

Analyzing the two dimensional Ising model with conformal field theory

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Abstract

The purpose of this report is to explain the topics discussed during my talk as part of the theoretical proseminar titled “Conformal field theory and string theory”, which was organized by Prof. Matthias Gaberdiel at the Federal Institute of Technology (ETH) in Zürich during the spring semester 2013. In the following analysis the mathematics of conformal field theory and their application to the statistical mechanical Ising model of ferromagnetism is presented. This approach allows to highlight the connections between phenomena at criticality and quantum field theory, in particular the model of a free fermion. Starting from a classical picture, the Ising model is quantized and a series of clever transformations are applied to obtain a system whose mathematical structure can be linked to that of quantum field theory. The symmetries of the derived Hamiltonian and in particular conformal invariance can then be exploited to extract the values of the critical exponents of physical quantities - such as the magnetization - matching the empirical values. In this way conformal field theory allows for an effective connection between theoretical predictions and phenomenological measurements.

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Chapter 0

Outline

This report is structured in four chapters. In the first chapter a review of classical statistical mechanics will be presented and the Ising model will be described and analyzed. In chapter two a connection between a classical and a quantum Ising model will be made. In chapter three the quantum Ising model will be mapped to a system of fermions via Jordan-Wigner transformation and the continuum limit will lead to a description in terms of free fermionic field. In chapter four the mathematics of conformal field theory will be briefly reviewed and then exploited to link the values of the conformal dimensions of a primary field describing the spins of the classical Ising model to the critical exponents of the connected correlation function. Finally, a few appendices will cover some mathematical gaps left open in the main text.

Chapter 1

Classical statistical mechanics

1.1 The partition function in classical statistics

Let us consider a system with a very high number of degree of freedoms, as for an ideal gas or a crystal lattice of spins. Given the enormous number of particles N in the system (usually around $\sim 10^{24}$) it is computationally impossible to perform exact calculations on their trajectories, even assuming that the initial position and momentum of each particle $(\mathbf{x}_i(0), \mathbf{p}_i(0))$ are known. Instead of knowing the microscopic properties of each particle, the system can be described macroscopically with functions of state, representing a statistical average calculated over a suitable ensemble of configurations of the system. It is important to note that in the thermodynamical limit $N \rightarrow \infty$ the variance of the macroscopic quantities vanishes¹ and their values become exact. In statistical mechanics the microscopic configurations (microstates) are distinguished from macroscopic states given by average quantities (macrostates): to a given macrostate described by a given set of functions of state correspond many microstates in terms of phase coordinates. Through the use of combinatorics and the *ergodic hypothesis*² one can calculate the probability that a specific microstate S is the actual state of the system. This is given by the Boltzmann distribution:

$$\mathbb{P}_i = \frac{1}{Z} e^{-\beta E_i} \quad (1.1)$$

where $\beta = \frac{1}{T}$ (the Boltzmann constant is set to $k_B = 1$ *de facto* being assimilated in T , which has now units of energy) and Z is a normalizing factor called *partition function*, which ensures that the probabilities sum up to one:

$$Z = \sum_i e^{-\beta E_i} \quad (1.2)$$

The partition function is of central meaning in statistical mechanics, since macroscopic quantities are generally related to its derivatives. For example, one may wish to calculate the inner energy U of the system:

$$U = \frac{1}{Z} \sum_i E_i e^{-\beta E_i} = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} \quad (1.3)$$

Another example is given by the *free energy* of the system, describing the energy needed to generate a system up to thermal equilibrium at temperature T :

$$F = U - TS = -T \log Z \quad (1.4)$$

¹By the law of large numbers the variance of quantity Q is proportional to $\frac{1}{\sqrt{N}}$ and hence the statistical fluctuations are negligible in the limit $N \rightarrow \infty$.

²The ergodic hypothesis states that the time average of a quantity over the time evolution of a specific microstate is equal to the average of the same quantity, at fixed time, over some statistical ensemble of microstates.

where S is the *entropy* of the system. Many physical observables are derived from the free energy. For instance, in the presence of a magnetic field h contributing to the free energy, one can calculate the magnetization M of the system as:

$$M = - \left. \frac{1}{N} \frac{\partial F}{\partial h} \right|_T \quad (1.5)$$

As seen the partition function is the generating function of many thermodynamic quantities.

1.2 The partition function in quantum statistics

If the quantum mechanical character of a system has to be taken into account, then canonical quantization of the relevant quantities has to be performed: quantum states $|\psi\rangle$ replace the classical phase coordinates, observables are represented by operators acting on the Hilbert space of such states and the classical structure given by the Poisson brackets is replaced by a quantum structure given by commutators $[\cdot, \cdot]$. Following this approach, the energy E_i is then replaced by the eigenvalue of Hamiltonian operator \hat{H} corresponding to the state $|\psi_i\rangle$ (i.e. $\hat{H}|\psi_i\rangle = E_i|\psi_i\rangle$) and the partition function Z can be regarded as a trace over eigenstates:

$$Z = \sum_n e^{-\beta E_n} = \sum_n \langle \psi_n | e^{-\beta \hat{H}} | \psi_n \rangle = \text{Tr } \rho \quad (1.6)$$

where the density operator $\rho = e^{-\beta \hat{H}}$ was introduced. The analogous quantity in Euclidean quantum field theory is the generating function:

$$Z_Q = \int \text{D}\phi e^{-S_E[\phi]/\hbar} \quad (1.7)$$

where S_E is the action depending on a set of local fields $[\phi]$. The similarity between this density operator and the time evolution operator $e^{-it\hat{H}}$ can lead to a path integral formulation of the partition function by substituting $t \rightarrow -i\tau$ (Wick rotation), where τ is a real variable going from 0 to β , and by replacing the sum with an integral:

$$Z = \int dx \rho(x_f, x_i) = \int_{(x_i, 0)}^{(x_f, \beta)} [dx] e^{-S_E[x]} \quad (1.8)$$

where $\rho(x_f, x_i)$ is the kernel of the density operator and the integration is taken over all trajectories from the initial coordinates $(x_i, 0)$ to the final coordinates (x_f, β) . This process qualitatively illustrates the connection between quantum statistical mechanics and quantum field theory, which will be discussed by simpler means not involving the path integral formalism in section 3. The connection between the two generating functions and between statistical mechanics and quantum field theory (in particular the free Majorana fermion) will be the key to connect the statistical mechanical Ising model to conformal field theory, whose language is naturally that of quantum field theory.

1.3 The Ising model

In order to deepen the relation between statistical mechanics and quantum field theory - hence allowing the use of the machinery of conformal field theory - it is useful to consider a practical example and work out the details of the transformation of a model into the other. Specifically, the Ising model provides an excellent starting point for the development of this connection. It

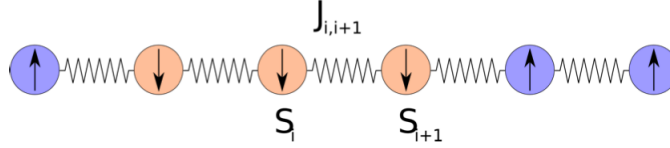


Figure 1.1: Schematic depiction of the one dimensional Ising model (Ising chain). In the image notation, S_i is the spin at the i -th position, and J_{ij} is the (site dependent) coupling between two adjacent spins [1].

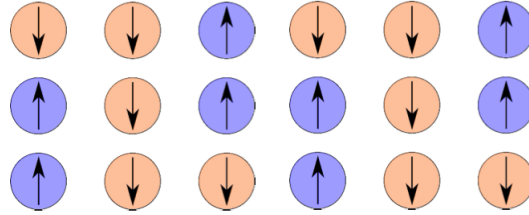


Figure 1.2: Illustration of the two dimensional Ising model on a rectangular lattice. [1].

is simple enough to be treated analytically, but complex enough to allow a simplified description of physical phenomena - such as ferromagnetism - and predict observed physical behaviors such as phase transitions at critical points. Furthermore, it can be formulated in various dimensions and generalized to more detailed models. Depending on the dimension of the model, different analytic approaches and various inherently fascinating mathematical tools can be used. Also, the Ising model can be algorithmically implemented, leading to interesting simulations used to test the solutions.

The Ising model consists of a multidimensional array of *spins* σ_i , whose value can be either 1 or -1 . These spins can be for instance regarded as atom cores, whose only property of interest is their magnetic dipole moment: in this case the model offers a simple description of ferromagnetism through exchange interaction between localized electrons of the valence band. Other applications to lattice gases consider instead an array of *bits* B_i taking value 0 or 1 depending on whether the position i is occupied or not. In this report focus will be put on the one and two dimensional spin Ising models, and other dimensionalities will only be briefly mentioned. In one dimension the aforementioned array is simply a chain of interacting spins (see figure 1). For a chain consisting of N sites there are then 2^N different spin configurations.

In the two dimensional case the spins are arranged in a rectangular lattice (figure 2). Unless otherwise indicated, a square lattice is used and i refers to the lattice site, usually numbered along the rows (for instance in a 3×3 -lattice, the central spin would be indexed by the number 5). With this indexing, for an array of $N \times N$ spins the number of different spin configurations is 2^{N^2} .

In order to determine the partition function of this system and derive other physical quantities the energy of each microstate (or configuration) σ must be calculated. In absence of any external interference, the energy between two spins must depend only on their value ($+1$ or -1) and a coupling J_{ij} describing their relative interaction as a function of their position:

$$E[\sigma] = - \sum_{i,j} J_{ij} \sigma_i \sigma_j \quad (1.9)$$

where $J_{ij} > 0$ in the ferromagnetic case, and a bound state occurs when the spins are aligned along the same direction, i.e. $\sigma_i \sigma_j = 1$. To further simplify the problem, the couplings will be assumed to be independent on the localization of the spins ($J_{ij} = J$) and the interaction will be limited to closest neighbors only. However, it is also often interesting to subject the system to an external

magnetic field h and analyze the behavior of the magnetization as h is varied, since a magnetic dipole tends to align itself along the direction of the field. The energy then acquires an extra term:

$$E[\sigma] = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \quad (1.10)$$

The notation $\langle ij \rangle$ indicates that the summation is taken only over pairs of nearest neighbor lattice site. It can be noted that in the zero field case, the lowest energy configuration is doubly degenerate: both cases when the spins are all up or all down will lead to the same energy. Only the presence of an external field lifts the degeneracy.

The main thermodynamical quantity of interest is of course the magnetization M , which is defined as the expectation value of a single magnetic dipole moment (in this case the spin). Because of translation invariance, any spin will lead to the same magnetization. Assuming a Boltzmann distribution for the probability of a single configuration the magnetization can be therefore computed as:

$$M = \langle \sigma_j \rangle = \frac{1}{Z} \sum_{[\sigma]} \left\{ \frac{1}{N} \sum_i \sigma_i \right\} e^{-\beta E[\sigma]} \quad (1.11)$$

Another important quantity of the Ising model is the *magnetic susceptibility* χ , describing the magnetic response of the system when placed in an external magnetic field h :

$$\begin{aligned} \chi &= \left. \frac{\partial M}{\partial h} \right|_{h=0} = \left. \frac{\partial}{\partial h} \frac{1}{Z} \sum_{[\sigma]} \left\{ \frac{1}{N} \sum_i \sigma_i \right\} e^{-\beta E[\sigma]} \right|_{h=0} = \\ &= \left[\frac{-1}{Z^2} \frac{\partial Z}{\partial h} \sum_{[\sigma]} \left\{ \frac{1}{N} \sum_i \sigma_i \right\} e^{-\beta E[\sigma]} + \frac{1}{Z} \sum_{[\sigma]} \left\{ \frac{1}{N} \sum_i \sigma_i \right\} e^{-\beta E[\sigma]} \left(-\beta \frac{\partial E[\sigma]}{\partial h} \right) \right]_{h=0} = \\ &= \left[-\beta \frac{1}{N} \left(\frac{1}{Z} \sum_{[\sigma]} \left\{ \sum_i \sigma_i \right\} e^{-\beta E[\sigma]} \right)^2 + \beta \frac{1}{NZ} \sum_{[\sigma]} \left\{ \sum_i \sigma_i \right\}^2 e^{-\beta E[\sigma]} \right]_{h=0} = \\ &= \frac{\beta}{N} \left[\langle \sigma_{\text{tot}}^2 \rangle - \langle \sigma_{\text{tot}} \rangle^2 \right] = \frac{1}{NT} \left[\langle \sigma_{\text{tot}}^2 \rangle - \langle \sigma_{\text{tot}} \rangle^2 \right] = \frac{1}{NT} \text{Var}(\sigma_{\text{tot}}) \end{aligned} \quad (1.12)$$

where $\sigma_{\text{tot}} = \sum_i \sigma_i$. Summarizing:

$$\boxed{\chi \propto \text{Var}(\sigma_{\text{tot}})} \quad (1.13)$$

The susceptibility is proportional to the variance of the total spin and can be thought of a measure of the fluctuations of the total magnetic dipole moment; we can therefore relate the susceptibility to the *connected pair correlation function* $\Gamma_c(i-j)$, which is a measure of the statistical dependence of two quantities:

$$\Gamma_c(i-j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \quad (1.14)$$

Using this statistical function the susceptibility can be rewritten as:

$$\chi = \beta \sum_i \Gamma_c(i) \quad (1.15)$$

The fact that the susceptibility can be expressed with correlation functions hints again to the possibility of connecting classical statistical system such as the Ising model to quantum field theories, in which the main physical quantities of interest are also correlation functions between fields, which for one degree of freedom are given by the expectation value of the time ordering operator \mathcal{T} ³ :

$$\langle x(t_1)x(t_2)\cdots x(t_n) \rangle = \langle 0 | \mathcal{T} (\hat{x}(t_1)\cdots\hat{x}(t_n)) | 0 \rangle \quad (1.16)$$

1.4 Generalizations

One of the important features of the Ising model is that it easily lends itself to generalizations. Because of the invariance of the Boltzmann distribution under a constant shift of the energy $\sigma_i\sigma_j = 2\delta_{\sigma_i,\sigma_j} - 1$ and translation invariance the configuration energy can be rewritten, up to a constant, as:

$$E[\sigma] = -2J \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j} - h \sum_i \sigma_i \quad (1.17)$$

This form can be further generalized to allow the spins σ_i to take q different non-negative integer values: $\sigma_i = 0, 1, \dots, q-1$. This multivalued spin array is then referred to as *q-state Potts model*. A $(q-1)$ -dimensional unit vector $\mathbf{d}(\sigma)$ is then associated to each possible value of σ , such that $\sum_{\sigma}^q \mathbf{d}(\sigma) = 0$ and $\mathbf{d}(\sigma)$ plays the role of the magnetic dipole moment associated with the spin value σ . The configuration energy changes to:

$$E[\sigma] = -\alpha \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j} - \mathbf{h} \cdot \sum_i \mathbf{d}(\sigma_i) \quad (1.18)$$

where α replaces the constant term J describing the type of interaction between spins.

