Logarithmic Conformal Field Theory

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Basics

- Conformal group for $\mathbb{R}^{p,q}$ isomorphic to $SO(p+1, q+1)$.
  $\rightarrow$ finite group implies finitely many conserved quantities, which help to determine correlators.

- 2D CFT with global conformal invariance: Work with $SL(2, \mathbb{C})$.

- Only L_{-1}, L_0 and L_1 compatible with global conformal invariance. Generators of $\mathfrak{sl}(2, \mathbb{C})$.

- Conformal invariance helps to determine correlators.

Alternative way to compute correlators

- Recall: $\phi_h \to \phi_h + \delta_n \phi_h$ under infinitesimal conformal transformation. $\delta_n \phi_h = \left(z^{n+1}\partial_z + h(n+1)z^n\right)\phi_h = [L_n, \phi_h]$.

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- Given the N point function $G_N = \langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\cdots\phi_{h_N}(z_N)\rangle$ impose $G_N \to G_N$ under global conformal transformation (i.e. $\delta_n G_N = 0,\ n \in \{-1, 0, 1\}$).

Global conformal Ward identities

$$L_{-1} \quad : \quad \sum_{i=1}^{N} \partial_{z_i} G_N(z_1, ..., z_N) = 0$$

$$L_0 \quad : \quad \sum_{i=1}^{N} (z_i \partial_{z_i} + h_i) G_N(z_1, ..., z_N) = 0$$

$$L_1 \quad : \quad \sum_{i=1}^{N} (z_i^2 \partial_{z_i} + 2 h_i z_i) G_N(z_1, ..., z_N) = 0$$

Two und three point functions

- Two point function: $\langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\rangle = \frac{D_{12}\delta_{h_1,h_2}}{(z_1-z_2)^{2h_1}}$
  with D_{12} a constant (determined by normalization of fields).

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- Three point function: $\langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)\rangle = \frac{C_{123}}{(z_{12})^{h_1+h_2-h_3}(z_{13})^{h_1+h_3-h_2}(z_{23})^{h_2+h_3-h_1}}$ with $z_{ij} := z_i - z_j$ and C_{123} a constant (not determined by conformal invariance).

Problem: Four point function

- $\langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)\phi_{h_4}(z_4)\rangle = F(x)\prod_{i>j} z_{ij}^{\frac{H}{3}-h_i-h_j}$.
- $H = \sum_{i=1}^{4} h_i$.
- $F(x)$ an arbitrary function (but conformal invariant).
- Anharmonic ratio $x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$.

Null vectors

- Some states in Verma module decouple from all other states in the Hilbert space: Null vectors
- Null vectors are manifestation of local conformal invariance.
- Example: Level 2 degeneracy
  $|\chi\rangle = \left(L_{-2} + aL^2_{-1}\right)|h\rangle$

  $a = -\frac{3}{2(2h+1)}$

  $h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right)$ or $h = 0$

Null vectors

Continuation of the level 2 example

- From null vectors to differential equations:

$$\left(\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2\right)\langle\phi(z)X\rangle = 0$$

$$\left[\sum_{i=1}^{N}\left(\frac{1}{z-z_i}\partial_{z_i} + \frac{h_i}{(z-z_i)^2}\right) - \frac{3}{2(2h+1)}\partial_z^2\right]\langle\phi(z)X\rangle = 0$$

The theory set up

- Central charge fixed to be $c = -2$.
- There is a primary field μ which is degenerate to the second level.
- Degeneracy implies $h_\mu \in \left\{-\frac{1}{8}, 1\right\}$. We choose $h_\mu = -\frac{1}{8}$.

Computation of four point function

- $\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle = (z_1 - z_3)^{\frac{1}{4}}(z_2 - z_4)^{\frac{1}{4}}[x(1-x)]^{\frac{1}{4}}F(x)$

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- $\langle \mu(z_1)\mu(z_2)\mu(z_3)\left(L_{-2} - 2L_{-1}^2\right)\mu(z_4)\rangle = 0$
- $\left[\sum_{i=1}^{3}\left(\frac{1}{z_4 - z_i}\frac{\partial}{\partial z_i} - \frac{1}{8(z_4 - z_i)^2}\right) + \frac{\partial^2}{\partial z_4^2}\right] G_4(z_1, \ldots, z_4) = 0$

The differential equation

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- Ansatz to solve differential equation:
  $$F(x) = x^s \sum_{n=0}^{\infty} a_n x^n \quad \text{with } a_0 \neq 0$$
- Determine s by solving "indicial equation" $(x \to 0)$:
  $s^2 = 0 \quad \Rightarrow \quad$ both roots are zero.

The solution and its properties

- $F(x) = a_0 \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} \right]^2 x^n$ is the expansion of the analytic continuation of the Elliptic Integral of the first kind:
  $F(x) \propto G(x) = \int_0^{\frac{\pi}{2}} \frac{d\Theta}{\sqrt{1 - x \sin^2(\Theta)}} \quad \text{for } |x| < 1$

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- Behavior of the solution for $x \to 1$:
$$\lim_{x \to 1} G(x) = \lim_{\alpha \to \pi/2} \ln \left| \tan\left(\frac{\Theta}{2} + \frac{1}{4} \right) \right| \Bigg|_0^{\alpha} \quad \to \text{log. div.!}$$

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- One finds $b_n = a_n$ and the second solution simplifies to $G(1-x) = \log(x) G(x) + H(x)$ with H(x) regular at $x = 0$.

