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and String Theory

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Fusion Rules and the Verlinde Formula

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Abstract:
Fusion rules determine which fields appear in the operator product expansion (OPE) of two primary fields. The derivation of differential equations for the correlation functions of the primary fields yields constraints on the possible conformal weights and the central charge. These restrictions result in the fusion rules. In the case of minimal models, we find a finite set of conformal families. Introducing the fusion algebra, which expresses the fusion rules on a more abstract level than the OPE, we detect a relation between the modular transformation $\mathcal{S} : \tau \to -\frac{1}{\tau}$ and the fusion rules. The so called Verlinde formula connects local and global properties of conformal field theory and is therefore one of the deepest and most important gains of this theory.

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1 Introduction

This report is based on a talk that I held in context of the *Proseminar Conformal Field Theory and String Theory* taking place at the Federal Institute of Technology Zurich in the spring semester 2013. Since the participants of the proseminar including myself did not have much prior knowledge of conformal field theory (CFT), the aim of my talk was to give an introductory overview of minimal models, fusion rules and the Verlinde formula.

In a conformal field theory we wish to compute correlation functions as they give us the expectation values of the action of different field operators on the vacuum. Using the operator product expansion (OPE) of two fields, we can reduce every n-point correlation function to a $(n-1)$-point correlation function. This is very helpful, since we know the form of the two-point function up to a constant:

$$\langle \Phi_1(w_1)\Phi_2(w_2)\rangle = \begin{cases} \frac{C_{12}}{(w_1-w_2)^{2h}}, & \text{if } h_1 = h_2 = h \\ 0, & \text{if } h_1 \neq h_2 \end{cases} \tag{1}$$

Therefore, it makes sense to look at the OPE of the fields of our theory in order to determine higher point correlation functions. But we can also look at this the other way around: Put the case that we assume the OPE of two fields Φ_1, Φ_2 to be of the form $\sum_j c_{12}^j \Phi_j$ without knowing which fields effectively appear in the sum (i.e. which coefficients c_{12}^j are not equal to zero). Then we can gain information by inserting this general expression into the correlator, yielding:

$$\langle \Phi_1\Phi_2\Phi_3\rangle = \sum_j c_{12}^j \langle \Phi_j\Phi_3\rangle \tag{2}$$

From (1) and (2) we can conclude that a fusion from Φ_1 and Φ_2 onto Φ_3 is possible if their correlation function does not vanish.

The so called fusion rules determine which conformal families appear in the OPE of two conformal fields. As we will see later, these fusion rules give constraints on the central charge and the conformal dimension of the CFT. As a consequence of these constraints it turns out that some CFT's consist of only a finite number of conformal fields. These theories are called rational CFT's.

In section 2, a quick overview of the Virasora algebra is given. The next section deals with minimal models, which are a special kind of rational CFT's. Putting emphasis on the case of the minimal models, we explain how to get from differential equations for the correlators to the fusion rules for the conformal families (section 4). In section 5, we introduce the fusion algebra which abstracts the concept of the OPE. Finally, we look at the connection of the fusion algebra to the modular transformations which is expressed in the famous Verlinde Formula. All these concepts are illustrated by a simple example for two-dimensional CFT's, the Ising model.

The main reference for this text is [1].

2 The Virasoro algebra

The Virasoro Algebra plays a very important role in two-dimensional CFT's as it describes the infinite local conformal symmetry of those field theories. The generators L_n, $\bar{L}_n$ ($n \in \mathbb{Z}$) of the Virasoro Algebra with central charge c can be expressed as expansion coeffiecients of the energy-momentum tensor:

$$L_n = \frac{1}{2\pi i} \oint z^{n+1} T(z) dz \tag{3}$$

$$\bar{L}_n = \frac{1}{2\pi i} \oint \bar{z}^{n+1} \bar{T}(\bar{z}) d\bar{z} \tag{4}$$

With these definitions we can write the Laurent expansion of the energy momentum tensor as