Another, more realistic treatment of classical ferromagnetism is obtained by considering a system involving a continuum of degrees of freedom. The spins are then replaced with unit vectors \mathbf{n}_i with configuration energy:

$$E[\mathbf{n}] = J \sum_{\langle ij \rangle} \mathbf{n}_i \cdot \mathbf{n}_j - \sum_i \mathbf{h} \cdot \mathbf{n}_i \quad (1.19)$$

This is known as the classical *Heisenberg model* or the classical $O(n)$ model is the vector \mathbf{n} is taken to have n components. When dealing with critical phenomena is also more convenient to replace the lattice by a continuum, so that the above hamiltonian can be written in integral form:

$$E[\mathbf{n}] = \int d^m x \{ J \partial_k \mathbf{n} \cdot \partial_k \mathbf{n} - \mathbf{h} \cdot \mathbf{n} \} \quad (1.20)$$

in which case m is the system's dimension and the gradient term is equivalent to the nearest-neighbor interaction of the discrete case. Each vector $\mathbf{n}(x)$ is now a continuous function of position, but it is still taken to fulfill the unity requirement:

$$\mathbf{n}^2(x) = 1 \quad (1.21)$$

³The time ordering operator sorts the factors that follow in chronological order from right to left:

$$\mathcal{T} (\hat{x}(t_1)\cdots\hat{x}(t_n)) = x(t_1)x(t_2)\cdots x(t_n), \quad \text{if } t_1 > t_2 > \cdots > t_n.$$

For correlation functions, however, there exists another definition, involving path integration and the action S , which is equivalent to the former. In the context of conformal field theory the latter definition is used:

$$\langle x(t_1)x(t_2)\cdots x(t_n) \rangle = \frac{1}{Z} \int [d\mathbf{x}] x(t_1)\cdots x(t_n) e^{-S[\mathbf{x}]}.$$

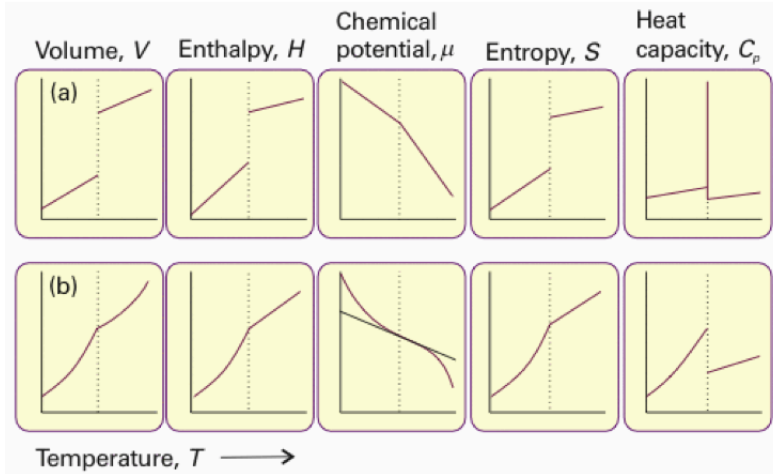


Figure 1.3: Qualitative example of phase transitions for various macroscopic quantities approaching criticality[2]. The first row refers to first order phase transitions, and the macroscopic quantities are discontinuous at the critical point. The second row represents the behavior of the same quantities under a second order phase transition, for which the derivatives of such quantities are discontinuous.

at every position x . Depending on how this constraint is implemented one can obtain various models. In particular, by introducing a quartic potential $V(|\mathbf{n}|)$ having a minimum at $|\mathbf{n}| = 1$ and after rescaling the function \mathbf{n} , the energy functional may be rewritten as:

$$E[\mathbf{n}] = \int d^m x \left\{ \frac{1}{2} \partial_k \mathbf{n} \cdot \partial_k \mathbf{n} - \frac{1}{2} \mu^2 \mathbf{n}^2 + \frac{1}{4} u (\mathbf{n}^2)^2 \right\} \quad (1.22)$$

If \mathbf{n} has a single component ϕ the model is called ϕ^4 model and in the case $u = 0$ is exactly solvable. These models are fully analogous to m-dimensional quantum field theories and can hence be studied using the Euclidean path integration formalism.

1.5 Critical phenomena and phase transitions

One of the most interesting physical features of the Ising model is that it presents a phase transition at a critical value T_c of the temperature (recall that in the convention used in this report $\beta = \frac{1}{T}$ and T is expressed in units of energy). In statistical mechanics phase transitions represent sudden changes in the macroscopic properties of the thermodynamic system (such as volume, or magnetization) as a control parameter (for instance the temperature in the two-dimensional Ising model) is varied; they are analyzed because of their common occurrence in nature and their phenomenological consequences on the physics of the underlying model. The most common example of a phase transition is offered by condensation from gaseous to liquid state, in which the volume of a substance is abruptly and discontinuously reduced. Condensation, along with evaporation, sublimation and other phase transition between solid, liquid and gaseous states of matter are termed *first order phase transitions* and usually involve a finite jump in the inner energy U of the system, which corresponds to the *latent heat* necessary to undergo the structural physical change in the system (i.e. break the weak bindings between molecules). Another type of phase transition do not involve latent heat nor discontinuities in the average value of microscopic quantities; instead, the derivatives of these quantities are discontinuous or display some kind of singular behavior (see figure 3).

In the specific case of the two-dimensional Ising model the magnetization at the critical value T_c is

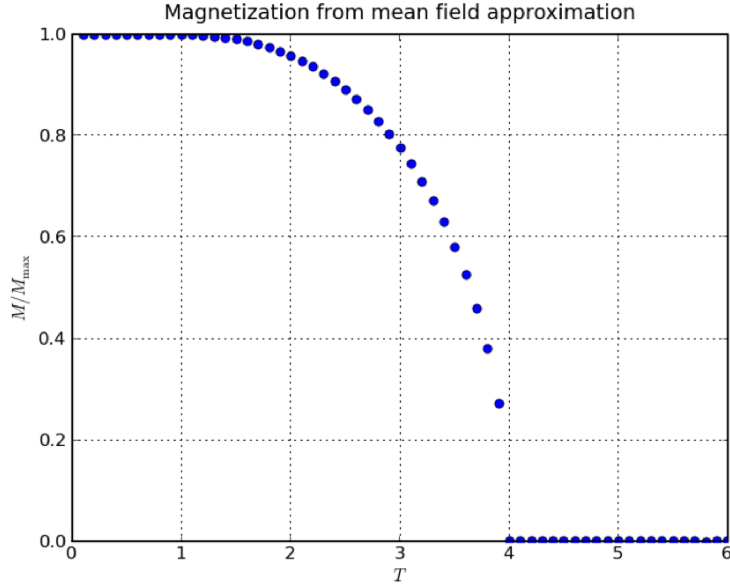


Figure 1.4: Qualitative phase diagram for the spontaneous magnetization in the Ising model. To the left of the curve the system is an ordered ferromagnetic phase, while to its right the system is in a symmetric paramagnetic state with vanishing spontaneous magnetization. It can be noted that at criticality ($T = 4$, in arbitrary units of temperature) the magnetization drops to zero continuously, but its first derivative (the susceptibility) becomes infinite, i.e. the tangent to the curve is vertical) [3].

not discontinuous, but its derivative (the magnetic susceptibility χ) exhibits such kind of divergence, as it can be gained from figure 4.

The critical temperature T_c is related to the coupling J between the spins by the *Kramers-Wannier duality relation*⁴:

$$\sinh\left(\frac{2J}{T_c}\right) = 1 \quad (1.23)$$

Explicitly the critical temperature T_c is then a function of J :

$$T_c = \frac{2J}{\log(1 + \sqrt{2})} \quad (1.24)$$

Above T_c the spontaneous magnetization (at zero field h) vanishes, whereas below T_c it takes a nonzero value, inducing the typical behavior of a ferromagnet below the Curie temperature. As $T \rightarrow T_c$ the magnetization tends towards 0 according to the power law:

$$M \sim (T_c - T)^\beta \quad (1.25)$$

with $\beta = \frac{1}{8}$. Since the physical properties of the systems and hence its phase states depend on the value for the magnetization, this quantity is termed *order parameter*: an *ordered phase* in which

⁴In the context of this report this formula will not be derived and the curious reader might want to look at the details in the original paper by Kramers and Wannier (1941). The intuitive idea behind the symmetry is to connect the edges of the finite lattice to form a torus and then apply an involutive transform on it. By considering the dual (in essence a Fourier transform) of this torus, one finds an equivalent model in which the control parameter (the temperature) is inverted, because the dual of a discrete torus is itself. The critical point will then be found when the direct and indirect temperatures cross.

the spins are aligned appears when $M \neq 0$, while $M = 0$ implies a *disordered* (or *symmetric*) phase. In this context it can also be noted that the critical point between ordered and disordered phase coincides with the breaking of the \mathbb{Z}_2 symmetry in the lattice. With “breaking of the symmetry” one should understand in this case the lifting of the symmetry of the configuration energy $E[\sigma]$ which ceases to be symmetric with respect to reversal of the spin value $\sigma_i \rightarrow -\sigma_i$. This \mathbb{Z}_2 symmetry is broken when the expectation value of a given quantity Q , which is not invariant under this symmetry operation, has a non vanishing expectation value $\langle Q \rangle \neq 0$. The magnetization $M = \langle \sigma_i \rangle$ is the simplest of such quantities and can hence be taken as the order parameter of the system, dividing ordered from symmetric phase. In contrast with the continuity of the magnetization at T_c , for the susceptibility χ one finds a divergent behavior according to the power law

$$\chi = \frac{\partial M}{\partial h} \sim (T - T_c)^{-\gamma} \quad (1.26)$$

with $\gamma = \frac{7}{4}$. Another very important parameter which describes the system around criticality is the so called *correlation length* ξ . Away from T_c the parameter ξ expresses the characteristic length at which the value of correlation function $\Gamma(i)$ has decayed to e^{-1} :

$$\Gamma(i - j) \sim e^{-\frac{|i-j|}{\xi(T)}}, \quad |i - j| \gg 1 \quad (1.27)$$

As T approaches its critical value, though, the correlation length also becomes singular:

$$\xi(T) \sim \frac{1}{|T - T_c|} \rightarrow \infty, \quad \text{as } T \rightarrow T_c \quad (1.28)$$

The divergence of the correlation length is a typical features of continuous phase transitions and a system which exhibits this type of behavior is said to undergo a *critical phenomenon*. In particular, for the case of the two-dimensional Ising model, the condition $\xi \gg a$ - where a is the lattice spacing - implies that spins at all levels of “zooming” are statistically correlated and can arrange themselves in droplets of equally oriented spins at all size scales: the system shows self-similarity and the spins fluctuate over all length scales between a and ξ . At T_c (or sufficiently close to it) the correlation length even exceeds the physical dimension L of the system: the free energy then depends only on the physical boundaries of the system. Under these circumstances the pair correlation function is forced, for lack of room, to decay algebraically with a power law

$$\Gamma(n) \sim \frac{1}{|n|^{d-2+\eta}} \quad (1.29)$$

where d is the dimension of the space. For the two-dimensional Ising model the dependence becomes then:

$$\Gamma(n) \sim |n|^{-\eta} \quad (1.30)$$

The exponents in relations (1.25), (1.26), (1.28) and (1.29) are called *critical exponents*⁵ and in the case of the two-dimensional Ising model can be obtained by calculations which can follow different paths. Onsager derived them by means of renormalization group action, which will be mentioned in subsection 1.7. In section the values for the critical exponents will be derived using conformal field theory. Table 1 shows the values of the critical exponents for the most relevant quantities in the two-dimensional Ising model.

⁵More exactly the *undetermined* numbers β , γ etc. are called critical exponents, and their effective values are called *Ising values*. This is because (see section 1.6) the same power law can describe various cases with different values of the critical exponents and the combination of the relevant critical exponents form universality classes which can describe different and possibly physically unrelated phenomena.

Exponent	Definition	Ising value
α	$C \propto (T - T_c)^{-\alpha}$	0
β	$M \propto (T - T_c)^\beta$	1/8
γ	$\chi \propto (T - T_c)^{-\gamma}$	7/4
δ	$M \propto h^{\frac{1}{\delta}}$	15
ν	$\xi \propto (T - T_c)^{-\nu}$	1
η	$\Gamma(n) \propto n ^{2-d-\eta}$	1/4

Table 1.1: Definitions of the most common critical exponents and their exact values within the two-dimensional Ising model. Here d is the dimension of space.[7]

1.6 Scaling and universality

Following the formulation of its first proponent⁶, the notion of *universality* can be formulated as follows:

Proposition (Universality). *The closer a system is to its critical point the less the microscopic dynamical details of the system influence the value of the order parameter.*

In other words, the concept of universality is the assertion that the physical properties (such as the magnetization) stop depending on the dynamical details of the system once it has reached a value of the control parameter (such as the temperature) close enough to its critical point. Universality is displayed by systems in a scaling limit, when an increasingly greater number of system components come to interact. Since macroscopic properties cease to depend on the microscopic details of the model, it is feasible that close to criticality various models come to share the same set of features and phase transitions and in practice this actually (and maybe surprisingly often) happens. Surprisingly, unrelated phenomena such as coffee percolation or liquid opalescence appear to have the same set of critical exponents as the Ising model. The property of universality in a system can be proven by means of renormalization group theory, in which, under a reduction $\{s_i\} \rightarrow \{\tilde{s}_i\}$ of the state variables, a certain function Z (usually the partition function or the hamiltonian) can be rewritten in terms of the new state variables $\{\tilde{s}_i\}$ simply by transforming the coupling constants $\{J_k\} \rightarrow \{\tilde{J}_k\}$ (see 1.7). The most important application of universality for the Ising model is however invariance with respect to “regrouping” of spins, i.e. invariance with respect to the underlying size of the lattice. This invariance is a direct consequence of the so called *Widom’s scaling hypothesis*, which can be formulated in the following way:

Scaling Hypothesis (Widom). *The free energy density (or the free energy per site, in the discrete case) near the critical point is a homogeneous function of its parameters, the external field h and the reduced temperature $t = \frac{T}{T_c} - 1$. In particular, there are exponents a and b such that:*

$$f(\lambda^a t, \lambda^b h) = \lambda f(t, h) \quad (1.31)$$

Widom’s hypothesis is of course a particular case of universality, since the control parameters h and t are not microscopic quantities and the energy density f is the macroscopic quantity which becomes

⁶Leo Kadanoff in its 1966 paper “Scaling Laws for Ising Models Near Critical Points” [?] actually formulated this assertion in terms of relevant, irrelevant and marginal observables. Any microscopic phenomenon may be categorized into a set of universality classes in which only the behavior of relevant observables fully describes the characteristic features of each class and the difference between physical systems is attributable to irrelevant observables.

independent of the dynamical details at the critical point. Widom's hypothesis can and will be proven in section 1.7, by considering the invariance of the Hamiltonian as the spins are regrouped into larger blocks (this transformation leads to the aforementioned renormalization group). In this section the attention will be concentrated on the consequences on critical exponents stemming from this scaling.

