Low Energy Effective Actions from String Theory

Sam Guns

ETH Zurich

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Outline

1. Introduction
   - Motivation
2. Two-dimensional CFT and the Weyl Anomaly
   - The Polyakov Action Revisited
   - Symmetries of the Polyakov Action
   - The Weyl Anomaly
3. The Weyl Anomaly Conditions
   - Expanding the Action
   - Calculating the Weyl Anomaly
   - The Emergence of Gravity

- How is Gravity incorporated in String Theory?
- What is the connection with Conformal Field Theory?

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The Polyakov Action

$$S_P = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)$$

- Equivalent to Nambu-Goto action
- Quadratic in derivatives
- Key ingredients:
  - γ^{ab}: Worldsheet Metric
  - $X^\mu(\sigma, \tau)$: Spacetime Coordinate functions
  - $G_{\mu\nu}$: Target Space Metric

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Symmetries of S_P

- Worldsheetreparametrization invariance:

$$(\sigma, \tau) \to (\tilde{\sigma}(\sigma, \tau), \tilde{\tau}(\sigma, \tau))$$

- Invariance to Weyl transformations in the Worldsheet

$$\gamma_{ab} \to e^{\phi(\sigma,\tau)} \gamma_{ab}$$

- Use reparametrization and Weyl invariance to fix degrees of freedom

Symmetries of S_P

String physics contained in embedding X^μ and should not depend on γ^{ab}. Equation of motion for the worldsheet metric

$$0 = \frac{4\pi}{\sqrt{\gamma}} \frac{\delta S_P}{\delta \gamma^{ab}} \equiv T_{ab}$$

Weyl invariance implies $T_a^a = 0$:

$$0 = \frac{\delta S}{\delta \phi} = T_a^a$$

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Anomalies in Quantum Field Theory

- Question: is a symmetry of classical physics necessarily a symmetry of quantum physics?
- Quantum fluctuations can break classical symmetries: Anomaly
- Example in the Standard Model: Chiral Anomaly

The Weyl Anomaly

- Quantizing the Polyakov action breaks Weyl invariance of classical theory.

$$\langle T^a_a \rangle \neq 0$$

- However, theory can be fixed by imposing constraints that set $\langle T^a_a \rangle$ to zero

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A More General Action

- Are there other terms we can add to the action that are
  - Reparametrisation invariant
  - Renormalisable
- Answer: three possibilities

$$S = S_P + S_{AS} + S_T + S_D$$

- Of these, only consider the Antisymmetric Tensor action:

$$S_{AS} = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma \mathrm{d}\tau \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}$$

  - ϵ^{ab} two-dimensional Levi-Civita tensor density
  - $B_{\mu\nu}$ antisymmetric spacetime coupling (generalisation of Electromagnetism to strings)

Calculating Expectation Values

The quantum Expectation Value of an operator O:

$$\langle O \rangle = < \Omega \,|\, \mathrm{T}\,(O)\, \mathrm{e}^{iS} \,|\, \Omega >$$

Expand Action in powers of small parameter
$\rightarrow$ Perturbative Expansion

Problem: choice of parameter? Can separate fields in 'background' part and small 'quantum fluctuation'

$$X^\mu = X_0^\mu + \eta^\mu$$

Some Concepts from General Relativity

In a general curved space, the ordinary partial derivative ∂_μ does not transform properly under general coordinate transformations.

$\rightarrow$ new notion of derivation to take curvature into account

- $\nabla_\mu \phi = \partial_\mu \phi$
- $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma$

$\Gamma^\nu_{\mu\sigma}$ = Christoffel Symbols or Connection Coefficients

Riemann Normal Coordinates

There exists a unique geodesic $\lambda^\mu(t)$ that connects $X_0^\mu = \lambda^\mu(0)$ with a point in the neighborhood $X_0^\mu + \pi^\mu = \lambda^\mu(1)$. The geodesic equation reads

$$\ddot{\lambda}^\mu(t) + \Gamma^\mu_{\nu\sigma}\dot{\lambda}^\nu(t)\,\dot{\lambda}^\sigma(t) = 0$$

We define

$$\eta^\mu = \dot{\lambda}^\mu(0)$$

The η^μ define a coordinate system called Riemann Normal Coordinates. For a small enough neighborhood, $\lambda^\mu(t) = \eta^\mu t$ so in RNC

$$\bar{\Gamma}^\mu_{\nu\sigma} = 0$$

The Riemann Curvature Tensor

The curvature tensor describes how two covariant derivatives commute

$$R^{\mu}_{\ \nu\alpha\beta} v^{\nu} = \left[\nabla_{\alpha} \nabla_{\beta}\right] v^{\mu}$$

and can be expressed in terms of the connection coefficients and their derivatives:

$$R^{\mu}_{\ \nu\alpha\beta} = \partial_{\alpha} \Gamma^{\mu}_{\beta\nu} - \partial_{\beta} \Gamma^{\mu}_{\alpha\nu} + \Gamma^{\mu}_{\alpha\lambda} \Gamma^{\lambda}_{\beta\nu} - \Gamma^{\mu}_{\beta\lambda} \Gamma^{\mu}_{\alpha\nu}$$

In RNC this expression can be inverted:

$$\partial_{\nu} \bar{\Gamma}^{\mu}_{\sigma\rho} = \frac{1}{3} \left(\bar{R}^{\mu}_{\sigma\nu\rho} + \bar{R}^{\mu}_{\rho\nu\sigma} \right)$$