The full solution and its problems

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  Does it break conformal invariance? NO!
  Are other correlators always finite? NO!

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- What's bad about the logarithm?
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  Are other correlators always finite? NO!
- Problem arises in the OPE...

The OPE for the first solution

- General OPE:
  $\phi_{h_n}(z)\phi_{h_m}(0) = \sum_p \sum_{\{k\}} C_{nm}^{p,\{k\}} z^{h_p - h_n - h_m + \sum_i k_i} \phi_p^{\{k\}}(0)$

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- In our case: $h_n = h_m = -1/8$ and $[\phi_p] = [\mathbf{I}] \Rightarrow h_p = 0$. Thus:
  $$\mu(z)\mu(0)|0\rangle \sim z^{\frac{1}{4}} \sum_{k=0}^{\infty} z^k \underbrace{C\beta^k \mathbf{I}^{\{k\}}(0)|0\rangle}_{=:|\mathbf{I},k\rangle} = z^{\frac{1}{4}} \underbrace{\sum_{k=0}^{\infty} z^k |\mathbf{I}, k\rangle}_{=:|\mathbf{I},z\rangle}$$

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- Final result: $\mu(z)\mu(0)|0\rangle = z^{\frac{1}{4}} |\mathbf{I}, z\rangle$

What about the OPE for the second solution?

- The usual OPE can't produce the $\log(z)$ appearing in the second solution!
- Usual OPE inconsistent with second solution of differential equation.
- We have to modify the OPE and introduce a new kind of operator:
  $\mu(z)\mu(0)|0\rangle = z^{\frac{1}{4}}\left(\log(z)|\mathbf{I}, z\rangle + |\mathbf{I}_1, z\rangle\right)$
- The operator $\mathbf{I}_1$ is called the logarithmic partner of the primary operator $\mathbf{I}$.

The properties of the logarithmic partner

- The properties of the primary operator:

$$L_0|\mathbf{I}, n\rangle = n|\mathbf{I}, n\rangle$$
$$L_k|\mathbf{I}, n+k\rangle = (n + (k-1)h_\mu)|\mathbf{I}, n\rangle \quad ; \; k > 0$$

- The properties of the logarithmic partner:

$$L_0|\mathbf{I}_1, n\rangle = |\mathbf{I}, n\rangle + n|\mathbf{I}_1, n\rangle$$
$$L_k|\mathbf{I}_1, n+k\rangle = |\mathbf{I}, n\rangle + (n + (k-1)h_\mu)|\mathbf{I}_1, n\rangle \quad ; \; k > 0$$

Consequence number one

Because of $L_0|\mathbf{I}_1, n\rangle = |\mathbf{I}, n\rangle + n|\mathbf{I}_1, n\rangle$
L_0 not diagonalizable anymore, takes only Jordan block form.

Affects representation theory: Leads to reducible but indecomposable representations.

Definition:
A representation X is called indecomposable if $X \neq 0$ and $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$.

Consequence number two

Usually, the full correlater is given by $\sum_k G_k(z)G_k(\bar{z})$.

In LCFT we get a "non-diagonal" mixing of holomorphic and anti-holomorphic parts. For example: $G_1(z)G_2(\bar{z}) + G_1(\bar{z})G_2(z)$.

Consequence number three

Modification of null vectors due to off-diagonal terms produced by the action of the Virasoro modes:
If $|\chi_\phi\rangle$ is a null vector in the Verma module of the primary field ϕ, the replacement $\phi \to \psi$ (where ψ is the logarithmic partner of ϕ) does not lead to another null vector.

A new formalism is needed.

Consequence number four

Transformation properties of the fields under local conformal transformations:

$$\Phi_h(z) = \left(\frac{\partial f(z)}{\partial z}\right)^h \Phi_h(f(z)) \quad \text{for } \Phi_h \text{ primary}$$

$$\Psi_h(z) = \left(\frac{\partial f(z)}{\partial z}\right)^h [\Psi_h(f(z)) + \log(\partial_z f(z))\Phi_h(f(z))]$$

Consequence number five

Modified Ward identities:
Let ϕ_i be either Φ_{h_i} (primary) or Ψ_{h_i} (logarithmic partner).

$$L_{-1} \quad : \quad \sum_{i=1}^{N} \partial_{z_i} \langle \phi_1 \cdots \phi_k \rangle = 0$$

$$L_0 \quad : \quad \sum_{i=1}^{N} (z_i \partial_{z_i} + (h_i + \Delta_{h_i})) \langle \phi_1 \cdots \phi_k \rangle = 0$$

$$L_1 \quad : \quad \sum_{i=1}^{N} (z_i^2 \partial_{z_i} + 2 z_i (h_i + \Delta_{h_i})) \langle \phi_1 \cdots \phi_k \rangle = 0$$

With $(\Delta_{h_i})^2 = 0$ and $\Delta_{h_i} \Phi_{h_i} = 0 \quad , \Delta_{h_i} \Psi_{h_j} = \Phi_{h_j} \delta_{h_i, h_j}$

An example

The two point function for two logarithmic operators:

$$\langle \Psi_{h_1}(z_1)\Psi_{h_2}(z_2)\rangle = \delta_{h_1,h_2}\frac{B - 2A\log(z_1 - z_2)}{(z_1 - z_2)^{2h_1}}$$

Applications

- Turbulences in 2D
- Fractional quantum hall effect ($\nu = 5/2$ case can only be explained by a $c = -2$ LCFT)
- Polymers
- Percolation (critical system from statistical physics)