First of all the homogeneity of equation (1.31) implies the invariance of the function $t^{-\frac{1}{a}}f$ under the scalings $t \rightarrow \lambda^a t$ and $h \rightarrow \lambda^b h$. The free energy can therefore be rewritten as

$$f(t, h) = t^{\frac{1}{a}} g(y), \quad y = ht^{-\frac{b}{a}} \quad (1.32)$$

where g is some function. From this form the magnetization and the magnetic susceptibility can be easily computed to be:

$$M = -\left. \frac{\partial f}{\partial h} \right|_{h=0} = t^{\frac{1-b}{a}} g'(0) \quad (1.33)$$

$$\chi = \left. \frac{\partial^2 f}{\partial h^2} \right|_{h=0} = t^{\frac{1-2b}{a}} g''(0) \quad (1.34)$$

Furthermore, the behavior of M at criticality is given by the power law in table 1 and this implies an analogous proportionality for the function $g(y)$:

$$M \sim h^{\frac{1}{\delta}}, \quad \text{as } t \rightarrow 0 \quad \Rightarrow \quad g(y) \sim y^{\frac{1}{\delta}}, \quad \text{as } y \rightarrow \infty \quad (1.35)$$

By comparing these asymptotic relations with the power laws in table 1 the critical exponents can be linked to the values a and b of the scalings through a set of four constraints:

$$\begin{aligned} \alpha &= 2 - \frac{1}{a} \\ \beta &= \frac{1-b}{a} \\ \gamma &= -\frac{1-2b}{a} \\ \delta &= \frac{b}{1-b} \end{aligned} \quad (1.36)$$

If it were somehow possible to express the scaling exponents a and b in terms of the critical exponents ν and η , the behavior of the directly physically measurable quantities M and χ at criticality could be determined by the power laws governing the behavior of the correlation length and the correlation function. As it happens, it *is* possible to relate scaling and critical exponents; this procedure comes about naturally by proving Widom's scaling hypothesis, which will be done in the next section.

1.7 Block spins and renormalization group

As repeatedly stated in the previous sections the energy of a spin configuration is given by:

$$E[\sigma] = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \quad (1.37)$$

In order to obtain the scaling factor λ and its powers an effective scaling is introduced in the system by regrouping the spins into blocks of side r containing r^d spins⁷. Accordingly one can define a

⁷In the case of an hyper cubic lattice there are r^d spins in a hypercube of side r . In two dimensions this translates for instance into 2×2 or 3×3 groupings and so on.

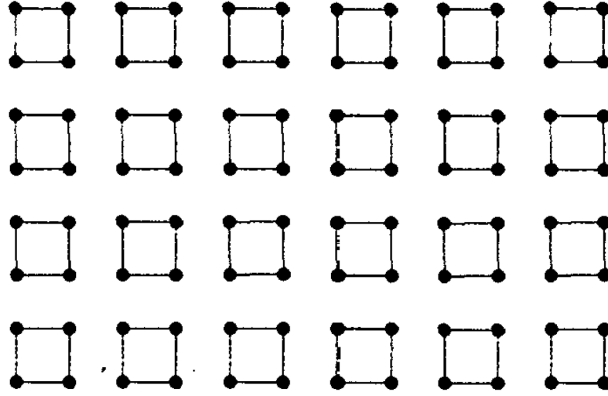


Figure 1.5: Example of block spin regrouping in the Ising model. The numbers of degrees of freedom are reduced by grouping the spins into blocks of side r to form a single site variable. [7].

block spin variable Σ_I which encompasses all spins in the block, but which is again normalized to be either 1 or -1 :

$$\Sigma_I = \frac{1}{R} \sum_{i \in I} \sigma_i \quad (1.38)$$

Here the factor R is introduced to guarantee that the block spin has a value $\Sigma_I = \pm 1$.

Assuming that the macroscopic behavior near the critical point can be accounted equally well by a description in terms of block spins, then the nearest-block Hamiltonian H' must have the same mathematical form as the original Hamiltonian, only with new couplings J' and h' :

$$H' = -J' \sum_{\langle IJ \rangle} \Sigma_I \Sigma_J - h' \sum_I \Sigma_I \quad (1.39)$$

The correlation length of the blocks is also reduced by a factor r : $\xi' = \frac{\xi}{r}$, which implies a scaling of the reduced temperature by a factor $r^{\frac{1}{\nu}}$ (recall the proportionality relation (1.28) at criticality, which links the correlation length to the reduced temperature):

$$t' = r^{\frac{1}{\nu}} t \quad (1.40)$$

Also, the interaction energy with the external field should not change when going from the separate spin to the block spin description:

$$h \sum_i \sigma_i = h' \sum_I \Sigma_I = h' R^{-1} \sum_i \sigma_i \quad (1.41)$$

which implies:

$$h' = Rh \quad (1.42)$$

In addition to the invariance of the interaction energy under grouping procedure, the total free energy should remain unaffected too. Nevertheless, the free energy *per site* will increase r^d times:

$$f(t', h') = r^d f(t, h) \quad (1.43)$$

or equivalently, inserting the newfound parameters t' and h' :

Rushbrooke's law	$\alpha + 2\beta + \gamma = 2$
Widom's law	$\gamma = \beta(\delta - 1)$
Fisher's law	$\gamma = \nu(2 - \eta)$
Josephson's law	$\nu d = 2 - \alpha$

Table 1.2: Summary of the scaling laws [7].

$$f(t, h) = r^{-d} f(r^{\frac{1}{\nu}} t, Rh) \quad (1.44)$$

To complete the picture and find the scaling factor for the external field parameter h one can look at the block-spin correlation function:

$$\begin{aligned} \Gamma'(n) &= \langle \Sigma_I \Sigma_J \rangle - \langle \Sigma_I \rangle \langle \Sigma_J \rangle = R^{-2} \sum_{i \in I} \sum_{j \in J} \{ \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \} = \\ &= R^{-2} r^{2d} \Gamma(rn) = \frac{R^{-2} r^{2d}}{|rn|^{d-2+\eta}} = \frac{R^{-2} r^{d+2-\eta}}{|n|^{d-2+\eta}} \end{aligned} \quad (1.45)$$

$$R = r^{(d+2-\eta)/2} \quad \Rightarrow \quad h' = r^{(d+2-\eta)/2} h \quad (1.46)$$

In terms of the critical exponents ν and η Widom's scaling hypothesis now reads:

$$\boxed{f(r^{\frac{1}{\nu}} t, r^{\frac{d+2-\eta}{2}} h) = r^d f(t, h)} \quad (1.47)$$

Letting $r = \lambda^{\frac{1}{d}}$:

$$a = \frac{1}{\nu d}, \quad b = \frac{d+2-\eta}{2d} \quad (1.48)$$

Eventually, with the help of Widom's scaling, all the critical exponents α through δ could be expressed in terms of the critical exponents pertaining to the correlation length and to the connected correlation function (see table 2). This is a very important result, because the correlation functions are the quantities which can directly related to quantum field theory and whose value can be obtained using conformal field theory and operator product expansions.

Chapter 2

Quantum statistical mechanics

Chapter 1 presented the main features of the classical Ising model and there it was discovered that the behavior of all macroscopical quantities near criticality can be related to the power laws governing the form of the correlation length and the correlation function. The aim of chapter 2 is to further deepen this analysis and reduce a classical $(d+1)$ -dimensional to a quantum d -dimensional Ising model, which will then be transformed into an equivalent quantum field theory in chapter 3 and whose correlation functions will be determined by exploiting conformal invariance. Section 2.1 will introduce the quantum mechanical Ising chain in terms of Pauli matrices, whose properties will then be summarized in section 2.2. Since the procedure linking classical to quantum models in arbitrary dimensions is mathematically involved, it will be performed only in two classical (respectively one quantum) dimensions, and the general result will just be inducted from this particular case. It should be noted that, technically speaking, the result which is being chased after in this chapter is a connection *from* the classical two-dimensional Ising model *to* the quantum one-dimensional one, since the aim of the transformation is to describe the *classical* Ising model at criticality with the tools of conformal field theory. Nevertheless, this report will follow a different and mathematically less demanding approach: in section ?? the one-dimensional quantum system will be used as a starting point to be connected to a classical anisotropic two-dimensional Ising lattice. A converse to this proposition will then just be inferred at the end of this chapter and briefly proved in appendix B.

2.1 The quantum mechanical 1D Ising model

The quantum mechanical counterpart of the classical Ising model can be obtained from the canonical quantization mentioned in chapter 1. Following the guidelines for the quantization of the configuration energy, the Hamiltonian for two interacting spins will have the same form as equation (1.10):

$$\hat{H}_{ij} = -J\hat{\sigma}_i \cdot \hat{\sigma}_j \quad (2.1)$$

but now the observables are replaced by operators, which have a representation in terms of *Pauli spin matrices* σ_i :

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.2)$$

The total Ising system in the presence of an external field h will then be governed by the quantum Hamiltonian $\mathcal{H} = H_0 + H_1$, where:

$$H_0 = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z \quad (2.3)$$

$$H_1 = -h \sum_i \sigma_i^x$$

$$\mathcal{H} = H_0 + H_1 = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x \quad (2.4)$$

Because of the action of the external magnetic field h on the Pauli matrix for the x-component of the spin, the underlying model of this Hamiltonian is often called *quantum Ising chain in a transverse field*.

2.2 Pauli matrices

Before analyzing the possibilities of linking the classical Ising model to its quantum counterpart, it is a good idea to refresh the basic algebraic properties of the Pauli matrices, since they will come in handy in the next section. The Pauli matrices obey the following commutation and anticommutation relations, which can be quickly proven by simple matrix multiplication¹:

$$\begin{aligned} [\sigma^a, \sigma^b] &= 2i \epsilon_{abc} \sigma^c \\ \{\sigma^a, \sigma^b\} &= 2\delta_{ab} \mathbf{1} \end{aligned} \quad (2.5)$$

Furthermore, the Pauli matrices are involutory:

$$(\sigma^x)^2 = (\sigma^y)^2 = (\sigma^z)^2 = \mathbf{1} \quad (2.6)$$

Since the determinant of the Pauli matrices is always -1 and the trace is always zero, their eigenvalues are eigenvalues are quickly determined to be ± 1 :

$$\begin{aligned} \det(\sigma^a) &= \lambda_- \lambda_+ = -1 \\ \text{Tr}(\sigma^a) &= \lambda_- + \lambda_+ = 0 \\ \Rightarrow \quad \lambda_+ &= 1, \quad \lambda_- = -1 \end{aligned} \quad (2.7)$$

The associated eigenfunctions are:

$$\begin{aligned} |+_x\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & |-_x\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ |+_y\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, & |-_y\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\ |+_z\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & |-_z\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \quad (2.8)$$

Lastly, by associating the eigenstate $|S^a\rangle$ to the Pauli matrix σ^a , one gets the completeness relations:

$$\sum_{S^a=\pm 1} |S^a\rangle \langle S^a| = \mathbf{1} \quad (2.9)$$

In particular for the z -component this means:

$$|+_z\rangle \langle +_z| + |-_z\rangle \langle -_z| = \mathbf{1} \quad (2.10)$$

¹Note that throughout this report the Einstein summation convention is always tacitly assumed.

This completeness relation for the z -component will be of fundamental importance in the analysis performed in the next section.

2.3 Quantum to classical correspondence

When confronted with the need of evaluating the partition function Z for the quantum Ising model, one has to find a way to circumvent the obstacle of the non-commuting terms H_0 and H_1 in the total Hamiltonian, which make the terms in the exponential hard to separate. Instead of unlimbering the heavy artillery offered by the Baker-Campbell-Hausdorff formula², one could opt for a more modest but equally powerful trick, which consists in slicing the inverse temperature β in the Hamiltonian into L equal parts of length $\Delta\tau$.

$$Z = \text{Tr} e^{-\beta\mathcal{H}} = \text{Tr} [e^{-\Delta\tau\mathcal{H}} e^{-\Delta\tau\mathcal{H}} \dots e^{-\Delta\tau\mathcal{H}}] \quad (2.11)$$

Note that there are L exponentials in the product and the inverse temperature $\beta = L\Delta\tau$ defines the extent of the slicing. If the temperature is not zero (i.e. $\beta \rightarrow \infty$) the slicing can be made so thin that the non-commutation of the operators can be neglected. In the following this idea will be developed more specifically. Since the Pauli matrices σ_i^z define a complete set of eigenstates $|S_i^z\rangle$:

$$\prod_{i=1}^N \left[\sum_{S_i^z=\pm 1} |S_i^z\rangle \langle S_i^z| \right] \equiv \sum_{\{S_i^z\}} |S^z\rangle \langle S^z| = 1 \quad (2.12)$$

equation (2.11) can be expanded to:

$$Z = \sum_{\{S_{i,l}=\pm 1\}} \langle S_1^z | e^{-\Delta\tau\mathcal{H}} | S_L^z \rangle \langle S_L^z | e^{-\Delta\tau\mathcal{H}} | S_{L-1}^z \rangle \dots \langle S_2^z | e^{-\Delta\tau\mathcal{H}} | S_1^z \rangle \quad (2.13)$$

Note that in equation (2.12) there is a set of complete eigenstates for every site. Since this insertion is made at several places along the string of exponentials, a new index l will be used to label that imaginary time position:

$$\sum_{\{S_{i,l}^z\}} |S_{i,l}^z\rangle \langle S_{i,l}^z| = 1 \quad (2.14)$$

It now remains to evaluate the matrix elements of the imaginary time evolution operator. As mentioned above the Hamiltonian operators H_0 and H_1 do not commute and the usual exponential law

$$e^{H_0+H_1} = e^{H_0} e^{H_1} \quad (2.15)$$

does not hold and must be approximated. If the slicing were infinitesimally thin, that is in the limit $L \rightarrow \infty$, then one could bypass the lack of commutation by using the *Lie-Trotter formula*, which applies to any pair of square matrices A and B (this formula is proven in appendix A):

$$e^{A+B} = \lim_{L \rightarrow \infty} \left(e^{A/L} e^{B/L} \right)^L \quad (2.16)$$

However, since the slicing truncates after finitely many steps L , this formula can not be applied directly. Instead, an estimate of the error committed by the truncation must be taken into account, leading to the *Suzuki-Trotter* approximation:

²The Baker-Campbell-Hausdorff formula offers a solution to the equation $Z = \log(e^X e^Y)$ in terms of an infinite sum of nested commutators for the case of non-commutative operators X and Y

$$e^{\delta A + \delta B} = e^{\delta A} e^{\delta B} + \mathcal{O}(\epsilon) \quad (2.17)$$

where $\mathcal{O}(\epsilon) = \mathcal{O}([\delta A, \delta B])$ is the Trotter error and is of order δ^2 . The Suzuki-Trotter approximation applied to the above Hamiltonian $\mathcal{H} = H_0 + H_1$ is:

$$e^{-\Delta\tau H_0 - \Delta\tau H_1} = e^{-\Delta\tau H_0} e^{-\Delta\tau H_1} + \mathcal{O}((\Delta\tau)^2 [H_0, H_1]) \quad (2.18)$$