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n \tag{5}$$

$$\bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n \tag{6}$$

By deforming the contours of the integrals (3) and (4) and by making use of the operator product expansion, the commutator of the Virasoro generators can be computed (see [1] for a derivation):

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0} \tag{7}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m) \bar{L}_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0} \tag{8}$$

$$[L_n, \bar{L}_m] = 0 \tag{9}$$

Thus, the Virasoro algebra can be split into a holomorphic and an anti-holomorphic part, which are independent from each other. In the following, we will construct a representation of the holomorphic part (7). The representation of the antiholomorphic part can be found with the same method and the overall representation is then obtained by taking the tensor product of the two indepedent components.

Following the concept of constructing a heighest weight representation that we know from the $\mathfrak{su}(2)$ representation of the angular momentum in quantum mechanics, we define a highest weight state $|h\rangle$ with the properties:

$$L_0 |h\rangle = h |h\rangle \tag{10}$$

$$L_n |h\rangle = 0, n > 0 \tag{11}$$

This so called asymptotic state is created by the action of a primary field on the vacuum. A primary field $\phi(z)$ is defined to be a field that transforms as

$$\phi'(w) = (\frac{dw}{dz})^{-h} \phi(z) \tag{12}$$

under any local two-dimensional conformal transformation $z \to w(z)$.

Making use of the commutation relation (7) we can compute

$$[L_0, L_m] = -L_m \tag{13}$$

and because $|h\rangle$ is an eigenstate of the L_0 operator we get on the other hand:

$$[L_0, L_m] = (L_0 L_m - h)\,|h\rangle \tag{14}$$

By combining (13) and (14) we can conclude that for m>0 the operator L_m lowers the eigenvalue of the eigenstate of the L_0 operator by m:

$$L_0(L_m\,|h\rangle) = (h-m)L_m\,|h\rangle \tag{15}$$

Similarly, $L_{-m}(m > 0)$ acts as a raising operator.

So how does the representation of the Virasoro algebra look like? A basis of the representation space is built by the descendant states:

$$L_{-k_1} \cdots L_{-k_n}\,|h\rangle \qquad (1 \leq k_1 \leq \cdots \leq k_n) \tag{16}$$

By induction we can see that states of this form (16) are eigenstates of L_0 with eigenvalue

$$h' = h + k_1 + k_2 + ...k_n \tag{17}$$

Here we call $N := \sum_{i=1}^{n} k_i$ the level of the desecendant.

The space of the highest weight state and all its descendants is called *Verma module* ($V(c,h)$) and is obviously mapped onto itself by the Virasoro algebra. Therefore it is a representation of the Virasoro algebra.

The hermitian conjugate of a Virasoro generator L_n is given by $L_n^\dagger = L_{-n}$ which leads to the definition of the inner product of two states $L_{-k_1} \cdots L_{-k_m}\,|h\rangle$ and $L_{-l_1} \cdots L_{-l_n}\,|h\rangle$ as:

$$\langle h|\, L_{k_m} \cdots L_{k_1} L_{-l_1} \cdots L_{-l_n}\,|h\rangle \tag{18}$$

This expression can be evaluated by using the commutation relation (7). By doing so, the $L_{k_j}(k_j > 0)$ will at some point act on the heighest weight state $|h\rangle$ and annihilate it, unless the states are at the same level. This can also be seen by the fact that two eigenspaces of the Hermitian operator L_0 with different eigenvalues are orthogonal.

Note that the term "inner product" is not applied correctly for expression (18), since this definition admits negative and zero norm states. So the requirement for an inner product to be positive definite is not fulfilled. In the next section we will see that the states with vanishing norm play an important role regarding the representation of the Virasoro algebra.

3 Minimal Models

The minimal models are an important subclass of conformal field theories. A minimal model has finitely many primary fields, so there exists only a finite number of representations of the Virasoro algebra. Due to the limited number of conformal families, i.e. primary fields and their descendants, this model is completely solvable. Here we mean solvable in the sense that all correlation functions can be calculated, eventhough this turns out to be rather technical. In practice, minimal models describe discrete statistical models such as the Ising model, which is presented in section 6.

