r}$$

$$G_r=\sum_{n\in\mathbb{Z}}\alpha_{-n}b_{r+n}.$$

By comparison with the bosonic string theory, new fermionic modes, F_m and G_r, arise from the supercurrent J_α, but this is not the unique difference from the previous theory. In fact, the bosonic modes acquire new terms, L_m^d and L_m^b, coming from the fermionic contribution to the stress-energy tensor.

The super-Virasoro constraints $T_{++} = T_{--} = J_+ = J_- = 0$ of the classical system in term of oscillators become $L_m = F_m = G_r = 0$ for $m, r \in \mathbb{Z}$. These equations are also know as physical constraint equations, since a consistent string theory can only be achieved by imposing these constraints. In the next section, the important role played by the infinite dimensional superconformal algebra, arising from the mode expansions of the charge density of superconformal transformations, is examined.

3 Quantization

The classical system can be quantized through different procedures. In this section two methods will be analyzed; they will give different perspectives on the theory but eventually they will agree.

In the *covariant quantization*, the system is quantized transforming all the Poisson brackets in canonical commutaion relations and promoting the bosonic and fermionic coordiantes to operators. Anomalies in the generators of the superconformal algebra will arise. Subsequently, the constraint equations are imposed as operator equations on the physical states of the system. This method of quantization reflects the Gupta-Bleuler method used in QED. Once quantized the system, the physical constraints give the right tools to eliminate ghosts and find a physical space with only positive definite norm states.

The alternative approach, *light-cone gauge quantization*, consists in getting rid of all the unphysical states, first, and then quantize only the physical space. The non-covariant gauge choice, made in order to find the physical space, brings about in the quantized system an anomaly term in the Lorentz algebra.

Eventually, a consistent string theory is obtained through both methods forcing the mass-shift term a and the critical spacetime dimension D to particular values.

3.1 Covariant quantization

The quantization of the system in the covariant quantization starts from the gauge-fixed action (1). The Poisson brackets are transformed into canonical commutation relations and, as usual in a two-dimensional free field theory, X^μ and ψ^μ are promoted to operators, obeying the following canonical commutation relations

$$[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau,)] = -i\pi\eta^{\mu\nu}\delta(\sigma-\sigma'),$$

$$\{\psi^\mu_A(\sigma,\tau), \psi^\nu_B(\sigma',\tau)\} = \pi\eta^{\mu\nu}\delta_{AB}\delta(\sigma-\sigma').$$

These equations imply the following relations for the oscillators:

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}$$

$$\{d^\mu_m, d^\nu_n\} = \delta_{m+n}\eta^{\mu\nu}$$

$$\{b^\mu_r, b^\nu_s\} = \delta_{r+s}\eta^{\mu\nu},$$

where $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$. The following discussion will be restricted to the description of open strings or the right-moving component of closed strings. In the case of left moving component, the analysis is identical if the oscillators $\tilde{\alpha}^\mu_m, \tilde{d}^\mu_m, \tilde{b}^\mu_r$ are considered.

The oscillating modes are related to properly normalized harmonic oscillators, e.g. $\alpha^\mu_m = \sqrt{m}a^\mu_m$ for $m > 0$ and $\alpha^\mu_m = \sqrt{m}a^{\dagger\mu}_m$ for $m < 0$, this last equality is given by the reality condition imposed on Majorana spinor. Quantizing the system, the oscillating modes split into creation and annihilation operators. Oscillating modes become annihilation operators when the index is positive

$$\begin{aligned} \alpha^\mu_m |0\rangle = b^\mu_r |0\rangle = 0 \quad m, r > 0 \qquad (NS) \\ \alpha^\mu_m |0\rangle = d^\mu_m |0\rangle = 0 \qquad m > 0 \qquad (R) \end{aligned} \tag{25}$$

and creation operators when the index is negative. For $m, r < 0$, α_m^μ, d_m^μ and b_r^μ increase the eigenvalue of M^2 by $2m$ and $2r$ units, respectively[14].