By using the Suzuki-Trotter approximation the matrix element becomes:

$$\langle S_{l+1}^z | e^{-\Delta\tau \mathcal{H}} | S_l^z \rangle = \langle S_{l+1}^z | e^{-\Delta\tau H_1 - \Delta\tau H_0} | S_l^z \rangle \approx \langle S_{l+1}^z | e^{-\Delta\tau H_1} e^{-\Delta\tau H_0} | S_l^z \rangle + \mathcal{O}(\epsilon) | S_l^z \rangle \quad (2.19)$$

where in this case the Trotter error is asymptotically smaller than $(\Delta\tau)^2 Jh$. This approximation holds as long as the error remains small, i.e.:

$$\begin{aligned} (\Delta\tau)^2 Jh &\ll 1 \\ L^2 &\gg \beta^2 Jh \\ L &\gg \beta\sqrt{Jh} \end{aligned} \quad (2.20)$$

which is certainly feasible, since the parameters in the last inequality are all finite at a given non-zero temperature $T = \frac{1}{\beta}$ and L can be made as large as required. Following this reasoning, the Trotter error can be ignored and the decoupled matrix element in (2.19) can be evaluated by noting that the exponential of H_0 acts on the σ^z eigenstate on the right, giving:

$$\begin{aligned} \langle S_{l+1}^z | e^{-\Delta\tau H_1} e^{-\Delta\tau H_0} | S_l^z \rangle &= \langle S_{l+1}^z | e^{-\Delta\tau H_1} e^{-\Delta\tau J \sum_{i=1}^N \sigma_{i,l}^z \sigma_{i+1,l}^z} | S_l^z \rangle \\ &= e^{-\Delta\tau J \sum_{i=1}^N S_{i,l}^z S_{i+1,l}^z} \langle S_{l+1}^z | e^{-\Delta\tau h \sum_{i=1}^N \sigma_i^x} | S_l^z \rangle \end{aligned} \quad (2.21)$$

To calculate the remaining matrix element of the σ^x exponential, one can make use of the fact that $(\sigma_i^x)^2 = \mathbb{1}$, which implies:

$$e^{\Delta\tau h \sigma_i^x} = \mathbb{1} \cosh(\Delta\tau h) + \sigma_i^x \sinh(\Delta\tau h) \quad (2.22)$$

Hyperbolic functions come about because of the absence of the imaginary unit in the exponential.

Now, to relate the quantum mechanical matrix element of the time-sliced Pauli matrix to a classical Hamiltonian, it should be required that the former be expressed in the form:

$$\langle \tilde{S}^z | e^{\Delta\tau h \sigma_i^x} | S^z \rangle \equiv \Lambda e^{\gamma \tilde{S}^z S^z} \quad (2.23)$$

This expression implicitly defines the variables Λ and γ , which can be determined simply by plugging in the two eigenstates $|+\rangle$ and $|-\rangle$, and noting that the action of the Pauli matrix σ^x along the x-direction reverses the spin eigenstate along the z-direction:

$$\begin{aligned} \langle + | e^{\Delta\tau h \sigma_i^x} | + \rangle &= \cosh(\Delta\tau h) = \Lambda e^\gamma \\ \langle - | e^{\Delta\tau h \sigma_i^x} | + \rangle &= \sinh(\Delta\tau h) = \Lambda e^{-\gamma} \end{aligned} \quad (2.24)$$

from which the parameters Λ and γ can be obtained explicitly:

$$\begin{aligned} \gamma &= -\frac{1}{2} \log(\tanh(\Delta\tau h)) \\ \Lambda^2 &= \sinh(\Delta\tau h) \cosh(\Delta\tau h) \end{aligned} \quad (2.25)$$

Equation (2.21) therefore becomes:

$$\langle S_{l+1}^z | e^{-\Delta\tau H_1} e^{-\Delta\tau H_0} | S_l^z \rangle = \Lambda^N e^{\Delta\tau J \sum_{i=1}^N S_{i,l}^z S_{i+1,l}^z + \gamma \sum_{i=1}^N S_{i,l}^z S_{i,l+1}^z} \quad (2.26)$$

It is interesting to note that the ‘‘chain coupling’’ term of the Hamiltonian H_0 led to Ising spin couplings at the same imaginary time slice, but between neighboring spatial sites, while the ‘‘field coupling’’ term H_1 led to couplings between Ising spins on the same spatial site but at different (consecutive) imaginary time slices. The final form of the partition function can be obtained by substituting the last equation into (2.11):

$$Z = \Lambda^{NL} \sum_{\{S_{i,l}^z = \pm 1\}} e^{\Delta\tau J \sum_{i=1}^N \sum_{l=1}^L S_{i,l}^z S_{i+1,l}^z + \gamma \sum_{i=1}^N \sum_{l=1}^L S_{i,l}^z S_{i,l+1}^z} \quad (2.27)$$

The result obtained for the partition function seems very complicated indeed and one might wonder why going through so much trouble to arrive at a form involving sums of exponentials of double sums. The reason appears evident if the z superscripts of the spins are dropped and if one realizes that the prefactor Λ^{NL} carries no physical significance since it does not affect the spins (it would cancel out anyway when averaged over in order to compute macroscopic quantities): equation (2.27) is nothing but the classical two-dimensional Ising model at zero field with different couplings in the i and l directions, i.e. an *anisotropic two-dimensional classical Ising model!*

$$Z_{cl} = \Lambda^{NL} \sum_{\{\sigma_{i,l}^z = \pm 1\}} e^{\tilde{\beta} J_x \sum_{i=1}^{N_x} \sum_{l=1}^{N_y} \sigma_{i,l} \sigma_{i+1,l} + \tilde{\beta} J_y \sum_{i=1}^{N_x} \sum_{l=1}^{N_y} \sigma_{i,l} \sigma_{i,l+1}} \quad (2.28)$$

Note that in this last equation σ refers to the value of the Ising spin ± 1 and *not* to the Pauli matrix (notation as in section 1.3), i.e. the classical spins are identified with the eigenvalue of the Pauli matrices:

$$\sigma_{i,l} = S_{i,l}^z \quad (2.29)$$

The spatial and imaginary time direction are identified with the classical x and y directions, thus the dimensions of the system are identified as:

$$\begin{aligned} N_x &= N \\ N_y &= L \end{aligned} \quad (2.30)$$

Furthermore, the classical inverse temperature $\tilde{\beta}$ together with the classical couplings J_x and J_y are identified as:

$$\begin{aligned} \tilde{\beta} J_x &= \Delta\tau J \\ \tilde{\beta} J_y &= \gamma \end{aligned} \quad (2.31)$$

With the identifications above, the one-dimensional quantum Ising model in a transverse field can be mapped onto a classical two-dimensional Ising model with anisotropic couplings. It is interesting how the quantum effect of the transverse field is transmuted through imaginary time-slicing into an additional classical spatial dimension, leaving the resulting classical two-dimensional Ising model field-free.

The mapping from quantum Ising model onto classical Ising model does not imply *per se* that the converse is also true. In principle, following the derivation above, there is no certainty that a given classical two-dimensional Ising model can be reduced to an equivalent quantum system. Nevertheless, this mapping is also possible and follows the same core principle of regarding the transfer from one site to another as an imaginary time evolution. This approach involves the definition of the so called *transfer matrices* (see appendix B), which originate from a clever reformulation of

the partition function Z in terms of a trace over a matrix product. These transfer matrices can then be easily generalized to be quantum operators of the form of an evolution operator, and the quantization naturally arises. The mapping from classical to quantum Ising model will also be discussed in appendix B.

As introduced at the beginning of chapter 3, the results derived in this section for the quantum Ising chain and the classical two-dimensional Ising lattice do not hold only for these specific cases, but can in fact be generalized to arbitrary dimensions: a quantum d -dimensional Ising model can be mapped to a classical $(d + 1)$ -dimensional Ising model and vice versa. This important theorem will not be proven in the framework of this report, but it is an interesting fact that is worth being aware of.

Chapter 3

Quantum field theory correspondence

In order to fully exploit the machinery of conformal field theory for statistical models at criticality, having reduced a two-dimensional classical system to a one-dimensional quantum mechanical one is not enough. Conformal field theory is naturally expressed in the language of the path-integral formulation of quantum field theory, in which correlation functions of free fields can be efficiently calculated by operator product expansions (OPE). Luckily enough, the quantum Ising chain in a transverse field discussed in section 2.3 can be mapped to one of the aforementioned free fields, which turns out to be the free Majorana fermion. However, the Ising model cannot be directly mapped to a free fermion, but it must first be mapped to a system of spinless fermions, which then in the continuum limit assumes the form of a quantum field theory. This mapping is termed *Jordan-Wigner transformation* and will be presented in the next section. If one wishes to continue to follow the exact representation in terms of spinless fermions, it is necessary to perform another transformation to a new set of fermionic operators whose number is conserved, because the Jordan-Wigner transformation outputs a Hamiltonian which does not have this property. This mapping is known as *Bogoliubov transformation* and will be analyzed in section 3.2. The continuum limit which relates the Ising model at criticality to the free fermion will then be discussed in section 3.3.

3.1 The Jordan-Wigner transformation

Before starting to talk of any transformation it is worth it to reexamine the quantum Hamiltonian for the Ising chain:

$$H_I = -J \sum_i (g \hat{\sigma}_i^x + \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z) \quad (3.1)$$

where $h = Jg$ was replaced in favor of the new coupling g , whose insertion does not affect the underlying physics of the problem but it is a simple substitution which simplifies the upcoming analysis. This Hamiltonian should be now transformed or at least connected to the Hamiltonian of a quantum field, more specifically that of a free fermion. The essential step in connecting a chain of spin- $\frac{1}{2}$ particles like the quantum one-dimensional Ising model to the free fermion is the observation that the Hilbert space of a system with a spin- $\frac{1}{2}$ degree of freedom per site can be mapped to that of spinless fermions jumping along a chain of single-orbital sites. The spin-up state of the Ising spin can then for instance be associated with an empty orbital and the spin-down state with an occupied orbital. In fact, the set of operators of both systems form a Lie algebra equivalent to $\mathfrak{su}(2)$ and their generators are respectively:

$$\hat{\sigma}_i^+ \equiv \frac{1}{2}(\hat{\sigma}_i^x + i\hat{\sigma}_i^y), \quad \hat{\sigma}_i^- \equiv \frac{1}{2}(\hat{\sigma}_i^x - i\hat{\sigma}_i^y), \quad \hat{\sigma}_i^z \quad (3.2)$$

and

$$c_i, \quad c_i^\dagger, \quad n_i \equiv c_i^\dagger c_i \quad (3.3)$$

To obtain a Lie algebra isomorphism between the two representations it is sufficient to map each of the generators for the Hilbert space of the Ising chain to another generator for the Hilbert space of the spinless fermions and check that the Lie brackets are preserved under this isomorphism. For a single site this is not a hard task: if the canonical fermion operator c_i annihilates a spinless fermion on a site i , then the qualitative picture presented above implies the following operator relation:

$$\hat{\sigma}_i^z = 1 - 2c_i^\dagger c_i \quad (3.4)$$

The operation of flipping the spin from down to up encoded in $\hat{\sigma}_i^+$ is then equivalent to the operation of the annihilation operator and creating a fermion by applying c_i^\dagger is equivalent to lowering the spin with $\hat{\sigma}_i^-$:

$$\begin{aligned} \hat{\sigma}_i^+ &= c_i \\ \hat{\sigma}_i^- &= c_i^\dagger \end{aligned} \quad (3.5)$$

With these identifications one can check that the Lie bracket for the fermionic algebra (the anti-commutator $\{\cdot, \cdot\}$) is preserved:

$$\{c_i^\dagger, c_i\} = \{\hat{\sigma}_i^-, \hat{\sigma}_i^+\} = 1 \quad (3.6)$$

Although this equivalence is possible for a single chain site, a naive generalization to the full Ising chain will fail in preserving the anticommutation relations of the Fermi algebra: while two fermionic operators on different sites anticommute (this is the fundamental requirement in constructing the Hilbert space for fermions, which are per definition antisymmetric with respect to parity), the corresponding two spin operators commute. Therefore, following the intuition of Jordan and Wigner, a new representation in terms of “strings” of operators replaces the simple single-site correspondence:

$$\begin{aligned} \hat{\sigma}_i^+ &= \prod_{j<i} (1 - 2c_j^\dagger c_j) c_i \\ \hat{\sigma}_i^- &= \prod_{j<i} (1 - 2c_j^\dagger c_j) c_i^\dagger \end{aligned} \quad (3.7)$$

Note that this is a highly nonlocal representation of the spins in terms of the fermion operators, involving all the sites to the left (conventionally) of the i -th position. This transformation can be inductively inverted by noting that the term in brackets is just the single-site spin $\hat{\sigma}_j^z$ defined above and by exploiting the involution of the Pauli operators. For instance, for the first two sites $i = 1, 2$:

$$\begin{aligned} \hat{\sigma}_1^+ &= c_1 \\ \hat{\sigma}_2^+ &= (1 - 2c_1^\dagger c_1) c_2 = \hat{\sigma}_1^z c_2 \\ \Rightarrow \hat{\sigma}_1^z \hat{\sigma}_2^+ &= \hat{\sigma}_1^z \hat{\sigma}_1^z c_2 = \mathbb{1} c_2 = c_2 \end{aligned} \quad (3.8)$$

and analogously for c_1^\dagger, c_2^\dagger . Thus the general formula is:

$$\begin{aligned} c_i &= \left(\prod_{j<i} \hat{\sigma}_j^z \right) \hat{\sigma}_i^+ \\ c_i^\dagger &= \left(\prod_{j<i} \hat{\sigma}_j^z \right) \hat{\sigma}_i^- \end{aligned} \quad (3.9)$$

It can be verified that equations (3.4), (3.7) and (3.9) are consistent with the commutation and anticommutation relations for the operators:

$$\begin{aligned} \{c_i, c_j^\dagger\} &= \delta_{ij}, & \{c_i, c_j\} &= \{c_i^\dagger, c_j^\dagger\} = 0 \\ [\hat{\sigma}_i^+, \hat{\sigma}_j^-] &= \delta_{ij} \hat{\sigma}_i^z, & [\hat{\sigma}_i^z, \hat{\sigma}_j^\pm] &= \pm 2\delta_{ij} \hat{\sigma}_i^\pm \end{aligned} \quad (3.10)$$