3.1 Null Submodules of Reducible Verma Modules

Depending on the values that the conformal weight h and the central charge c obtain, the representation of the Virasoro algebra may be reducible. In order to show that the representation is reducible we need to find a proper submodule of the Verma module that is invariant under the action of the Virasoro algebra. Such a subrepresentation is generated by a state $|\chi\rangle \in V(c,h)$ that fulfills $L_n |\chi\rangle = 0$ for all positive n and that is of course not equal to the highest weight state$|h\rangle$. Such a state is called *singular vector*. The important characteristic of a singular vector is that it is a so called null state. Null states are orthogonal to the whole Verma module: Let $|\chi\rangle$ be a singular vector and $L_{-k_1} \cdots L_{-k_n} |h\rangle$ a basis state. Then we can calculate the inner product of these two states to be zero, considering that $L_n |\chi\rangle = 0$for all $n > 0$:

$$\langle\chi| L_{-k_1} \cdots L_{-k_n} |h\rangle = \langle h| L_{k_n} \cdots L_{k_1} |\chi\rangle^* = 0 \tag{19}$$

Especially, we can conclude that the singular vector has zero norm. Singular vectors are of special importance since they can be used to derive differential equations for the correlation functions as we will see in section 4. Singular vectors are not the only null states, as their descendants are orthogonal to the whole Verma module too. We have seen before, that states having different levels are orthogonal to each other. Thus, we only need to show that the inner product of any descendant $L_{-k_1} \cdots L_{-k_n} |\chi\rangle$ of a singular vector $|\chi\rangle$ and any state $L_{-b_1} \cdots L_{-b_m} |h\rangle$ belonging to the same level vanishes. As a consequence from the conditon for the states to have the same level and from the fact that the level of the singular vector $|\chi\rangle$ is not equal to zero, we note that $\sum_i b_i > \sum_i k_i$. This means that we can commute all the L_{b_i} over the the L_{-k_i} until they hit the singular vector. As a result we obtain:

$$\langle h| L_{b_m} \cdots L_{b_1} L_{-k_1} \cdots L_{-k_n} |\chi\rangle = 0 \tag{20}$$

In order to illustrate the existence of singular vectors, we consider a theory with conformal weight $h = 0$ and look at the subspace generated by $L_{-1} |0\rangle$. To see that $L_{-1} |0\rangle$ is indeed a singular vector, we need to show that this state is annihilated by all generators $L_m, m > 0$:

$$\begin{aligned} L_m L_{-1} |0\rangle &= ([L_m, L_{-1}] - L_{-1} L_m) |0\rangle \\ &= (m+1) L_{m-1} |0\rangle \\ &= 0 \qquad \forall m > 0 \end{aligned} \tag{21}$$

Obviously, this subspace consisting of states of the form

$$L_{-n_1} \cdots L_{-n_l} L_{-1} |0\rangle \tag{22}$$

is invariant under the action of the Virasoro generators. But why is it a proper subspace of the Verma module? To prove this, we want to show that the highest weight state $|0\rangle$ itself is not included in this subspace: The only possibility to reduce the conformal weight of $L_{-1} |0\rangle$ by 1 consists in applying the L_1 operator on this state. But from (21) we already know that $L_1 L_{-1} |0\rangle = 0 \neq |0\rangle$. Hence we have found a proper subrepresentation.

By quotienting out of the Verma module $V(c,h)$ all the null submodules generated by the contained singular vectors, the representation of the Virasoro algebra is made irreducible. Quotienting a null submodule $N(c,h)$ out of $V(c,h)$ means that we define an equivalence relation on $V(c,h)$ by identifying two states $|x\rangle, |y\rangle \in V(c,h)$, i.e. $|x\rangle \sim |y\rangle$, if they differ by a state of zero norm, i.e. if $|x\rangle - |y\rangle \in N(c,h)$.