The ground state in theNS-sector is completely determined by the first relation in (25), since half integer modes uniquely specify a non-degenerate ground state, that can be identified as spin zero state. In contrast, in the R-sector the ground state is not uniquely defined. The zero-modes commute with the mass operator ($[d_0^\mu, M^2] = 0$) and, consequently, the ground state is degenerate. Furthermore, d_0^μ obey the Clifford algebra, i.e.

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}.$$

From this expression is clear that d_0^μ are Dirac matrices up to a normalisation factor, $\gamma^\mu = i\sqrt{2}d_0^\mu$ where $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$. Since the irreducible representation of the Clifford algebra corresponds to spinors of $SO(D-1,1)$, the boundary condition that gives rise to integer modes produces fermionic states.

In conclusion, the boundary conditions, previously analyzed, lead to spacetime bosons in the NS-sector and spacetime fermions in the R-sector. As a result, for open string the NS-sector gives rise to bosonic strings while the R-sector to fermionic strings. In the case of closed string the four sectors are tensor products of two different sectors, being the two spinorial component independent from each other, thus $(NS - NS)$ and $(R - R)$ determine bosons in spacetime while $(NS - R)$ and $(R - NS)$ are spacetime fermions.

3.1.1 Super-Virasoro algebra

The super-Virasoro modes in the quantized system acquire the form

$$\begin{aligned}
&(R)\\
L_m &= L_m^\alpha + L_m^d \qquad (26)\\
L_m^\alpha &= \tfrac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{-n}\alpha_{m+n}:\\
L_m^d &= \tfrac{1}{2}\sum_{n\in\mathbb{Z}} (n + \tfrac{1}{2}m) :d_{-n}d_{m+n}:\\
F_m &= \sum_{n\in\mathbb{Z}} \alpha_{-n}d_{m+n}
\end{aligned}$$

for the fermionic sector,

$$\begin{aligned}
&(NS)\\
L_m &= L_m^\alpha + L_m^b \qquad (27)\\
L_m^\alpha &= \tfrac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{-n}\alpha_{m+n}:
\end{aligned}$$

[14]The factor of 2 comes from having set $\alpha' = 1/2$ and the definition of mass operator given in section 2.5, footnote 7.

$$L_m^b = \frac{1}{2} \sum_{r\in\mathbb{Z}+1/2} (r + \tfrac{1}{2}m) : b_{-r} b_{m+r} :$$

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_{-n} b_{r+n}$$

for the bosonic sector.

The only difference with the classical expression (26), (27) is the appearance of normal ordering products at the quantum level. The passage from the classical to the quantum system, in fact, brings about ordering amibiguites in the products of oscillators whose commutation or anticommutaion relations are not zero.

Quantizing the system, the algebra of the Fourier modes acquires a central extension and becomes the super-Virasoro algebra [15] .

$$(NS)$$

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n},$$
$$[L_m, G_r] = \left(\tfrac{1}{2}m - r\right) L_{m+r},$$
$$\{G_r, G_s\} = 2L_{r+s} + B(r)\delta_{r+s}.$$

The anomaly terms are

$$A(m) = \tfrac{1}{8}D(m^3 - m),$$
$$B(r) = \tfrac{1}{2}D(r^2 - \tfrac{1}{4}).$$

While imposing periodicity conditions

$$(R)$$

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n},$$
$$[L_m, F_n] = \left(\tfrac{1}{2}m - n\right) L_{m+n},$$
$$\{F_m, F_n\} = 2L_{m+n} + B(m)\delta_{m+n},$$

with anomaly terms

$$A(m) = \tfrac{1}{8}Dm^3,$$
$$B(m) = \tfrac{1}{2}Dm^2.$$

Shifting L_0 by a constant, the linear and the constant term in the anomalies can be changed and the anomalies in the different sectors will coincide[16]. The anomaly terms, written above, are found via the super-Jacoby identity [15] .