This is the conventional formulation of the Jordan-Wigner transformation. For its particular application on the Ising model, however, it is convenient to rotate the spin axes by 90 degrees about the y -axis, so that:

$$\hat{\sigma}_i^z \rightarrow \hat{\sigma}_i^x, \quad \hat{\sigma}_i^x \rightarrow -\hat{\sigma}_i^z \quad (3.11)$$

In this frame, $\hat{\sigma}_i^z = -(\hat{\sigma}_i^+ + \hat{\sigma}_i^-)$ and the transformation becomes:

$$\boxed{\begin{aligned} \hat{\sigma}_i^x &= 1 - 2c_i^\dagger c_i \\ \hat{\sigma}_i^z &= -\prod_{j<i} (1 - 2c_j^\dagger c_j)(c_i + c_i^\dagger) \end{aligned}} \quad (3.12)$$

With this newfound expressions for the spin operators, the Hamiltonian of the quantum Ising chain can be expressed in terms of fermionic operators. Recalling that the fermionic operators are nilpotent because of the anticommutation relations, one can evaluate the new Hamiltonian as:

$$\begin{aligned} H_I &= -J \sum_i (g \hat{\sigma}_i^x + \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z) = \\ &= -J \sum_i \left[g(1 - 2c_i^\dagger c_i) + \prod_{j<i} (1 - 2c_j^\dagger c_j)(c_i + c_i^\dagger) \prod_{j<i+1} (1 - 2c_j^\dagger c_j)(c_{i+1} + c_{i+1}^\dagger) \right] \\ &= -J \sum_i \left[g(1 - 2c_i^\dagger c_i) + \prod_{j<i} (1 - 2c_j^\dagger c_j)(c_i + c_i^\dagger) \prod_{j<i} (1 - 2c_j^\dagger c_j)(1 - 2c_i^\dagger c_i)(c_{i+1} + c_{i+1}^\dagger) \right] \\ &= -J \sum_i \left[g(1 - 2c_i^\dagger c_i) + \prod_{j<i} \hat{\sigma}_j^z (c_i + c_i^\dagger) \prod_{j<i} \hat{\sigma}_j^z (c_{i+1} + c_{i+1}^\dagger - 2c_i^\dagger c_i c_{i+1} - 2c_i^\dagger c_i c_{i+1}^\dagger) \right] \\ &= -J \sum_i \left[g(1 - 2c_i^\dagger c_i) + \underbrace{\left(\prod_{j<i} \hat{\sigma}_j^z \right)^2}_{\mathbb{1}^{i-1}} (c_i + c_i^\dagger)(c_{i+1} + c_{i+1}^\dagger - 2c_i^\dagger c_i c_{i+1} - 2c_i^\dagger c_i c_{i+1}^\dagger) \right] \\ &= -J \sum_i \left[g(1 - 2c_i^\dagger c_i) + (c_i c_{i+1} + c_i c_{i+1}^\dagger + c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger - 2c_i c_i^\dagger c_{i+1} - 2c_i c_i^\dagger c_{i+1}^\dagger) \right] \\ &= -J \sum_i \left[g(1 - 2c_i^\dagger c_i) + (c_i c_{i+1} + c_i c_{i+1}^\dagger + c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger - 2c_i c_{i+1} - 2c_i c_{i+1}^\dagger) \right] \\ &= -J \sum_i \left[g(1 - 2c_i^\dagger c_i) + (-c_i c_{i+1} - c_i c_{i+1}^\dagger + c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger) \right] \\ &= -J \sum_i \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i - 2g c_i^\dagger c_i + g \right) \end{aligned} \quad (3.13)$$

or, reintroducing the field coupling h :

$$\Rightarrow \boxed{H_I = - \sum_i \left[J \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i \right) - 2hc_i^\dagger c_i + h \right]} \quad (3.14)$$

The resulting Hamiltonian is quadratic in the fermionic operators but presents terms such as $c^\dagger c^\dagger$ that violate fermion number conservation (i.e. only create or only annihilate particles, changing the total number of fermions in the systems). In other words, this means that $\sum_i \hat{\sigma}_i^x$ is not conserved and $|+_x\rangle$ spins can be flipped in pairs under time evolution. Nevertheless, since the additional terms are still quadratic in the fermionic operators, H_I can be diagonalized.

3.2 Bogoliubov transformation and exact solution

As mentioned in the previous section, although all terms in equation (3.14) are quadratic, the fermion number is not conserved. In order to circumvent this problem the idea is to apply a transformation on the fermionic operators so that the extra quadratic terms cancel out. First of all, it is necessary to diagonalize (3.14) the Hamiltonian. To that end one can apply a discrete Fourier transformation to momentum space:

$$c_k = \frac{1}{\sqrt{N}} \sum_j c_j e^{ikx} \quad (3.15)$$

where N is the number of sites in the chain, a is the lattice spacing, $x_i = ia$ is the position of the i -th site and $k = \frac{2\pi}{Na}n$ is the wave number. Depending on the boundary conditions n can take different values. For periodic boundary conditions n takes following values:

$$\begin{aligned} n &= -\frac{N-1}{2}, -\frac{N-3}{2}, \dots, 0, \dots, \frac{N-1}{2} && \text{if } N \text{ is odd} \\ n &= -\frac{N}{2} + 1, -\frac{N}{2} + 2, \dots, 0, \dots, \frac{N}{2} && \text{if } N \text{ is even} \end{aligned} \quad (3.16)$$

For antiperiodic boundary conditions the role of N is interchanged:

$$\begin{aligned} n &= -\frac{N}{2} + 1, -\frac{N}{2} + 2, \dots, 0, \dots, \frac{N}{2} && \text{if } N \text{ is odd} \\ n &= -\frac{N-1}{2}, -\frac{N-3}{2}, \dots, 0, \dots, \frac{N-1}{2} && \text{if } N \text{ is even} \end{aligned} \quad (3.17)$$

The algebraic calculation behind the transformation of the Hamiltonian to momentum space is a bit lengthy and here only the main results will be presented. The following relations hold:

$$\begin{aligned} \sum_j c_j^\dagger c_j &= \sum_k c_k^\dagger c_k \\ \sum_j c_j^\dagger c_{j+1}^\dagger &= \sum_k c_k^\dagger c_{-k}^\dagger e^{-ik} \\ \sum_j c_j c_{j+1} &= \sum_k c_k c_{-k} e^{ik} \\ \sum_j c_j^\dagger c_{j+1} &= \sum_k c_k^\dagger c_k e^{-ik} \\ \sum_j c_j c_{j+1}^\dagger &= - \sum_k c_k^\dagger c_k^\dagger e^{ik} \end{aligned} \quad (3.18)$$

As a consequence, the Hamiltonian becomes:

$$H_I = \sum_k (2[h - J \cos(ka)]c_k^\dagger c_k + iJ \sin(ka)[c_{-k}^\dagger c_k^\dagger + c_{-k}c_k] - h) \quad (3.19)$$

The next step is to map into a new set of fermionic operators $\{\gamma_k\}$ whose number is conserved. These new operators are defined by a unitary transformation, called *Bogoliubov transformation*, which is induced by an isomorphism of the anticommutation relation algebra:

$$\gamma_k = u_k c_k - i v_k c_{-k}^\dagger \quad (3.20)$$

where u_k and v_k are real numbers satisfying $u_k^2 + v_k^2 = 1$, $u_{-k} = u_k$ and $v_{-k} = -v_k$. From these conditions one has the freedom to express u_k and v_k through trigonometric functions:

$$u_k = \cos\left(\frac{\theta_k}{2}\right), \quad v_k = \sin\left(\frac{\theta_k}{2}\right) \quad (3.21)$$

Furthermore, it can be checked that this unitary transformation preserves the anticommutation relations, hence:

$$\{\gamma_k, \gamma_l^\dagger\} = \delta_{kl}, \quad \{\gamma_k^\dagger, \gamma_l^\dagger\} = \{\gamma_k, \gamma_l\} = 0 \quad (3.22)$$

By choosing

$$\tan(\theta_k) = \frac{\sin(ka)}{h/J - \cos(ka)} \quad (3.23)$$

the requirements of the Bogoliubov transformation are satisfied and the final form of the Hamiltonian is:

$$H_I = \sum_k \epsilon_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right) \quad (3.24)$$

where

$$\epsilon_k = 2(J^2 + h^2 - 2hJ \cos k)^{\frac{1}{2}} \quad (3.25)$$

takes the form of a single particle energy. Since this energy is never negative, the ground state $|0\rangle$ of the Hamiltonian has no γ fermions and therefore satisfies $\gamma_k |0\rangle = 0$ for all k . The ground state of the Ising model in an arbitrary transverse field is equivalent to the vacuum state of a spinless fermionic system. An n -particle state can then be obtained by repeatedly applying distinct creation operators and has the form $\gamma_{k_1}^\dagger \gamma_{k_2}^\dagger \cdots \gamma_{k_n}^\dagger |0\rangle$. For $g \ll 1$, the ground state is a ferromagnet (at least in the short-range order for a finite temperature) and the fermionic particles can be visualized as domain walls between the two ground states (recall that the ground state is degenerate: the spins can be all up or all down). For $g \gg 1$, where the spins are oriented along the positive x -direction in the ground state, excitations are represented by spins flipped to the negative x -direction.

3.3 Continuum limit

The calculations of the previous section showed how it is possible to transform the Hamiltonian of the quantum Ising chain into a Hamiltonian of spinless fermions. By analyzing the excitation energy ϵ_k of these fermions it is possible to find a universal continuum quantum field theory that describes the critical properties in its vicinity and obtain a description of the Ising model at criticality in

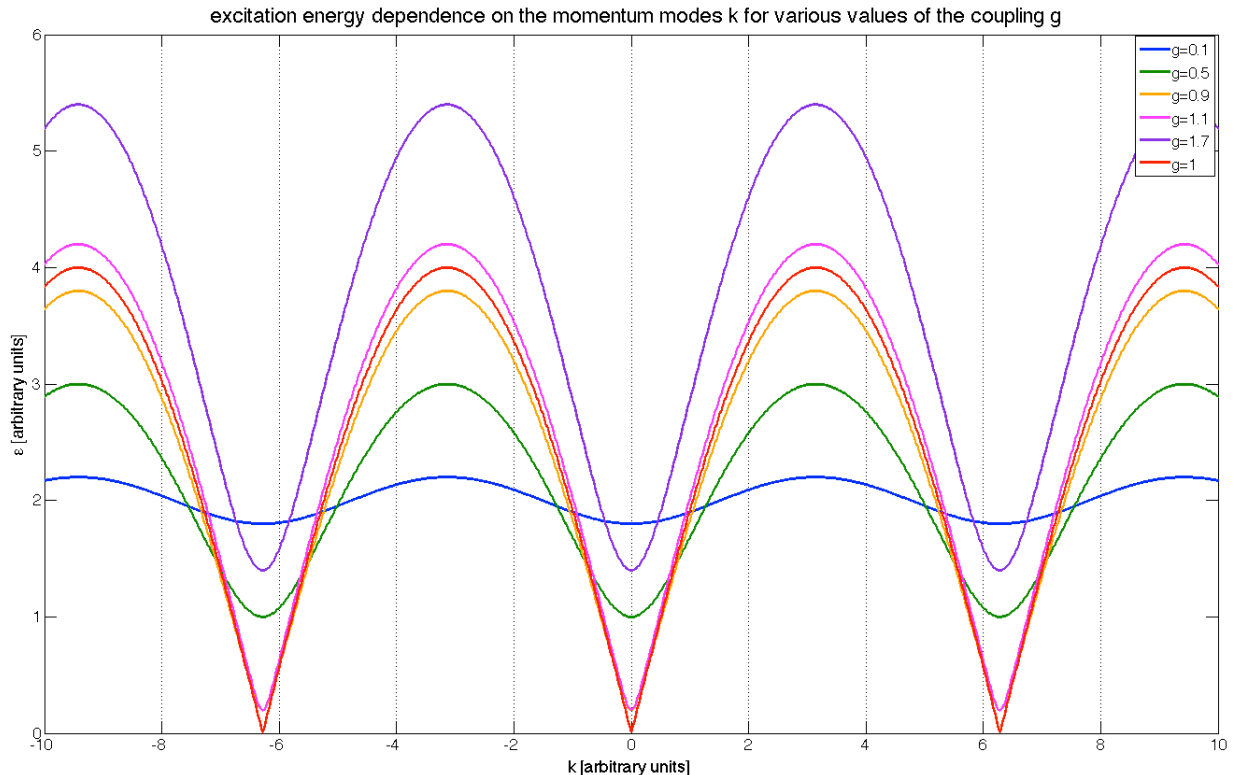


Figure 3.1: Illustration of the excitation energy for various values of the dimensionless coupling g for a one-dimensional system of spinless fermions. In this graph J was set to 1.

terms of a free fermion. First of all, it is useful to reintroduce the dimensionless parameter $g = h/J$ already used in section 3.1. The excitation energy can then be described by a g -dependence:

$$\epsilon_k = 2J\sqrt{(1 + g^2 - 2g \cos k)} \quad (3.26)$$

As it can be gained from figure 3.1, the excitation energy in equation (3.25) is always positive as long as $g \neq 1$, for which it goes to zero. The energy gap or the minimum of the excitation energy always occurs at $k = 0$ and equals:

$$\epsilon_{min} = 2J\sqrt{(1 + g^2 - 2g \cos 0)} = 2J\sqrt{((1 - g)^2)} = 2J|1 - g| \quad (3.27)$$