3.2 The Kac Determinant

What are the conditions for a state to have a vanishing norm? To anwer this question, we just have to make some linear algebra considerations: Assuming our Verma module $V(c,h)$ to be finite dimensional with basis vectors $\{|i\rangle\}$, we look at the *Gram matrix* $M_{ij} = \langle i|j\rangle$ of all inner products of the basis states. We know that for two states at different levels the corresponding matrix element vanishes, which allows us to write the Gram matrix in a block diagonal form, with each block $M^{(N)}$ corresponding to one level N. Furthermore the Gram matrix is Hermitian, hence it can be diagonalized by a unitary matrix $U : M = U^\dagger \Lambda U$. Let Λ_i denote the eigenvalues of the Gram matrix and let $|x\rangle = \sum_i x_i |i\rangle$ be an arbitrary element of $V(c,h)$. Then we have

$$\begin{aligned} \langle x|x\rangle &= x^\dagger M x && (23)\\ &= \sum_i \Lambda_i \left|(Ux)_i\right|^2 && (24) \end{aligned}$$

from which we conclude that there will be a state of zero norm if one of the eigenvalues Λ_i vanishes. Since a state of zero norm is either a singular state or one of its descendants, we can conclude that the Verma module is reducible if one or more eigenvalues of the Gram matrix are zero. As the determinant of a matrix is equal to the product of its eigenvalues, it makes sense to calculate the determinant of the Gram block matrices.

In order to illustrate this procedure, we calculate the matrices $M^{(N)}(c,h)$ for the lowest levels $N = 1, 2$: There exists only one basis state at level $N = 1$, namely $L_{-1}|h\rangle$, hence we find using the Virasoro algebra (7):

$$\begin{aligned} M^{(1)}(c,h) &= \langle h| L_1 L_{-1} |h\rangle \\ &= \langle h| 2L_0 |h\rangle \\ &= 2h \end{aligned}$$

In this case the determinant is trivially given by

$$\det M^{(1)}(c,h) = 2h \tag{25}$$

Thus, at level $N = 1$ we have a singular vector if and only if $h = 0$.

At level $N = 2$ we have the two different basis vectors $L_{-2}|h\rangle$ and $L_{-1}L_{-1}|h\rangle$ in the Verma module. We calculate the entries of $M^{(2)}(c,h)$ making repeatedly use of the Virasoro algebra:

$$\begin{aligned} \langle h| L_2 L_{-2} |h\rangle &= \langle h| \frac{c}{2} + 4L_0 |h\rangle = 4h + \frac{c}{2} && (26)\\ \langle h| L_1 L_1 L_{-2} |h\rangle &= \langle h| L_1 3L_{-1} |h\rangle = 6h && (27)\\ \langle h| L_1 L_1 L_{-1} L_{-1} |h\rangle &= \langle h| L_1 [L_1, L_{-1}] L_{-1} |h\rangle + \langle h| L_1 L_{-1} L_1, L_{-1} |h\rangle && (28)\\ &= \langle h| L_1 2L_0 L_{-1} |h\rangle + \langle h| [L_1, L_{-1}] [L_1, L_{-1}] |h\rangle && (29)\\ &= 2\langle h| L_1 [L_0, L_{-1}] |h\rangle + 8\langle h| L_0^2 |h\rangle && (30)\\ &= 4h + 8h^2 && (31) \end{aligned}$$

So for the determinant at level $N = 2$ block we have

$$\begin{aligned} \det M^{(2)}(c,h) &= \det \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h(2h+1) \end{pmatrix} && (32)\\ &= 32(h - h_{1,1}(c))(h - h_{1,2}(c))(h - h_{2,1}(c)) && (33) \end{aligned}$$

where the roots are given by:

$$h_{1,1} = 0 \tag{34}$$

$$h_{1,2} = \frac{1}{16}(5 - c - \sqrt{(1-c)(25-c)} \tag{35}$$

$$h_{2,1} = \frac{1}{16}(5 - c + \sqrt{(1-c)(25-c)} \tag{36}$$

This means that there exist three states with zero norm at level $N = 2$. Note that the first root $h_{1,1} = 0$ results from the descendant state of the singular vector at level one.