Concentrating on the R-sector the super-Jacoby identity reads

$$[\{F_r, F_s\}, L_m] + \{[L_m, F_r], F_s\} + \{[L_m, F_s], F_r\} = 0,$$

[15] A superalgebra is a generalization of a Lie algebra that includes a $\mathbb{Z}_2$ grading. Given three elements of the superalgebra g, h, l, the Lie superbracket is defined $[g,h] = gh - (-1)^{|g||h|}hg$, this obeys the super skew-symmetry, $[g,h] = -(-1)^{|g||h|}[h,g]$, and the super Jacobi identity, $[g,[h,l]] = [[g,h],l] + (-1)^{|g||h|}[h,[g,l]]$.

[16] For further details the reader could refer to [2, p. 149].

using the parity of $B(m)$, i.e. $B(m) = B(-m)$, and the commutation relations of the super-Virasoro algebra, the following recursion relation is found

$$B(m+s) = \frac{1}{\frac{1}{2}m - s}\Big[-(s+\tfrac{3}{2}m)B(s) + 2A(m)\Big],$$

which gives, together with the value of the $A(m)$ anomaly, the demanded result $B(m) = \frac{1}{2}Dm^2$. As an aside remark to the super-Virasoro algebra, in the bosonic sector the generators $L_1, L_0, L_{-1}, G_{1/2}, G_{-1/2}$ form a closed superalgebra, while in the fermionic sector adding F_0 to the three bosonic generators, the infinite dimensional superalgebra follows directly.

3.1.2 Physical states

Once performed the quantization of the system, the Fock space, created by the action of creation operators on the ground states, has to be restricted to a space containing only physical states. The Fock space built up by the oscillators α^μ_m, d^μ_m and

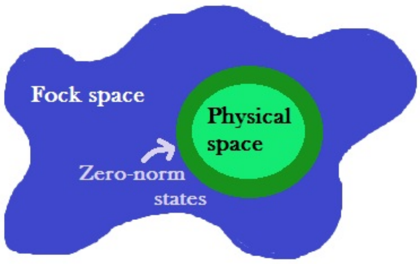

Figure 2: The Fock space is composed by a physical space (light green), a space with extra-zero norm states on the verge of developing ghosts (dark green), and a space which includes ghosts (blu).

b^μ_r, contains negative and zero norm states that need to decouple from the theory. The negative norm states arise from the presence of the Minkowski metric in the commutation relations between oscillators.

Fortunately, the super-Virasoro algebra, described in the previous section, is an infinite-dimensional algebra whose constraint equations gives the possibility to eliminate all the negative-norm states. In a quantum system the classical statement of vanishing modes take a weaker form. The super-Virasoro operators are required to have vanishing matrix elements when evaluated between two physical states $\langle\chi| L_n |\phi\rangle = \langle\chi| G_r |\phi\rangle = \langle\chi| F_n |\phi\rangle = 0$ for $n, r \in \mathbb{Z}$. However, since $L_{-m} = L^\dagger_m, F_{-m} = F^\dagger_m, G_{-r} = G^\dagger_r$ the physical constraints reduces to the demand that the positive-frequency components annihilate physical states.

Considering for the moment the bosonic sector, the physical constraints are

$$L_n |\phi\rangle = 0 \quad n > 0$$

$$(L_0 - a) |\phi\rangle = 0$$

$$G_r |\phi\rangle = 0 \quad r > 0.$$

In the NS-sector the mass shell condition for the ground state is, as in the bosonic case, $L_0 = a$ from which the momentum of the string is related to the parameter a through the relation $k^2/2 = a$ [17]. The first excited state can be written as $G_{-1/2}\,|0;k\rangle$, in this case the on-shell condition is reflected by the request $k^2/2 = (a-1/2)$[18]. From these calculations a space free of ghosts is obtained when $a \leq 1/2$. The interesting result appears when

$$a = \tfrac{1}{2},$$

with which condition the first excited state becomes a massless vector particle and the scalar ground state a tachyon[19]. Choosing the mass shift parameter a equal to 1/2 an infinite family of zero-norm sates appears. This is due to the presence of extra zero-norm states on the boundary between the free-ghosts space and the one with ghosts. The numerous zero-norm states delineate a region with an enlarged gauge symmetry that gives the possibility to decouple completely the zero-norm states from the theory. Until now, we have faced the problem of negative-norm states, however zero-norm states represent unphysical degrees of freedom that need to decouple from the system. Considering $a = 1/2$, another family of zero-norm states is found at a certain value of the spacetime dimension. The states $|\phi\rangle = (G_{-3/2} + \lambda G_{-1/2}L_{-1})\,|\chi\rangle$ where $|\chi\rangle$ is a physical state, are zero-norm states if [20]