Covariant Taylor Expansion of the Action

Covariant expansion of an arbitrary tensor in powers of η:

$$\bar{T}_{\mu\nu}(X_0+\eta) = \bar{T}_{\mu\nu}(X_0) + \frac{1}{2}\partial_\lambda \bar{T}_{\mu\nu}(X_0)\,\eta^\lambda + \partial_\lambda\partial_\sigma \bar{T}_{\mu\nu}(X_0)\,\eta^\lambda\eta^\sigma + \ldots$$

$$= \bar{T}_{\mu\nu}(X_0) + \nabla_\lambda \bar{T}_{\mu\nu}(X_0)\,\eta^\lambda + \frac{1}{2}\{\nabla_\lambda\nabla_\sigma \bar{T}_{\mu\nu}(X_0) - \frac{1}{3}\bar{R}^\rho{}_{\lambda\mu\sigma}\bar{T}_{\rho\nu}(X_0) - \frac{1}{3}\bar{R}^\rho{}_{\lambda\nu\sigma}\bar{T}_{\mu\rho}(X_0)\}\eta^\lambda\eta^\sigma + \ldots$$

Covariant Taylor Expansion of the Action

$$S_P[X_0+\pi] = S_P[X_0] + \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau \sqrt{\gamma}\gamma^{ab}\left(G_{\mu\nu}\nabla_a\eta^\mu\nabla_b\eta^\nu + R_{\mu\lambda\nu\sigma}\partial_a X_0^\mu\partial_b X_0^\nu\eta^\lambda\eta^\sigma\right) + \text{higher orders in } \eta$$

$$S_{AS}[X_0+\pi] = S_{AS}[X_0] + \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\, \epsilon^{ab}\left(H_{\mu ij}\partial_a X_0^\mu\nabla_b\eta^i\eta^j + \frac{1}{2}\nabla_i H_{\mu\nu j}\partial_a X_0^\mu\partial_b X_0^\nu\eta^i\eta^j\right) + \text{higher orders}$$

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Expectation Value of the Energy-Momentum Tensor

In Light Cone Coordinates, $\nabla^a T_{ab} = 0$ reads

$$\partial_- T_{++} + \partial_+ T_{-+} = 0$$

Promoting the energy-momentum tensor to a quantum operator gives (in momentum space)

$$q_- \langle T_{++} \rangle + q_+ \langle T_{-+} \rangle = 0$$

The Weyl Anomaly is expressed by

$$\langle T_{-+} \rangle \neq 0$$

The S_P Term

Inserting T_{++} and the first term in S_P gives

$$< \Omega \,|\, \mathrm{T}\left(\partial_+\eta^i\partial_+\eta^i R_{\mu kj\nu}\gamma^{ab}\partial_a X_0^\mu \partial_b X_0^\nu \eta^k \eta^j\right)\,|\, \Omega >= -\frac{1}{4}\frac{q_+}{q_-}$$

Using the conservation equation $q_- \langle T_{++}\rangle + q_+ \langle T_{-+}\rangle = 0$,

$$\langle T_{-+}\rangle = \frac{1}{4} R_{\mu\nu}\partial_a X_0^\mu \partial_b X^\nu \gamma^{ab}$$

The S_{AS} Terms

Two diagrams derived from S_{AS} contribute at first order. The results are

$$\langle T_{-+} \rangle = -\frac{1}{16} H_{\mu\lambda\sigma} H_{\nu}^{\lambda\sigma} \partial_a X_0^\mu \partial_b X^\nu \gamma^{ab}$$
$$\langle T_{-+} \rangle = \frac{1}{8} \nabla^\lambda H_{\lambda\mu\nu} \partial_a X_0^\mu \partial_b X^\nu \epsilon^{ab}$$

And the total expectation value reads

$$\langle T_{-+} \rangle = \frac{1}{4} \left\{ R_{\mu\nu} - \frac{1}{4} H^2_{\mu\nu} \right\} \partial_a X_0^\mu \partial_b X^\nu \gamma^{ab} + \left\{ \nabla^\lambda H_{\lambda\mu\nu} \right\} \partial_a X_0^\mu \partial_b X^\nu \epsilon^{ab}$$

The Weyl Anomaly Conditions

The previous derivations hold for a flat worldsheet $\gamma^{ab} = \eta^{ab}$. On a curved worldsheet extra terms are generated:

$$\langle T_{-+}\rangle = \frac{1}{4}\left\{R_{\mu\nu} - \frac{1}{4}H^2_{\mu\nu} + \frac{1}{24}G_{\mu\nu}H^2 - \frac{1}{2}G_{\mu\nu}R\right\}\partial_a X_0^\mu \partial_b X_0^\nu \gamma^{ab} + \left\{\nabla^\lambda H_{\lambda\mu\nu}\right\}\partial_a X_0^\mu \partial_b X^\nu \epsilon^{ab}$$

Two independent equations for the (anti)symmetric parts:

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = \frac{1}{4}\left[H^2_{\mu\nu} - \frac{1}{6}G_{\mu\nu}H^2\right]$$

$$\nabla^\lambda H_{\lambda\mu\nu} = 0$$

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An Action Appears Out of Nowhere...

The two equations

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = \frac{1}{4}\left[H^2_{\mu\nu} - \frac{1}{6}G_{\mu\nu}H^2\right]$$

$$\nabla^\lambda H_{\lambda\mu\nu} = 0$$

are equations of motion for the action

$$\boxed{\mathbf{S} = \int \mathrm{d}^{\mathbf{D}}\mathbf{X}\sqrt{\mathbf{G}}\left\{\mathbf{R} - \frac{\mathbf{1}}{\mathbf{12}}\mathbf{H}^{\mathbf{2}}\right\}}$$

Einstein-Hilbert action of General Relativity along with a kinetic term for H