The mathematician Kac found a general formula for the determinant of the Gram matrix at an arbitrary level l, the *Kac determinant*:

$$\det M^{(l)} = \alpha_l \prod_{\substack{r,s \geq 1, \\ rs \leq l}} [h - h_{r,s}(c)]^{p(l-rs)} \tag{37}$$

where

$$h_{r,s}(c) = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 \tag{38}$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \tag{39}$$

$$h_0 = \frac{1}{24}(c-1) \tag{40}$$

and $p(l - rs)$ equals the number of partitions of the integer $l - rs$. This exponent reflects that with increasing level l the multiplicity of the roots of the Kac determinant increase due to the descending states. Moreover it shows that the first singular state in the reducible Verma module $V(c, h_{r,s})$ occurs at level $l = rs$.

The important information that we get from the Kac formula is that a representation with highest weight h is reducible if and only if h is of the form form (38) for some non negative integers r, s. Note that the formula for the Kac determinant is not obvious at all!

3.3 Kac Determinant for Minimal Models

With a further analysis of the formulae (38) to (40) it is possible to find out that if there exist two coprime positive integers p and p', $p > p'$ such that $p\alpha_- + p'\alpha_+ = 0$, then we can write the roots of the Kac determinant as

$$h_{r,s} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'} \tag{41}$$

and the central charge as

$$c = 1 - \frac{6(p-p')^2}{pp'} \tag{42}$$

in terms of these integers.

Having a closer look at the expression for the conformal weight (41) we detect some important properties:

$$h_{r,s} = h_{r+p',s+p} \qquad \text{(periodicity)} \tag{43}$$

$$h_{r,s} = h_{p'-r,p-s} \qquad \text{(symmetry)} \tag{44}$$

$$h_{r,s} + rs = h_{p'+r,p-s} = h_{p'-r,p+s} \tag{45}$$

$$h_{r,s} + (p'-r)(p-s) = h_{r,2p-s} = h_{2p'-r,s} \tag{46}$$

These properties contain a lot of information about the structure of the Verma module: From the formula for the Kac determinant (37) we know that in the reducible Verma module $V(c, h_{rs})$ the first singular vector appears at level $l = rs$. But this singular vector is again the highest weight of a reducible Verma module $V(c, h_{p'+r,p-s})$, since using equation (45) its conformal weight can be rewritten as a root of the Kac determinant. Similarly, the singular vector at level $(p' - r)(p - s)$ generates a further reducible submodule $V(c, h_{r,2p-s})$. Repeatedly applying the periodicities given by the equations (44)-(46) we get:

$$\begin{aligned} h_{p'+r,p-s} + (p' + r)(p - s) &= h_{2p'-(-r),s} && (47)\\ &= h_{-r,2p-s} && (48)\\ &= h_{p'-r,3p-s} && (49)\\ &= hr, 2p - s + r(2p - s) && (50) \end{aligned}$$

Thus, the degenerate submodules $V(c, h_{p'+r,p-s})$ and $V(c, h_{r,2p-s})$ share again two submodules. Iterating this method, we find an infinite tower of submodules that contain infitely many null vectors. By quotienting all these null submodules out of the original Verma module $V(c, h_{r,s})$, we obtain a reducible representation. In section 4, we will see that each singular vector results in a differential equation that constraints the conformal weights and the central charge appearing in the correlation function. Due to the infinite number of singular vectors we have so many restrictions on the set of conformal weights $h_{r,s}$ such that after quotienting out the null sumbmodules a finite set of conformal families remains. This finite set is closed under fusion and the number of conformal families is limited by

$$1 \leq r < p' \qquad \text{and} \qquad 1 \leq s < p \tag{51}$$

and the symmetry property (44) such that there are $\frac{(p-1)(p'-1)}{2}$ different conformal families left. A model characterized by the coprime positive integers p and p' with a finite number of primary fields $\Phi_{r,s}$ restricted by (51) with conformal weight and central charge given by (41) and (42) is called *minimal model* and denoted by $M(p, p')$.