$$G_{1/2}\,|\phi\rangle = (2-\lambda)L_{-1}\,|\chi\rangle = 0$$

$$G_{3/2}\,|\phi\rangle = (D-2-4\lambda)\,|\chi\rangle = 0$$

and hence λ=2 and

$$D = 10.$$

In conclusion, a consistent quantization of the bosonic superstring constrains the parameters of the theory to precise values.

In the fermionic sector, the physical constraints are

$$L_n\,|\psi\rangle = 0 \quad n > 0$$

$$(F_0 - \mu)\,|\psi\rangle = 0$$

$$F_n\,|\psi\rangle = 0 \quad n > 0$$

where μ is an arbitrary constant, representing the mass shift for the fermionic ground state. Proceeding as before, the research for zero-norm states lead to the condition $\mu = 0$ [20] . This is reasonable since the super-Virasoro operators F_n do not have any ordering ambiguities in term of oscillators and, furthermore, the constant μ should be a commuting c-number that could not be add to an anticommuting variable. Another family of zero-norm states $|\psi\rangle = F_0 F_{-1}\,|\chi\rangle$ arises when

$$L_1\,|\psi\rangle = (\tfrac{1}{4}D - \tfrac{5}{2})\,|\chi\rangle = 0$$

[17] This results from the on-shell condition: $M^2 = -k^2 = \frac{1}{\alpha'}(\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n - a)$, where for the ground state there is no excited oscillator and for convenience $\alpha' = 1/2$.

[18] Since $(L_0 - a)G_{-1/2}\,|0;k\rangle = (\frac{1}{2}G_{-1/2} - aG_{-1/2})\,|0;k\rangle = (\frac{1}{2} - a)G_{-1/2}\,|0;k\rangle = 0$ this gives $k^2/2 = a - 1/2$.

[19] The arbitrary elimination of $a < 1/2$ is due to the appearance of problems with unitarity at one loop level, [1, p. 86].

[20] For more details the reader could refer to [1, p. 206].

The number of zero-norm states increases dramatically in $D = 10$ spacetime dimensions. Thus, for the fermionic sector, the mass shift constant turns out to be $\mu = 0$ and the critical spacetime dimension $D = 10$. After shifting the linear term in the anomaly, as seen in section 3.1.1, the mass shift constants are renormalized to the same value 1/2 for both sectors.

3.2 Light-Cone gauge

3.2.1 Residual gauge freedom

In the bosonic string theory, the gauge condition $h_{\alpha\beta} = \eta_{\alpha\beta}$ does not fix completely the gauge. The residual gauge freedom is reflected in the possibility of reparametrizing the light-cone coordinates, $\sigma^{\pm} \to \tilde{\sigma}^{\pm}(\sigma^{\pm})$. The action, in fact, remains invariant under conformal transformations.

The superconformal action (1) is the gauge-fixed expression of a locally supersymmetric action, that will be introduced in the next section. As in the bosonic theory, the gauge-fixed action maintains/keeps a residual gauge freedom. In order to completely fix the gauge, light-cone coordinates are used. They are defined as linear combinations of spacetime coordinates,

$$X^{\pm} \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}), \qquad \psi^{\pm} \equiv \frac{1}{\sqrt{2}}(\psi^0 \pm \psi^{D-1}.) \tag{28}$$

The scalar product between light-cone coordinates is $X \cdot Y = -X^+Y^- - X^-Y^+ + X^iY^i$, where the $\pm$-components are the longitudinal components while the i-components are the transverse components. The residual gauge freedom can be used to set a bosonic light-cone coordinate proportional to τ,

$$X^+(\sigma, \tau) = x^+ + p^+\tau.$$

For the fermionic coordinates the proper non-covariant gauge condition is

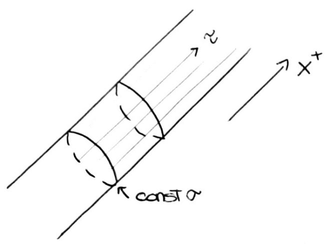

Figure 3: A closed string and two equal light-cone time slices.

$$\psi^+(\sigma, \tau) = 0.$$

As a check of consistency, under superconformal transformations the bosonic gauge condition is preserved, in fact $\delta X^+ = \bar{\epsilon}\psi^+ = 0$.