At $g = 1$, the energy gap vanishes and it is plausible to expect that this value of the field coupling marks the phase boundary between symmetric and ordered phase. In this regime, long wavelength excitations (i.e. excited fermions with low momenta) are possible with arbitrary low energies and therefore they must dominate the low-temperatures properties. It is hence possible to construct a continuum Hamiltonian to get rid of the irrelevant short wavelength degrees of freedom by expanding in spatial gradients. Upon taking this continuum limit, it turns out that the quantum Ising chain is also described by a *free* (also called *Gaussian*) *quantum field theory*, expressed in terms of fermions. The first step in taking this continuum limit is to replace the fermionic operators by continuum Fermi fields:

$$\Psi(x_i) = \frac{1}{\sqrt{a}}c_i \quad (3.28)$$

which in the continuum limit $a \rightarrow 0$ obey the continuum anticommutation relations, obtained simply by generalizing the Kronecker delta to a Dirac delta distribution:

$$\{\Psi(x), \Psi^\dagger(x')\} = \delta(x - x') \quad (3.29)$$

Substituting the continuum field (3.29) into the Hamiltonian and expanding it to first order in the spatial gradients one obtains:

$$\begin{aligned}
H_I &= - \sum_i \left[J \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i \right) - 2hc_i^\dagger c_i + h \right] \\
&= - \sum_i \left[a \left(J \left(\Psi^\dagger(x_i) \Psi(x_{i+1}) + \Psi^\dagger(x_{i+1}) \Psi(x_i) + \Psi^\dagger(x_i) \Psi^\dagger(x_{i+1}) + \Psi(x_{i+1}) \Psi(x_i) \right) + \right. \right. \\
&\quad \left. \left. - 2h \Psi^\dagger(x_i) \Psi(x_i) \right) + h \right] \\
&= - \int dx \left[\left(J \left(\Psi^\dagger(x) \Psi(x+a) + \Psi^\dagger(x+a) \Psi(x) + \Psi^\dagger(x) \Psi^\dagger(x+a) + \Psi(x+a) \Psi(x) \right) + \right. \right. \\
&\quad \left. \left. - 2h \Psi^\dagger(x) \Psi(x) \right) \right] + C \\
&= - \int dx \left[\left(J \left(\Psi^\dagger(x) \Psi(x) + \Psi^\dagger(x) \frac{\partial \Psi(x)}{\partial x} a + \Psi^\dagger(x) \Psi(x) + a \frac{\partial \Psi^\dagger(x)}{\partial x} \Psi(x) + \cancel{\Psi^\dagger(x) \Psi^\dagger(x)} + \right. \right. \right. \\
&\quad \left. \left. + \Psi^\dagger(x) \frac{\partial \Psi^\dagger(x)}{\partial x} a + \cancel{\Psi(x) \Psi(x)} + a \frac{\partial \Psi(x)}{\partial x} \Psi(x) \right) - 2h \Psi^\dagger(x) \Psi(x) \right] + C + \mathcal{O}(a^2) \\
&= - \int dx \left[2(J-h) \Psi^\dagger(x) \Psi(x) + a \frac{\partial}{\partial x} \left(\Psi^\dagger(x) \Psi(x) \right) + Ja \left(\Psi^\dagger(x) \frac{\partial \Psi^\dagger(x)}{\partial x} + \frac{\partial \Psi(x)}{\partial x} \Psi(x) \right) \right] + \\
&\quad C + \mathcal{O}(a^2) \\
&= E_0 - \int dx \left[2(J-h) \Psi^\dagger(x) \Psi(x) + Ja \left(\Psi^\dagger(x) \frac{\partial \Psi^\dagger(x)}{\partial x} - \Psi(x) \frac{\partial \Psi(x)}{\partial x} \right) \right] + \mathcal{O}(a^2)
\end{aligned} \quad (3.30)$$

Summarizing:

$$H_F = E_0 + \int dx \left[\frac{v}{2} \left(\Psi^\dagger \frac{\partial \Psi^\dagger}{\partial x} - \Psi \frac{\partial \Psi}{\partial x} \right) + \Delta \Psi^\dagger \Psi \right] + \mathcal{O}(a^2) \quad (3.31)$$

The coupling constants in H_F are:

$$\Delta = 2(J-h), \quad v = 2Ja \quad (3.32)$$

The coupling Δ represents the aforementioned energy gap, which vanishes for $J = h$ and distinguishes the magnetically ordered phase ($\Delta > 0$) from the symmetric paramagnetic phase ($\Delta < 0$). The equation $J = h$ is related to the Kramers-Wannier duality relation of the classical phase transition through the associations made in chapter 2 between the quantum and the classical Ising model. Note that the continuum limit $a \rightarrow 0$ is taken while holding Ψ , v and Δ fixed, which requires $J \rightarrow \infty$ and $g \rightarrow 1$, i.e. $h \rightarrow \infty$. In order to apply the machinery of conformal field theory in the next chapter, it is necessary to represent the dynamics of H_F in a path integral formalism. The derivation of the path integral formulation for a free fermion is quite involved and it requires the description of the Fock space of fermions in terms of coherent state which are inescapably related to anticommuting Grassman numbers. The full derivation will not be presented here and it can be found in [7]. In the path integral formulation the partition function is given as:

$$\mathcal{Z} = \text{Tre}^{-\frac{H_F}{T}} = \int \mathcal{D}\Psi \mathcal{D}\Psi^\dagger e^{-\int_0^{1/T} d\tau \int dx \mathcal{L}_I} \quad (3.33)$$

with the Lagrangian density:

$$\mathcal{L}_I = \Psi^\dagger \frac{\partial \Psi}{\partial \tau} + \frac{v}{2} \left(\Psi^\dagger \frac{\partial \Psi^\dagger}{\partial x} - \Psi \frac{\partial \Psi}{\partial x} \right) + \Delta \Psi^\dagger \Psi \quad (3.34)$$

Note that this Lagrangian is invariant under conformal transformations only if Δ vanishes, *de facto* connecting the system at criticality with a description in terms of conformal field theory. With a non-vanishing Δ term the Lagrangian is no longer invariant under the full conformal group, but as for the lattice model H_I , the continuum theory \mathcal{L}_I can be diagonalized to give an excitation energy in relativistic form:

$$\epsilon_k = (\Delta^2 + v^2 k^2)^{\frac{1}{2}} \quad (3.35)$$

which shows that $|\Delta|$ is the $T = 0$ energy gap and v is the velocity of the excitations (both are measurable quantities). The form of ϵ_k suggests that \mathcal{L}_I is invariant under Lorentz transformations and the action becomes what is known as the field theory of Majorana fermions of mass Δ/v^2 .

Chapter 4

Applied conformal field theory

Recall the power law for the connected correlation function of the two-dimensional Ising model at criticality:

$$\Gamma_c(r) \propto \frac{1}{r^{d-2+\eta}} = \frac{1}{r^\eta} \quad (4.1)$$

where now the letter r was chosen to symbolize a system with continuous degrees of freedom. From Onsager's exact solution of the Ising model the value of the critical exponent is known to be $\eta = \frac{1}{4}$. From the analysis conducted in chapters 2 and 3 it is known that the correlation function for the statistical mechanical model should behave as the two-point function $\langle \sigma(n)\sigma(0) \rangle$ of a primary field σ . The aim of this final chapter is to derive the matching field from the conformal field theory of the free fermion. In section 4.1 the concept of conformal invariance will be introduced and its main consequences on correlation functions will be presented. In section 4.2 these correlation functions will be described in terms of operator product expansions, which will then explicitly calculated in sections 4.3 and 4.4 for the free boson and the free fermion respectively. In the final section 4.5 all the conclusions of the previous sections will be wrapped up to allow for the primary fields describing the Ising spins to be determined along with their conformal dimensions.

4.1 Conformal invariance

A field theory has conformal symmetry if its action is invariant under conformal transformations. In d dimensions conformal transformations are those invertible mappings $\mathbf{x} \rightarrow \mathbf{x}'$ that leave the metric invariant up to a scale:

$$g'_{\mu\nu}(\mathbf{x}') = \Lambda(x)g_{\mu\nu}(x) \quad (4.2)$$

The set of conformal transformations forms a group that preserves angles and in two dimensions it can be identified as the set of all holomorphic and antiholomorphic maps, which motivates the use of complex coordinates z and \bar{z} . The requirement that an infinitesimal transformation $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(\mathbf{x})$ be conformal place it into one of the following categories:

translation :	$x'^\mu = x^\mu + a^\mu$	(4.3)
dilation :	$x'^\mu = \alpha x^\mu$	
rigid rotation :	$x'^\mu = M^\mu_\nu x^\nu$	
special conformal transformation :	$x'^\mu = \frac{x^\mu + b^\mu \mathbf{x}^2}{1 - 2\mathbf{b} \cdot \mathbf{x} + b^2 \mathbf{x}^2}$	

Under a conformal transformation $\mathbf{x} \rightarrow \mathbf{x}'$ a spinless field $\phi(\mathbf{x})$ transforms as

$$\phi(\mathbf{x}) \rightarrow \phi'(\mathbf{x}') = \left| \frac{\partial \mathbf{x}'}{\partial \mathbf{x}} \right|^{\Delta/d} \phi(\mathbf{x}) \quad (4.4)$$

and is called *quasi-primary*. The parameter Δ is called *scaling dimension* of the field. Conformal invariance has important consequences on the two-point functions of spinless quasi-primary fields, because it restricts their possible form. The two point function is defined as:

$$\langle \phi_1(\mathbf{x}_1) \phi_2(\mathbf{x}_2) \rangle = \frac{1}{Z} \int [d\Phi] \phi_1(\mathbf{x}_1) \phi_2(\mathbf{x}_2) e^{-\mathcal{S}[\Phi]} \quad (4.5)$$

where $\mathcal{S}[\Phi]$ is the action of the field theory. Assuming that the action and the functional integral measure are invariant under conformal transformations, then the correlation function transforms according to its quasi-primary fields:

$$\langle \phi_1(\mathbf{x}_1) \phi_2(\mathbf{x}_2) \rangle = \left| \frac{\partial \mathbf{x}'}{\partial \mathbf{x}} \right|_{x=x_1}^{\Delta_1/d} \left| \frac{\partial \mathbf{x}'}{\partial \mathbf{x}} \right|_{x=x_2}^{\Delta_2/d} \langle \phi'_1(\mathbf{x}_1) \phi'_2(\mathbf{x}_2) \rangle \quad (4.6)$$

Scale transformations $\mathbf{x} \rightarrow \lambda \mathbf{x}$ then imply:

$$\langle \phi_1(\mathbf{x}_1) \phi_2(\mathbf{x}_2) \rangle = \lambda^{\Delta_1 + \Delta_2} \langle \phi_1(\lambda \mathbf{x}_1) \phi_2(\lambda \mathbf{x}_2) \rangle \quad (4.7)$$

while invariance under rotations and translations, whose Jacobian is one, requires:

$$\langle \phi_1(\mathbf{x}_1) \phi_2(\mathbf{x}_2) \rangle = f(|\mathbf{x}_1 - \mathbf{x}_2|) \quad (4.8)$$

for some function f . Combining these two results the two-point function must then have the following form:

$$\langle \phi_1(\mathbf{x}_1) \phi_2(\mathbf{x}_2) \rangle = \frac{C_{12}}{(|\mathbf{x}_1 - \mathbf{x}_2|)^{\Delta_1 + \Delta_2}} \quad (4.9)$$

By requiring finally invariance under special conformal transformations the form of equation (4.9) is further restricted to:

$$\langle \phi_1(\mathbf{x}_1) \phi_2(\mathbf{x}_2) \rangle = \begin{cases} \frac{C_{12}}{(|\mathbf{x}_1 - \mathbf{x}_2|)^{2\Delta_1}}, & \text{if } \Delta_1 = \Delta_2 \\ 0 & \text{if } \Delta_1 \neq \Delta_2 \end{cases} \quad (4.10)$$

Thus, two fields are correlated only if they have the same scaling dimension. In two dimensions, when this happens, the two-point correlator becomes:

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \quad (4.11)$$

with $h = \frac{1}{2}(\Delta + s)$ *holomorphic conformal dimension* and $\bar{h} = \frac{1}{2}(\Delta - s)$ its antiholomorphic counterpart, which take into account also a possible planar spin s (not relevant for the case of the free fermion). In this two-dimensional case a quasi-primary field under a conformal map $z \rightarrow w(z)$, $\bar{z} \rightarrow \bar{w}(\bar{z})$ transforms as:

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz} \right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}} \right)^{-\bar{h}} \phi(z, \bar{z}) \quad (4.12)$$

If this holds for any *local* conformal transformation, which is certainly the case for the free fermion, the field is then termed *primary*. If the classical correlation function is to be described by a primary field σ , then by translation invariance the two-point function can be formulated as:

$$\langle \sigma(r)\sigma(0) \rangle = \frac{1}{r^{2(h_\sigma + \bar{h}_\sigma)}} \quad (4.13)$$

and by looking at equation (4.1) the following association between conformal dimension of the primary field and critical exponent at criticality in the classical model can be made:

$$\eta = 2(h_\sigma + \bar{h}_\sigma) \quad (4.14)$$

If the critical exponent of the correlation function has to take the value $\eta = \frac{1}{4}$, then the equation can be matched for $h_\sigma = \bar{h}_\sigma = \frac{1}{16}$.

4.2 Operator product expansions

In the two-point correlation functions defined in the previous section, or more specifically in their conformal dimension, is encoded the information that allows to relate statistical models at criticality to primary fields in quantum field theory. In order to derive the form of these fields it is necessary to expand the product of these fields in the correlators. This procedure is known as *operator product expansion* and will be briefly discussed in this section.

By Noether's theorem, with each symmetry of the Lagrangian is associated a conserved quantity (also called *conserved current*). The conserved current associated with invariance under an infinitesimal translation $x^\mu \rightarrow x^\mu + \omega_\alpha \frac{\partial x^\mu}{\partial \omega_\alpha}$ is given by the *energy momentum tensor*, whose components are the density and flux density of energy and momentum. If the conserved current is

$$j^\mu = \eta^{\mu\nu} \mathcal{L} \omega_\nu - \omega_\nu \partial^\nu \phi \frac{\mathcal{L}}{\partial(\partial_\mu \phi)} \quad (4.15)$$

the energy momentum tensor can be written as:

$$j^\mu = T^{\mu\nu} \omega_\nu \quad (4.16)$$

Here $\eta^{\mu\nu} = \text{diag}(1, 1)$ refers to the standard Euclidean metric. Conformal invariance with respect to translations implies:

$$\partial_\mu T^{\mu\nu} = 0 \quad (4.17)$$

while other conformal transformations force the tensor to be traceless: $T^\mu_\mu = 0$. In addition the energy-momentum tensor can be made symmetric: $T^{\rho\nu} = T^{\nu\rho}$. In two dimensions there are two parts of the energy-momentum tensor: a chiral part $2\pi T_{zz}(z, \bar{z}) = T(z)$ depending only on z and an antichiral part $2\pi \bar{T}_{\bar{z}\bar{z}}(z, \bar{z}) = \bar{T}(\bar{z})$. The energy-momentum tensor is an important field because from its operator product expansion (OPE) with the primary field $\phi(z, \bar{z})$ one can determine the conformal dimension h . The OPE is the representation of a product of operators (at position z and w respectively) by a sum of terms, each being a single operator well defined as $z \rightarrow w$, multiplied by a \mathbb{C} -number function of $z - w$. The OPE are derived from the *Ward identities*¹ and are meaningful only within correlation functions, where an appropriate time ordering is in place. Since the space and time variables are often compactified on the cylinder, the time variable taking the radial component, time ordering often translate into *radial ordering* $\mathcal{R}(\phi(z, \bar{z}), \psi(w, \bar{w}))$. For a single primary field ϕ of conformal dimensions h and \bar{h} its OPE with the energy-momentum tensor reads:

¹In quantum field theory the Ward identities can be thought of generalizations of Noether's theorem and relate important fields such as a primary field ϕ , its derivative $\partial_\mu \phi$ and the energy momentum tensor T^μ_ν to their correlation functions.

$$\begin{aligned}
T(z)\phi(w, \bar{w}) &\sim \frac{h}{(z-w)^2}\phi(w, \bar{w}) + \frac{1}{z-w}\partial_w\phi(w, \bar{w}) \\
\bar{T}(\bar{z})\phi(w, \bar{w}) &\sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w, \bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w, \bar{w})
\end{aligned}
\tag{4.18}$$

The OPE of the energy-momentum tensor T with itself is

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T}{z-w} \tag{4.19}$$

The constant c depends on the model under consideration and is called *central charge*. The central charge appears in the extension of the Witt algebra constructed by the normalized generators L_n of infinitesimal conformal transformations, which is called *Virasoro algebra* and is endowed with the following commutator playing the role of the algebra operation:

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \tag{4.20}$$

The central charge c is also known as *conformal anomaly* and is related to a conformal symmetry breaking by the introduction of a macroscopic scale into the system.