4 Fusion Rules

4.1 Differential Equations for the Correlation Functions

How do we get from the singular vectors of a reducible Verma module to the fusion rules of the primary fields? In section 1 we have already seen that we can extract information about possible fusions from the correlation functions. In the following, we will make use of the orthogonality of a singular vector to the whole Verma module in order to derive differential equations for the correlators. These differential equations in turn restrict the conformal weights of the fields that are created by fusion.

One important property that we will use here is the fact that to each descendant $L_{-n}|h\rangle$ of the highest weight state $|h\rangle$ there corresponds a descendant $\Phi^{(-n)}(w)$ of a primary field $\Phi(w)$ which is defined to be the field appearing in the operator product expansion of the primary with the energy momentum tensor $T(z)$:

$$T(z)\Phi(w) = \sum_{n\geq 0}(z-w)^{n-2}\Phi^{(-n)}(w) \tag{52}$$

By performing an integration with deformed contours the descendant field Φ^{-n} can be derived (see [1], chapter 6):

$$\Phi^{(-n)}(w) = \frac{1}{2\pi i}\oint_w dz \frac{1}{(z-w)^{n-1}}T(z)\Phi(w) \tag{53}$$

Assume that we have N primary fields $\Phi_1(w_1), \cdots, \Phi_N(w_N)$ with conformal weights $(h_i)_{i=1}^N$ and that we are interested in the correlation function of those primaries with an arbitrary descendant field$\Phi^{(-n)}(w)$. Then we can show that the correlator of these fields can be rewritten as a correlation function of primary fields, that is acted on by a differential operator

$$\langle \Phi^{(-n)}(w)\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle = \mathcal{L}_{-n}\langle \Phi(w)\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle \qquad (n\geq 1) \tag{54}$$

where the differential operator $\mathcal{L}_{-n}$ is of the form:

$$\mathcal{L}_{-n}(w) = \sum_i \left(\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right) \tag{55}$$

To derive these expressions, we first deform the contour around w to a contour circling around each of the w_i and pick up a minus sign due to the change of the orientation of the contour. In the next step we plug in the OPE of the energy momentum tensor with a primary field. In the last step we use the residue theorem.

$$\langle \Phi^{(-n)}(w)\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle \tag{56}$$

$$=\frac{1}{2\pi i}\oint_{C(w)} dz \frac{1}{(z-w)^{n-1}} \langle (T(z)\Phi(w))\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle \tag{57}$$

$$=-\frac{1}{2\pi i}\sum_{i=1}^{N}\oint_{C(w_i)} dz \frac{1}{(z-w)^{n-1}} \langle \Phi(w)\Phi_1(w_1)\cdots(T(z)\Phi_i(w_i))\cdots\Phi_N(w_N)\rangle \tag{58}$$

$$=-\frac{1}{2\pi i}\sum_{i=1}^{N}\oint_{C(w_i)} dz \frac{1}{(z-w)^{n-1}} \left(\frac{h_i}{(z-w_i)^2}+\frac{1}{z-w_i}\partial_{w_i}\right) \langle \Phi(w)\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle \tag{59}$$

$$=-\sum_{i=1}^{N}((1-n)(w_i-w)^{-n}h_i+(w_i-w)^{1-n}\partial_{w_i}) \langle \Phi(w)\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle \tag{60}$$