Substituting the light-cone coordinates $(X^{\pm}, X^i, \psi^{\pm}, \psi^i)$ in the super-Virasoro constraints, $T_{++} = T_{--} = J_+ = J_- = 0$, two differential equations are found

$$\psi_{\pm} \cdot \partial_{\pm} X = 0,$$

$$(\partial_\pm X)^2 + \tfrac{i}{2}\psi_\pm \cdot \partial_\pm \psi_\pm = 0.$$

From the definition of scalar product between light-cone coordinates and the gauge conditions, the equations become

$$\psi_\pm^- = \frac{2}{p^+}\psi_\pm^i \cdot \partial_\pm X^i,$$

$$\partial_\pm X^- = \frac{1}{p^+}(\partial_\pm X^i \cdot \partial_\pm X^i + \tfrac{i}{2}\psi_\pm^i \cdot \partial_\pm \psi_\pm^i).$$

From these expressions X^- and ψ^- are fixed and only the transverse oscillators, X^i, ψ^i, are left as free coordinates. The discussion will be restricted to the NS-sector, the R-sector can be reconstructed in a similar way with integer oscillating modes instead of half-integer one. Expanding in Fourier modes

$$\begin{aligned}
\psi_+^- &= \frac{2}{p^+}\psi_+^i \cdot \partial_+ X^i \\
\sum_r b_r^- e^{-ir(\tau+\sigma)} &= \frac{1}{p^+}\sum_s b_s^i e^{-is(\tau+\sigma)} \sum_n \alpha_n^i e^{-in(\tau+\sigma)} \\
b_r^- &= \frac{1}{p^+}\sum_{i=1}^{D-2}\sum_s \alpha_{r-s}^i b_s^i,
\end{aligned}$$

and from the second equation,

$$\alpha_n^- = \frac{1}{2p^+}\sum_{i=1}^{D-2}\Big(\sum_m \alpha_{n-m}^i \alpha_m^i + \sum_r (r - \tfrac{n}{2}) b_{n-r}^i b_r^i\Big).$$

3.2.2 Lorentz symmetry

Before concluding the classical discussion, it is useful to review the Lorentz symmetry of the system. The Lorentz invariance of the original system was due to the invariance of the action under the following transformations $X^\mu \to a_\nu^\mu X^\nu$, $\psi^\mu \to a_\nu^\mu \psi^\nu$. The conserved current arising from Noether's theorem is $J_\alpha^{\mu\nu} = \frac{1}{\pi}(X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu + i\bar{\psi}^\mu \rho_\alpha \psi^\nu)$, which leads to the conserved charge

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + K^{\mu\nu}$$

where

$$\begin{aligned}
l^{\mu\nu} &= x^\mu p^\nu - p^\mu x^\nu, \\
E^{\mu\nu} &= -i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu), \\
K^{\mu\nu} &= -i\sum_{r=1/2}^{\infty}(b_{-r}^\mu b_r^\nu - b_{-r}^\nu b_r^\mu), \qquad (NS) \\
K^{\mu\nu} &= -\frac{i}{2}[d_0^\mu, d_0^\nu] - i\sum_{n=1}^{\infty}(d_{-n}^\mu d_n^\nu - d_{-n}^\nu d_n^\mu). \qquad (R)
\end{aligned}$$

The Lorentz algebra is defined by the following Poisson bracket

$$\{J^{\mu\nu}, J^{\rho\lambda}\} = -\eta^{\nu\rho}J^{\mu\lambda} + \eta^{\mu\rho}J^{\nu\lambda} + \eta^{\nu\lambda}J^{\mu\rho} - \eta^{\mu\lambda}J^{\nu\rho}.$$