4.3 The free boson

As an example of the consequence of the OPE it is interesting to take a look at a basic but useful example: the free boson. The analysis performed on the free boson will also pave the way for the same application on the free fermion required to understand the Ising model.

The action of a free massless boson is given by:

$$S = \frac{1}{2}g \int dzd\bar{z} \{ \partial_\mu\phi(z, \bar{z})\partial^\mu\phi(z, \bar{z}) \} \tag{4.21}$$

The action can be equivalently rewritten as:

$$S = \frac{1}{2} \int d^2x d^2y \phi(\mathbf{x}) A(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) \tag{4.22}$$

with $A(\mathbf{x}, \mathbf{y}) = -g\delta(\mathbf{x} - \mathbf{y})\partial^2$ and $K(\mathbf{x}, \mathbf{y}) \equiv A(\mathbf{x}, \mathbf{y})^{-1}$.

The variational principle leads to equations of motion:

$$-g\partial_x^2 K(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \tag{4.23}$$

which can be integrated over x within a disk D of radius r centered around y to give:

$$2\pi g \int_0^r d\rho \rho \left(-\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho K'(\rho)) \right) = -2\pi g r K'(r) = 1 \tag{4.24}$$

The solution to this differential equation is, up to a constant:

$$K(r) = -\frac{1}{2\pi g} \log(r) \tag{4.25}$$

or, going back to the x and y notation:

$$\langle \phi(x), \phi(y) \rangle = -\frac{1}{4\pi g} \log(\mathbf{x} - \mathbf{y})^2 \tag{4.26}$$

In terms of complex coordinates this is:

$$\langle \phi(z, \bar{z}), \phi(w, \bar{w}) \rangle = -\frac{1}{4\pi g} \{ \log(z - w) + \log(\bar{z} - \bar{w}) \} \quad (4.27)$$

The OPE of the derivative of the primary field ϕ with itself is obtained from this expression by taking the derivatives with respect to z and \bar{z} :

$$\begin{aligned} \langle \partial_z \phi(z, \bar{z}), \partial_w \phi(w, \bar{w}) \rangle &= -\frac{1}{4\pi g} \frac{1}{(z - w)^2} \\ \langle \partial_{\bar{z}} \phi(z, \bar{z}), \partial_{\bar{w}} \phi(w, \bar{w}) \rangle &= -\frac{1}{4\pi g} \frac{1}{(\bar{z} - \bar{w})^2} \end{aligned} \quad (4.28)$$

The energy-momentum tensor associated with the free massless boson is:

$$T_{\mu\nu} = g(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \partial_\rho \phi \partial^\rho \phi) \quad (4.29)$$

and its quantum version in complex coordinates reads:

$$T(z) = -2\pi g : \partial\phi\partial\phi : \quad (4.30)$$

where the colons indicate normal ordering². The OPE of $T(z)$ with $\partial\phi$ can be calculated by Wick's theorem³ and for the holomorphic part it leads to the following result (the antiholomorphic part is obtained in the same way):

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{(z-w)} \quad (4.31)$$

Comparison of equation (4.31) with the general form in equation (4.18) allows to determine that $\partial\phi$ is a primary field with conformal dimension $h = 1$. The application of Wick's theorem to the OPE of the energy-momentum tensor with itself yields:

$$T(z)T(w) \sim \frac{\frac{1}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T}{z-w} \quad (4.32)$$

A quick comparison of this formula with the general case of equation (4.19) leads to the association $c = 1$ for the central charge of the free boson.

4.4 The free fermion

In two dimensions the Euclidean action of a free Majorana fermion is given by:

$$S = \int d^2x \mathcal{L} \quad (4.33)$$

where the Lagrangian is:

$$\mathcal{L} = \frac{1}{2} g \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi \quad (4.34)$$

²In quantum field theory a product of operators is said to be in normal ordering when all creation operators are to the left of all annihilation operators. This special ordering of field operators guarantees that the vacuum expectation value vanishes. Normal ordering is related to time ordering (and hence radial ordering) by Wick's theorem

³Wick's theorem states that the time-ordered product is equal to the normal-ordered product, plus all possible ways of contracting pairs of fields (i.e. omitting them from the normal order and replacing them with their two-point function) within it.

with the spinor $\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$ and the Dirac gamma matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (4.35)$$

which satisfies the Dirac algebra:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \quad (4.36)$$

The association with the holomorphic $\partial_0 = \partial_z$ and antiholomorphic $\partial_1 = \partial_{\bar{z}}$ derivatives can be done by writing the matrices in full:

$$\gamma^0(\gamma^0\partial_0 + \gamma^1\partial_1) = 2 \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \quad (4.37)$$

Therefore the action can be written as:

$$S = g \int d^2x \bar{\psi} \partial_z \psi + \psi \partial_{\bar{z}} \bar{\psi} \quad (4.38)$$

From this form one can obtain the classical equations of motions:

$$\partial_{\bar{z}} \psi = 0, \quad \partial_z \bar{\psi} = 0 \quad (4.39)$$

Any holomorphic (respectively antiholomorphic) satisfies the first (respectively the second) differential equation. To calculate the two-point correlation function $\langle \Psi_i(\mathbf{x}) \Psi_j(\mathbf{x}) \rangle$ the action can be expressed, in analogy with the free boson, as:

$$S = \frac{1}{2} \int d^2x d^2y \Psi_i(\mathbf{x}) A_{ij}(\mathbf{x}, \mathbf{y}) \Psi_j(\mathbf{y}) \quad (4.40)$$

with the kernel

$$A_{ij}(\mathbf{x}, \mathbf{y}) = g \delta(\mathbf{x} - \mathbf{y}) (\gamma^0 \gamma^\mu)_{ij} \partial_\mu \quad (4.41)$$

The derivation of the differential equation which determines the two-point function involves Gaussian integrals of Grassmann variables, whose understanding stretches beyond the scope of this report. Therefore, only the main result will be presented here:

$$g \delta(\mathbf{x} - \mathbf{y}) (\gamma^0 \gamma^\mu)_{ik} \frac{\partial}{\partial x^\mu} (A^{-1})_{ij}(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \delta_{ij} \quad (4.42)$$

The solution of this matrix equation provides an expression for the two-point correlators:

$$\begin{aligned} \langle \psi(z, \bar{z}) \psi(w, \bar{w}) \rangle &= \frac{1}{2\pi g} \frac{1}{z - w} \\ \langle \bar{\psi}(z, \bar{z}) \bar{\psi}(w, \bar{w}) \rangle &= \frac{1}{2\pi g} \frac{1}{\bar{z} - \bar{w}} \\ \langle \psi(z, \bar{z}) \bar{\psi}(w, \bar{w}) \rangle &= 0 \end{aligned} \quad (4.43)$$

and the OPE of the fermion with itself is (holomorphic components only):

$$\langle \psi(z) \psi(w) \rangle = \frac{1}{2\pi g} \frac{1}{z - w} \quad (4.44)$$

It is necessary to point out that this result was derived under the assumption of periodic boundary conditions for the time component defined on the cylinder; antiperiodic boundary conditions will

lead to a different result described in section 4.5. The other OPE's require the use of the energy-momentum tensor. As pointed out in the previous section, it relates to the Lagrangian by the conserved current given in Noether's theorem and its standard holomorphic component is given by its first diagonal term:

$$T(z) = -2\pi T_{zz} = -\frac{1}{2}\pi T^{\bar{z}\bar{z}} = -\frac{1}{2}\pi \frac{\partial \mathcal{L}}{\partial \bar{\partial} \Psi} \partial \Psi = -\pi g : \psi(z) \partial \psi(z) : \quad (4.45)$$

and the colons refer again to normal ordering. The OPE between T and the field ψ is calculated by using once again Wick's theorem:

$$T(z)\psi(w) = \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial \psi(w)}{z-w} \quad (4.46)$$

From this expression follows that ψ is a primary field of conformal dimension $h = \frac{1}{2}$. Wick's theorem allows also to compute the OPE of the energy-momentum tensor with itself:

$$T(z)T(w) = \frac{\frac{1}{4}}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \quad (4.47)$$

A comparison with the general form for a operator product expansion of the energy momentum tensor with itself allows to immediately determine the central charge of the corresponding Virasoro algebra: $c = \frac{1}{2}$.

4.5 The twist fields

In the previous section the holomorphic field of the free fermion ψ was found to be primary, i.e. of the form given in equation (4.12). As a holomorphic primary field, then, ψ can be expanded in a Laurent series:

$$i\psi(z) = \sum_n \psi_n z^{-n-h} \quad (4.48)$$

where the coefficients ψ_n of this expansion are given by Laurent's Theorem:

$$\psi_n = \oint \frac{dz}{2\pi i} z^{n-h} \psi(z) \quad (4.49)$$

The modes obey the anticommutation relations

$$\{\psi_n, \psi_m\} = \delta_{n,-m} \quad (4.50)$$

and act therefore as generators of the fermionic Witt's algebra represented by ψ_{-n} and ψ_n ($n > 0$), which build up a Hilbert space of state through the action of their fermionic creation and annihilation operators on the vacuum state:

$$\psi_n |0\rangle = 0, \quad \psi_{-n_1} \dots \psi_{-n_k} |0\rangle = |n_1, \dots, n_k\rangle \quad (4.51)$$

From radial quantization, which was a consequence of defining the independent variables on the cylinder, follows a periodicity on the complex plane, which allows the fermionic field to have either periodic or antiperiodic boundary conditions. This periodicity is reflected directly in the different mode indices n in the Laurent expansion of the field:

$$\begin{aligned}
\text{periodic BC :} \quad & \psi(e^{2\pi i} z) = \psi(z) \quad \Rightarrow \quad n \in \mathbb{Z} + \frac{1}{2} \\
\text{antiperiodic BC :} \quad & \psi(e^{2\pi i} z) = -\psi(z) \quad \Rightarrow \quad n \in \mathbb{Z}
\end{aligned} \tag{4.52}$$

The operators associated with the zero modes ψ_0 and $\bar{\psi}_0$ leave the conformal weight invariant when acting on a state. In case of antiperiodic boundary conditions they can be taken to represent the Virasoro algebra with defining anticommutation relations:

$$\{\psi_0, \bar{\psi}_0\}, \quad \{\psi_0, \psi_0\} = \{\bar{\psi}_0, \bar{\psi}_0\} = 1 \tag{4.53}$$

The smallest irreducible representation of this algebra consists of 2 ground states which can be labeled by the conformal dimension $h = \frac{1}{16}$: $|\frac{1}{16}\rangle_{\pm}$. The action of the corresponding operators can be represented by Pauli matrices, which fulfil up to a factor the same anticommutation relations. In this $|\frac{1}{16}\rangle_{\pm}$ basis the following connections can be then made:

$$\bar{\psi}_0 = \frac{1}{\sqrt{2}}\sigma^z, \quad \psi_0 = \frac{1}{\sqrt{2}}\sigma^x \tag{4.54}$$

and for the action:

$$\bar{\psi}_0 |1/16\rangle_{\pm} = \frac{1}{\sqrt{2}} |1/16\rangle_{\pm}, \quad \psi_0 |1/16\rangle_{\pm} = \pm \frac{1}{\sqrt{2}} |1/16\rangle_{\mp} \tag{4.55}$$

The two fields derived from $|\frac{1}{16}\rangle_{\pm}$ through the operator-state correspondence are called *twist fields* and are symbolized with σ and μ :

$$\begin{aligned}
\left| \frac{1}{16} \right\rangle_+ &= \sigma(0) |0\rangle \\
\left| \frac{1}{16} \right\rangle_- &= \mu(0) |0\rangle
\end{aligned} \tag{4.56}$$

To determine the conformal dimension of $\sigma(z)$ one may look at the vacuum expectation value of the energy momentum tensor $T(z) \frac{1}{2} : \sigma(z) \partial_z \sigma(z) :$ and use the OPE for the antiperiodic boundary conditions:

$$\begin{aligned}
\langle \psi(z) \psi(w) \rangle_A &= {}_A \langle 0 | \psi(z) \psi(w) | 0 \rangle_A = \\
&= {}_A \langle 0 | \sum_{n=0}^{\infty} z^{-n-\frac{1}{2}} \psi_n(z) \sum_{m=0}^{-\infty} w^{-m-\frac{1}{2}} \psi_m(w) | 0 \rangle_A = \\
&= \sum_{n,m=1}^{\infty} z^{-n-\frac{1}{2}} w^{m-\frac{1}{2}} \underbrace{{}_A \langle \psi_n(z) | \psi_m(w) \rangle_A}_{\delta_{m,n}} + \frac{1}{2\sqrt{zw}} = \\
&= -\frac{\frac{1}{2}(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}})}{z-w}
\end{aligned} \tag{4.57}$$

Applying this formula for the expectation value of the energy-momentum tensor one gets:

$$\frac{1}{2} \langle \sigma(z) \partial_w \sigma(w) \rangle_A = \frac{1}{2} \partial_w \langle \sigma(z) \sigma(w) \rangle_A = -\frac{1}{2(z-w)^2} + \frac{1}{16w^{\frac{3}{2}}z^{\frac{1}{2}}} \tag{4.58}$$

On the other hand the general OPE of the energy momentum tensor with a field σ is given by an expansion in terms of generators L_n of the Witt's algebra:

$$T(z)\sigma(w) = \sum_{n \geq 0} (z-w)^{n-2} L_n \sigma(w) \quad (4.59)$$

so that the vacuum expectation value for the energy-momentum tensor is

$$\langle T \rangle_A = \langle 1/16|_+ T(z) |1/16\rangle_+ = \langle 1/16|_+ \frac{1}{z^2} L_0 |1/16\rangle_+ = \frac{h_\sigma}{z^2} \quad (4.60)$$

By comparing equations (4.58) (in the limit $w \rightarrow z$) and (4.60) the conformal dimension comes out to be $h_\sigma = \frac{1}{16}$. This result is identical also for the antiholomorphic part of the fermion, so that $h_\sigma = \bar{h}_\sigma = \frac{1}{16}$, and this primary field corresponds to the quantum field theoretical description of the Ising spin operator, matching the required value of critical exponent η .

As shown in this chapter, the consequences of conformal invariance on correlation function and the operator product expansion for those correlators were able to reveal the form of the spin field $\sigma(z)$ associated with the free fermion description of the two-dimensional Ising model and therefore to match the critical exponent η of its classical correlation function Γ .