Repeating this calculation shows, that a correlator including a descendant of the form $\Phi^{(-k_1,\cdots,-k_n)}(w)$ that corresponds to the state $L_{-k_1}\cdots L_{-k_n}|h\rangle$ in the Verma module can be replaced by a correlation function of primaries acted on by a string of differential operators:

$$\langle \Phi^{(-k_1,\cdots,-k_n)}(w)\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle = \mathcal{L}_{-k_1}\cdots\mathcal{L}_{-k_n}\langle \Phi(w)\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle \tag{61}$$

Now we want to apply this important result by inserting the field corresponding to some singular vector of the reducible Verma module $V(c,h_0)$ into a correlator. So suppose that $|h_0+n_0\rangle = \sum_{Y,|Y|=n_0}\alpha_Y L_{-Y}|h_0\rangle$ is a singular vector at level n_0, where we used the following notation:

$$Y=\{r_1,\cdots,r_k\}\quad(1\le r_1\le\cdots\le r_k) \tag{62}$$

$$|Y|=r_1+\cdots+r_k \tag{63}$$

$$L_Y=L_{-r_1}\cdots L_{-r_k} \tag{64}$$

Quotienting this singular vector out of the Verma module means that we also set the corresponding field to zero. Let Φ_0 be the field that corresponds to the highest weight state $|h_0\rangle$. Of course, the correlation function of this nullfield with a chain of primary fields must also vanish. Using (61), the vanishing condition for the singular vector can be converted into a differential equation for the correlator of the primary fields:

$$0=\langle \sum_{Y,|Y|=n_0}\alpha_Y\Phi_0^{(-r_1,\cdots,-r_n)}(w_0)\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle \tag{65}$$

$$=\sum_{Y,|Y|=n_0}\alpha_Y\mathcal{L}_{-Y}\langle \Phi_0(w_0)\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle \tag{66}$$

How does such a differential equation restrict the conformal weights of the primaries appearing in the correlator? To illustrate this, we look at an example: Suppose we have a Verma module $V(c,h)$ and

want to find the singular vector at level two. A level two state can be written as a linear combination $|\chi\rangle = (L_{-2} + aL_{-1}^2)\,|h\rangle$ and to be a singular state it must fulfill the condition

$$L_n\,|\chi\rangle = 0 \tag{67}$$

for all positive n. But it is already enough to demand that this equation holds for $n = 1$ and $n = 2$ since the Virasoro algebra implies that $|\chi\rangle$ is also annihilated by the L_n-generators for $n \geq 3$. Hence we calculate, making repeatedly use of the Virasoro algebra:

$$\begin{aligned}
L_1\chi &= ([L_1, L_{-2}] + a[L_1, L_{-1}^2])\,|h\rangle && (68)\\
&= 3L_{-1}\,|h\rangle + a(L_{-1}[L_1, L_{-1}] + [L_1, L_{-1}]L_{-1})\,|h\rangle && (69)\\
&= 3L_{-1}\,|h\rangle + a(L_{-1}2L_0 + (2L_0)L_{-1})\,|h\rangle && (70)\\
&= 3L_{-1}\,|h\rangle + 2a([L_0, L_{-1}] + 2L_{-1}L_0)\,|h\rangle && (71)\\
&= 3L_{-1}\,|h\rangle + 2a(L_{-1} + 2hL_{-1})\,|h\rangle && (72)\\
&= (3 + 2a(2h+1))L_{-1}\,|h\rangle && (73)
\end{aligned}$$

So in order to make expression (73) vanish, we get the condition

$$a = -\frac{3}{2(2h+1)} \tag{74}$$

if $h \neq 0$. If $h = 0$, we do not have to impose any condition on a. To get a condition on the relation between the central charge c and the conformal weight h, we apply L_2 on $|\chi\rangle$ and set this state equal to zero:

$$\begin{aligned}
L_2\chi &= ([L_2, L_{-2}] + a[L_2, L_{-1}^2]\,|h\rangle && (75)\\
&= (4L_0 + \frac{c}{2})\,|h\rangle + a(L_{-1}[L_2, L_{-1}] + [L_2, L_{-1}]L_{-1})\,|h\rangle && (76)\\
&= (4h + \frac{c}{2})\,|h\rangle + a(L_{-1}3L_1 + (3L_1)L_{-1})\,|h\rangle && (77)\\
&= (4h + \frac{c}{2})\,|h\rangle + 3a([L_1, L_{-1}] + 2L_{-1}L_1)\,|h\rangle && (78)\\
&= (4h + \frac{c}{2})\,|h\rangle + 3a(2L_0)\,|h\rangle && (79)\\
&= (2h(2+3a) + \frac{c}{2})\,|h\rangle && (80)
\end{aligned}$$