The ligth-cone gauge does not preserve the Lorentz invariance of the theory and the generators of the Lorentz algebra that may change the gauge choice become complicated and non-linear functions of the transversal oscillators. In the NS-sector for instance J^{i-} is

$$J^{i-} = l^{i-} + E^{i-} + K^{i-} =$$
$$= x^i p^- - p^- x^i - i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^i_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^i_n) - i\sum_{r=1/2}^{\infty}(b^i_{-r}b^-_r - b^-_{-r}b^i_r),$$

Nevertheless, all the Poisson brackets are still satisfied. The classical system, even if the gauge condition is not Lorentz invariant, still possesses Lorentz symmetry. This will not hold in general after the process of quantization.

3.2.3 Quantization

To quantize the system, the Poisson brackets are replaced by commutation or anti-commutation relations through the rule[21] $\{\,,\,\} \to -i[\,,\,]$. The canonical commutation relations for the oscillating modes are

$$[\alpha^i_m, \alpha^j_n] = m\delta_{m+n}\delta^{ij},$$
$$\{d^i_m, d^j_n\} = \delta_{m+n}\delta^{ij},$$
$$\{b^i_r, b^j_s\} = \delta_{r+s}\delta^{ij}.$$

The commutation relations are restricted to the transverse modes as unique free coordinates in the physical space. The center of mass light-cone coordinates satisfy

$$[x^-, p^+] = -i,$$
$$[x^i, p^j] = i\delta^{ij}.$$

The fundamental canonical commutation relations arise from the negative longitudinal modes, α^-_n, b^-_r, whose explicit form in the quantum system is

$$\alpha^-_n = \frac{1}{2p^+}\sum_{i=1}^{D-2}\Big(\sum_m :\alpha^i_{n-m}\alpha^i_m: + \sum_r\big(r - \tfrac{n}{2}\big): b^i_{n-r}b^i_r:\Big) - \frac{a\delta_n}{2p^+},$$

$$b^-_r = \frac{1}{p^+}\sum_{i=1}^{D-2}\sum_s \alpha^i_{r-s}b^i_s.$$

These expressions resemble the super-Virasoro operators, indeed the canonical commutation relations

$$[p^+\alpha^-_m, p^+\alpha^-_n] = (m-n)p^+\alpha^-_{m+n} + \left[\frac{D-2}{8}(m^3 - m) + 2am\right]\delta_{m+n},$$

[21] In the case of anticommuting variables the reader could refer to [2, p. 141] for a generalization of the Poisson brackets.

$$\{p^+b^-_r, p^+b^-_s\} = p^+\alpha^-_{r+s} + \left[\frac{D-2}{2}(r^2 - \tfrac{1}{4}) + 2a\right]\delta_{r+s},$$

show that the longitudinal modes satisfy the super-Virasoro algebra. Instead of the usual coefficient D, in front of the first anomaly term, a factor of $D-2$ denotes the decrease of the degrees of freedom in the theory.

The anomaly terms in these commutation relations give rise to an anomaly term in the generators of the Lorentz algebra. As mentioned before, at the classical level, even if the residual gauge freedom has been fixed through a non-covariant gauge choice, the Lorentz symmetry is preserved. When the system is quantized the anomaly terms occurring in the commutation relations bring about an anomaly term also in the Lorentz algebra, breaking the symmetry of the system.

In order to determine the Lorentz anomaly, we should consider the generators of the Lorentz algebra that do not preserve the gauge condition. Certainly, all the transverse generators preserve the Lorentz algebra, since the system is manifestly invariant under the transversal Lorentz subgroup. However, the gauge condition is not invariant under transformations generated by J^{i-} or J^{+-}. It follows that J^{i-} may have non-trivial commutation relations

$$[J^{i-}, J^{j-}] \neq 0,$$

while for the Lorentz algebra

$$[J^{\mu\nu}, J^{\rho\lambda}] = -i\eta^{\nu\rho}J^{\mu\lambda} + i\eta^{\mu\rho}J^{\nu\lambda} + i\eta^{\nu\lambda}J^{\mu\rho} - i\eta^{\mu\lambda}J^{\nu\rho},$$

it should vanish.