Appendix A

Proof of the Lie-Trotter formula

In this appendix a complete proof of the Lie-Trotter product formula will be given for the case in which A and B are non-commuting matrices, i.e. $A, B \in \text{Mat}_n(\mathbb{C})$. The claim is the following:

$$e^{A+B} = \lim_{n \rightarrow \infty} \left(e^{A/n} e^{B/n} \right)^n \quad (\text{A.1})$$

or, equivalently:

$$\|e^{A+B} - (e^{A/n} e^{B/n})^n\| \leq \mathcal{O}(n^{-1}), \quad \text{for } n \rightarrow \infty. \quad (\text{A.2})$$

This assertion can be proven in three steps.

First of all, the exponential are rewritten in terms of new symbols:

$$X_n \equiv e^{\frac{A+B}{n}}, \quad Y_n \equiv e^{A/n} e^{B/n}. \quad (\text{A.3})$$

By using the definition of the matrix exponential $e^X = \sum_{i=0}^{\infty} \frac{X^i}{i!}$ the two exponential terms can be expanded as a Taylor series up to first order in n^{-1} :

$$X_n = 1 + \frac{A+B}{n} + \underbrace{\frac{(A+B)^2}{2n^2} + \frac{(A+B)^3}{3!n^3} + \dots}_{\mathcal{O}(n^{-2}) \text{ as } n \rightarrow \infty} \quad (\text{A.4})$$

$$\begin{aligned} Y_n &= \left(1 + \frac{A}{n} + \underbrace{\frac{A^2}{2n^2} + \dots}_{\mathcal{O}(n^{-2}) \text{ as } n \rightarrow \infty} \right) \left(1 + \frac{B}{n} + \underbrace{\frac{B^2}{2n^2} + \dots}_{\mathcal{O}(n^{-2}) \text{ as } n \rightarrow \infty} \right) = \\ &= \left(1 + \frac{A+B}{n} + \mathcal{O}(n^{-2}) \right) \end{aligned} \quad (\text{A.5})$$

From these expansions a bound for the difference $X_n - Y_n$ as $n \rightarrow \infty$ can be immediately be found:

$$X_n - Y_n = \mathcal{O}(n^{-2}) \quad (\text{A.6})$$

The second step consists in evaluating $X_n^n - Y_n^n = e^{A+B} - e^A e^B$ for a finite n . To this end the following inequality can be used:

$$\|X^n - Y^n\| \leq n \cdot [\max(\|X\|, \|Y\|)]^{n-1} \|X - Y\| \quad (\text{A.7})$$

where $\|\cdot\|$ denotes the norm of a matrix and $X, Y \in \text{Mat}_n(\mathbb{C})$ (the indices have been dropped in this formula to simplify the notation). To prove inequality (A.7) it suffices to expand the difference $X^n - Y^n$ into a telescopic sum:

$$\begin{aligned}
X^n - Y^n &= (X^n - X^{n-1}Y) + (X^{n-1}Y - X^{n-2}Y^2) + \dots + (XY^{n-1} - Y^n) = \\
&= X^{n-1}(X - Y) + X^{n-2}(X - Y)Y + \dots + (X - Y)Y^{n-1}
\end{aligned} \tag{A.8}$$

There are n terms in this telescopic sum and each term is bounded by $[\max(\|X\|, \|Y\|)]^{n-1} \cdot \|X - Y\|$, since

$$\|X^p(X - Y)Y^q\| \leq [\max(\|X\|, \|Y\|)]^{n-1} \cdot \|X - Y\|, \quad p + q = n - 1. \tag{A.9}$$

Application of the triangle inequality and use of the submultiplicative property of the matrix norm leads to the final result:

$$\begin{aligned}
\|X^n - Y^n\| &\leq [\max(\|X\|, \|Y\|)]^{n-1} \cdot \|X - Y\| + [\max(\|X\|, \|Y\|)]^{n-1} \cdot \|X - Y\| + \dots + \\
&\quad + [\max(\|X\|, \|Y\|)]^{n-1} \cdot \|X - Y\| \leq \\
&\leq n \cdot [\max(\|X\|, \|Y\|)]^{n-1} \cdot \|X - Y\|
\end{aligned} \tag{A.10}$$

The third and final step consists in using the above inequality for $X = X_n$, $Y = Y_n$ along with the bound found in step one. To evaluate (A.7) the norm of the X_n and Y_n matrices must be estimated. Since the exp function is monotonic and again by the triangle inequality ($\|A\|$ is a real number) one finds

$$\|X_n\| \leq \|e^{\frac{A+B}{n}}\| \leq e^{\frac{\|A+B\|}{n}} \leq e^{\frac{\|A\|+\|B\|}{n}} \tag{A.11}$$

respectively

$$\|Y_n\| \leq \|e^{\frac{A}{n}}e^{\frac{B}{n}}\| \leq \|e^{\frac{A}{n}}\| \cdot \|e^{\frac{B}{n}}\| \leq e^{\frac{\|A\|}{n}}e^{\frac{\|B\|}{n}} = e^{\frac{\|A\|+\|B\|}{n}} \tag{A.12}$$

where in the first inequality the submultiplicativity of the matrix norm was again exploited. Inequality (A.7) becomes therefore

$$\|e^{A+B} - e^{A/n}e^{B/n}\| = \|X_n^n - Y_n^n\| \leq n \cdot e^{\|A\|+\|B\|} \cdot \mathcal{O}(n^{-2}) = \mathcal{O}(n^{-1}) \tag{A.13}$$

which proves the Lie-Trotter formula (equation (A.1)).

Appendix B

Classical 1D to quantum 0D correspondence in the Ising model

In chapter 2 the equivalence between a classical $(d + 1)$ -dimensional and a quantum d -dimensional Ising model was stated to be true for arbitrary dimension d . The validity of this assertion was proven in one direction and for the particular case $d = 1$ by converting - through the use of imaginary time slicing - a quantum Ising chain in a transverse field to an isotropic classical Ising lattice. In this appendix the other direction of the equality will be proven for the particular case $d = 0$, i.e. a classical Ising chain will be mapped to a single quantum Ising spin in a transverse field.

B.1 The transfer matrix

The classical Ising model treated in chapter 1 is described by the partition function

$$Z = \sum_{\{\sigma_i = \pm 1\}} e^{-\beta H} \quad (\text{B.1})$$

where σ_i denotes the Ising spin at the i -th site and H is the energy of a given spin configuration:

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \quad (\text{B.2})$$

To simplify the notation, in the following analysis the inverse temperature β will be dropped (i.e. it will be set to $\beta = 1$). If the classical system is one dimensional, i.e. a linear chain, equation (B.2) can be rewritten in terms of one index only as

$$H = -J \sum_i^M \sigma_i \sigma_{i+1} - h \sum_i^M \sigma_i \quad (\text{B.3})$$

since the nearest neighbor interaction involves just two adjacent sites in the chain. Here M is the total number of spins in the chain, which is assumed to be large. Furthermore, the system can be assumed to be periodic with $\sigma_{M+1} = \sigma_1$, *de facto* closing itself up to form a ring. This conception has no influence on the dimensionality of the problem.

Following the steps traced by Ising in his original work, the partition function can be cleverly written in terms of a trace over a matrix product, which can then be directly canonically quantized by introducing Pauli spin matrices. Since the partition function involves an exponential of a sum of integers (recall that the spins can be either $+1$ or -1), it can be equally well written as a product of exponential:

$$Z = \sum_{\{\sigma_i = \pm 1\}} e^{J \sum_i^M \sigma_i \sigma_{i+1} + h \sum_i^M \sigma_i} = \sum_{\{\sigma_i = \pm 1\}} \prod_{i=1}^M e^{J \sigma_i \sigma_{i+1}} e^{h \sigma_i} \quad (\text{B.4})$$

By defining

$$T_1(\sigma_i, \sigma_{i+1}) \equiv e^{J \sigma_i \sigma_{i+1}}, \quad T_2(\sigma_i) \equiv e^{h \sigma_i} \quad (\text{B.5})$$

the partition function obtains precisely the structure of a matrix product:

$$Z = \sum_{\{\sigma_i = \pm 1\}} \prod_{i=1}^M T_1(\sigma_i, \sigma_{i+1}) T_2(\sigma_i) \quad (\text{B.6})$$

where the possible values of the spins $((+1, +1), (+1, -1), (-1, +1), (-1, -1)$ for T_1 and ± 1 for T_2) should be interpreted as indices labeling the rows and columns of the matrices T_1 and T_2 . In particular, since there is just one index describing T_2 , this matrix should be interpreted as a diagonal matrix:

$$T_1 \equiv \begin{pmatrix} e^J & e^{-J} \\ e^{-J} & e^J \end{pmatrix}, \quad T_2 \equiv \begin{pmatrix} e^h & 0 \\ 0 & e^{-h} \end{pmatrix}. \quad (\text{B.7})$$

With this association the summation over the configurations in the partition function can be fully converted into a trace (because of the periodic boundary condition introduced earlier) and thus Z becomes:

$$Z = \text{Tr} [(T_1 T_2)^M] \quad (\text{B.8})$$

The product of the two matrices $T \equiv T_1 T_2$ is termed *transfer matrix* of the Ising chain and as the name suggests it “transfers” the trace over the spins from each site to the next one. Because of the cyclic property of the trace and since T_2 is diagonal, the partition function can also be written as

$$Z = \text{Tr} [(T_2^{1/2} T_1 T_2^{1/2})^M] \quad (\text{B.9})$$

with the square root matrix

$$T_2^{1/2} = \begin{pmatrix} e^{h/2} & 0 \\ 0 & e^{-h/2} \end{pmatrix} \quad (\text{B.10})$$

which leads to a description in terms of the eigenvalues $\epsilon_{1,2} = e^J \cosh(h) \pm (e^{2J} \sinh^2(h) + e^{-2J})^{1/2}$ of the symmetric matrix

$$T_2^{1/2} T_1 T_2^{1/2} = \begin{pmatrix} e^{J+h} & e^{-J} \\ e^{-J} & e^{J-h} \end{pmatrix} \quad (\text{B.11})$$

i.e., since the trace is the sum of the eigenvalues, Z can be written as

$$Z = \epsilon_1^M + \epsilon_2^M \quad (\text{B.12})$$

With the help of this form the two-point spin correlator (i.e. the correlation function between spins) can in turn be written as

$$\Gamma(i - i') \equiv \langle \sigma_i \sigma_{i'} \rangle = \frac{1}{Z} \text{Tr} \left(T_1^{M-i'} \hat{\sigma}^z T_1^{i'-i} \hat{\sigma}^z T_1^i \right) = \frac{\epsilon_1^{M-i'+i} \epsilon_2^{i'-i} + \epsilon_2^{M-i'+i} \epsilon_1^{i'-i}}{\epsilon_1^M + \epsilon_2^M} \quad (\text{B.13})$$

which in the limit of an infinite chain (i.e. in the thermodynamical limit $M \rightarrow \infty$) becomes:

$$\Gamma(i - i') = (\tanh(J))^{i'-i} \quad (\text{B.14})$$

B.2 Classical to quantum correspondence

In this section the classical Ising chain will be mapped onto a system of a single quantum mechanical Ising spin. Once the language of the transfer matrices has been introduced, the correspondence to a quantum system comes in naturally by canonical quantization of the spins through Pauli matrices and some considerations about the scaling of the system. Since the spins are regularly spaced by a , the discrete index i can also be written in terms of the lattice spacing:

$$\tau = ia \quad (\text{B.15})$$

and τ substitutes the discrete labeling i in the spins:

$$\sigma(\tau) \equiv \sigma_i \quad (\text{B.16})$$

This is somehow reminiscent of the procedure invoked at the end of chapter 3 for the continuum limit of the Jordan-Wigner transformation. With this notation the correlation function becomes:

$$\Gamma(\tau) \equiv \langle \sigma(\tau)\sigma(0) \rangle = e^{-\frac{|\tau|}{\xi}} \quad (\text{B.17})$$

where the correlation length ξ can be obtained from equation (B.14) with $i' = 0$:

$$\frac{\xi}{a} = \frac{1}{\ln(\coth(J))} \quad (\text{B.18})$$

In the large- J limit the logarithm in the denominator can be expanded and the correlation length ξ becomes much larger than the lattice spacing a , as already pointed out in chapter 1:

$$\frac{\xi}{a} \approx \frac{1}{2} e^{2J} \gg 1, \quad J \gg 1 \quad (\text{B.19})$$

In this limit, the transfer matrices T_1 and T_2 can be written as

$$T_1 = e^J (\mathbf{1} + e^{-2J} \hat{\sigma}^x) \approx e^J (\mathbf{1} + (a/2\xi) \hat{\sigma}^x) \approx e^{a(-E_0 + (1/2\xi) \hat{\sigma}^x)} \quad (\text{B.20})$$

respectively

$$T_2 = e^{a\tilde{h}\hat{\sigma}^z} \quad (\text{B.21})$$

where $E_0 = -J/a$ and $\tilde{h} = h/a$ are the scaled couplings and the canonical quantization $\sigma(\tau) \rightarrow \hat{\sigma}$ was also introduced. Both transfer matrices have the form e^{aO} , with an a -independent operator O acting on the eigenstates of the z -Pauli matrix. By using the Suzuki-Trotter approximation $e^{aO_1} e^{aO_2} = e^{a(O_1+O_2)} (1 + \mathcal{O}(a^2))$ (see section 2.2 and appendix A) in the limit $a \rightarrow 0$ the transfer matrix T becomes:

$$T = T_1 T_2 \approx e^{-aH_Q} \quad (\text{B.22})$$

with

$$H_Q = E_0 - \frac{1}{2\xi} \hat{\sigma}^x - \frac{\tilde{h}}{a} \hat{\sigma}^z = E_0 - \frac{\Delta}{2} \hat{\sigma}^x - \tilde{h} \hat{\sigma}^z \quad (\text{B.23})$$

Thus, one can write the partition function as

$$Z = \text{Tr} [(T_1 T_2)]^M \approx \text{Tr} (e^{-L_\tau H_Q}) \quad (\text{B.24})$$

in terms of *quantum Hamiltonian* H_Q describing the dynamics of a *single* Ising spin, whose Hilbert space consists only of the two eigenstates of the z -Pauli operator. Note that H_Q depends on an irrelevant additive constant E_0 (which does not describe any underlying physical phenomenon), a longitudinal field \tilde{h} and a *transverse* field Δ (the longitudinal field takes over the role of the spin-spin coupling which is obviously absent for a single spin). It is also interesting to mention that the transfer matrix of the classical chain is the quantum evolution operator $e^{-H_Q \tau}$ over an imaginary time $\tau = a$: as for the case analyzed in chapter 2, the transfer from one physical site to the next in the classical chain corresponds to imaginary time evolution in the quantum system.

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