Defining $J^{i-} = L^{i-} + K^{i-}$ where $L^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu}$, the anomaly term is expected to be quadratic in oscillators

$$[J^{i-}, J^{j-}] = (p^+)^{-2}\sum_{m=1}^{\infty}\Delta_m(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m), \tag{31}$$

since quartic terms would appear also classically and hence they do not give any contribution to the anomaly term.

The commutation that gives rise to the anomaly term is

$$\begin{aligned}
[J^{i-}, J^{j-}] =& [L^{i-} + K^{i-}, L^{j-} + K^{j-}] \\
=& [x^i p^- - x^- p^i, x^j p^- - x^- p^j] + \\
& [x^i p^- - x^- p^i, E^{j-}] + [E^{i-}, x^j p^- - x^- p^j] + \\
& [E^{i-}, E^{j-}] + [K^{i-}, K^{j-}] + \\
& [L^{i-}, K^{j-}] + [L^{i-}, K^{j-}],
\end{aligned}$$

where $[L^{i-}, L^{j-}]$ can be written as

$$[L^{i-}, L^{j-}] = -\frac{1}{(p^+)^2}C^{ij}.$$

Defining $E^i = p^+E^{i-}$, and recalling the following commutation relations

$$\left[x^-, \frac{1}{p^+}\right] = i\frac{1}{(p^+)^2},$$

$$[x^i, E^j] = -iE^{ij}.$$

The first commutation relation in $[L^{i-}, L^{j-}]$ gives zero since the only terms that do not commute are

$$\begin{aligned}
[x^i p^- - x^- p^i, x^j p^- - x^- p^j] = & \\
&=[x^i p^-, -x^- p^j] + [-x^- p^i, x^j p^-] \\
&= - p^- x^- [x^i, -p^j] + -x^- p^- [p^i, x^j] \\
&=0.
\end{aligned}$$

The second term becomes

$$\begin{aligned}
[x^i p^- - x^- p^i, E^{j-}] = & \\
&=[x^i p^-, \tfrac{1}{p^+} E^j] - [x^- p^i, \tfrac{1}{p^+} E^j] \\
&= - i(p^-/p^+)E^{ij} - ip^i E^j (p^+)^{-2}.
\end{aligned}$$

Thus, C^{ij} takes the form

$$C^{ij} = 2ip^+ p^- E^{ij} - [E^i, E^j] - iE^i p^j + iE^j p^i. \tag{35}$$

Comparing with the expression (31) and using the commutation relations between the transversal oscillators, it is possible to find Δ_m as

$$\langle 0| \alpha^k_m C^{ij} \alpha^l_{-m} |0\rangle = m^2(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il})\Delta_m.$$

Furthermore,

$$[K^{i-}, K^{j-}] + [L^{i-}, K^{j-}] + [L^{i-}, K^{j-}] = (p^+)^{-2} \sum_{m=1}^{\infty} m(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m),$$

from

$$[K^{ij}_m, \alpha^-_n] = mK^{ij}_{m+n},$$

$$[K^{ij}_m, K^{kl}_n] = -i(K^{il}_{m+n}\delta^{jk} - K^{jl}_{m+n}\delta^{ik} - K^{ik}_{m+n}\delta^{jl} + K^{jk}_{m+n}\delta^{il}) + m(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})\delta_{m+n.}$$

The Δ_m is found to be

$$\Delta_m = m\left(1 - \frac{D-2}{8}\right) + \frac{1}{m}\left(\frac{D-2}{8} - 2a\right).$$

Thus, for general values of the parameters a and D the theory is not Lorentz invariant. Spacetime Lorentz symmetry is recovered constraining the two parameters to the values

$$D = 10 \qquad \wedge \qquad a = \tfrac{1}{2}.$$

4 Locally Supersymmetric Action

The NS-R model is described by the global supersymmetric action (1). As it has been already mentioned, however, a consistent string theory requires physical constraints that eliminate the unphysical states from the system. By analogy to the bosonic string theory, the super-Virasoro constraints should emerge as gauge condition from a gauge-invariant action. The fundamental action, therefore, needs to be invariant under local supersymmetry trasformations. The action before fixing the gauge turns out to be

$$\mathcal{S} = \mathcal{S}' + \mathcal{S}''$$

$$\mathcal{S}' = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma e\Big\{h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\Big\}$$

$$\mathcal{S}'' = -\frac{1}{\pi}\int \mathrm{d}^2\sigma e\Big\{\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \partial_\beta X_\mu + \tfrac{1}{4}\bar{\psi}^\mu \psi_\mu \bar{\chi}_\alpha \rho^\beta \rho^\alpha \chi_\beta\Big\},$$

where new fields are introduced, the gravitino χ_α and the zweibein e^a_α. A local supersymmetry transformation on the gauge-fixed action would bring about a term[22] proportional to the fermionic current J_α, whose presence can be annihilated with the introduction of the supersymmetry gauge field χ_α. Following the Noether method, a new term, corresponding to the first term inside $\mathcal{S}''$ cancels the variation of $\mathcal{S}'$ under superconformal transformation. The action of a local supersymmetry on this new term produces another term that is eliminated thanks to the second term in $\mathcal{S}''$.

Local supersymmetry transformations[23] correspond to

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \qquad \delta\psi^\mu = -i\rho^\alpha \epsilon(\partial_\alpha X^\mu - \bar{\psi}^\mu \chi_\alpha),$$

$$\delta e^a_\alpha = -2i\bar{\epsilon}\rho^a \chi_\alpha, \qquad \delta\chi_\alpha = \nabla_\alpha \epsilon.$$

Since the derivative of the gravitino does not appear in the action, its equation of motion is a constraint equation. In particular, it corresponds to the constraint equation related to the the fermionic current, as the equation of motion for the metric on the worldsheet represents the constraint equation for the stress-energy tensor.

The conserved currents of superconformal transformations are

- Supercurrent

$$J_\alpha \equiv -\frac{\pi}{2e}\frac{\delta \mathcal{S}}{\delta \chi^\alpha} = \tfrac{1}{2}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu.$$

- Stress-Energy Tensor

$$T_{\alpha\beta} \equiv -\frac{2}{\pi}\frac{1}{\sqrt{h}}\frac{\delta \mathcal{S}}{\delta h^{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu + \tfrac{i}{4}\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu + \tfrac{i}{4}\bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu - (trace).$$

The constraint equations $J_\alpha = 0$, $T_{\alpha\beta} = 0$, used in the quantization procedures, follow as gauge-invariant equations of motion of a gauge theory.

The symmetries of the action can be used to impose the so-called superconformal gauge. This amounts to setting the world-sheet metric equal to the Minkowski metric up to a conformal factor and setting the gravitino field to zero,

$$h^{\alpha\beta} = \eta^{\alpha\beta} \qquad \wedge \qquad \chi_\alpha = 0.$$

[22]Look at equation (7).

[23]For more details the reader could refer to [1, p. 228] or [2, p. 129].

5 Conclusion

The worldsheet theory in the NS-R model is a locally supersymmetric theory that includes bosonic and fermionic fields. The superstring action is the gauge-invariant version of the superconformal action, where the superconformal gauge amounts to set the worldsheet metric equal to the Minkowski metric up to a conformal factor and remove the gravitino field. The variation of the fundamental action with respect to the gauge fields gives rise to gauge-invariant equations of motion that impose constraints on the system. In particular, they represent the conserved charge densities of supersymmetry and conformal transformations, whose mode expansions are the super-Virasoro modes. The infinite dimensional algebra that emerges gives the opportunity to eliminate all the unphysical degrees of freedom in the theory. Through the process of quantization, unitarity problems or Lorentz anomaly appear. As a consequence, superstring theory becomes a consistent string theory only for particular values of the mass shift constant and critical spacetime dimension ($a = 1/2$, $D = 10$). The appeareance of tachyons is an issue that need to be solved and will find a solution in the next lecture via GSO projection.

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