Thus, the central charge is restricted to be of the form:

$$c = 2h\frac{5 - 8h}{2h+1} \tag{81}$$

which can be solved for h:

$$h = \frac{1}{16}(5 - c \pm \sqrt{(c-1)(c-25)}) \tag{82}$$

So plugging the corresponding nullfield into a correlator with a product $X = \Phi_1(w_1), \cdots, \Phi_N(w_N)$ of primary fields and using (54) we get a differential equation

$$(\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2)\,\langle\Phi(w)X\rangle = 0 \tag{83}$$

which can be written as

$$\left[\sum_{i=1}^{N}\left(\frac{1}{w-w_i}\partial_{w_i}+\frac{h_i}{(w_i-w)^2}\right)-\frac{3}{2(2h+1)}\partial_w^2\right]\langle\Phi(w)X\rangle=0 \tag{84}$$

inserting the expression for the differential operators $\mathcal{L}_n$ (eq. (55)). Here $\Phi(w)$ is the field corresponding to the highest weight state $|h\rangle$.

In order to find restrictions on the conformal weights of the primaries, it is very helpful that we know the form of correlation functions up to a constant. The two-point function does not vanish if and only if the conformal weights of the primaries are the same:

$$\langle\Phi_1(w_1)\Phi_2(w_2)\rangle=\begin{cases}\frac{C_{12}}{(w_1-w_2)^{2h}}, & \text{if } h_1=h_2=h\\ 0, & \text{if } h_1\neq h_2\end{cases} \tag{85}$$

But plugging only one primary field $X=\Phi(w_1)$ into (84) and then using the form of the two-point function (85), we do not get any new information, since the differential equation is then just trivially satisfied.

So let us plug in $X=\Phi(w_1)\Phi(w_2)$ and use the general form of the three-point function

$$\langle\Phi(w)\Phi_1(w_1)\Phi_2(w_2)\rangle=\frac{C_{h,h_1,h_2}}{(w-w_1)^{h+h_1-h_2}(w_1-w_2)^{h_1+h_2-h}(w-w_2)^{h+h_2-h_1}} \tag{86}$$

where $C(h,h_1,h_2)$ is a constant depending on the conformal weights. As a consequence of equation (84), we get the following constraint on the conformal weights:

$$h_2=\frac{1}{6}+\frac{h}{3}+h_1\pm\frac{2}{3}\sqrt{h^2+3hh_1-\frac{1}{2}h+\frac{3}{2}h_1+\frac{1}{16}} \tag{87}$$

If we choose for example $h=h_{2,1}(c)$ and $h_1=h_{r,s}(c)$ then formula (87) gives us two possible solutions for h_2. Comparing the result to the formula for the roots of the Kac determinant (38) we find that these solutions are precisely $[h_{r-1,s},h_{r+1,s}]$. So we have found our first fusion rule! We found out that the OPE of the fields $\Phi_{2,1}$ with an arbitrary primary field $\Phi_{r,s}$ in a minimal model may only contain the fields $\Phi_{r+1,s}$ and $\Phi_{r-1,s}$. We use the following notation to express this fusion rule:

$$[\Phi_{2,1}]\times[\Phi_{r,s}]=[\Phi_{r-1,s}]+[\Phi_{r+1,s}] \tag{88}$$

Here $[\Phi_{(r,s)}]$ denotes the conformal family consisting of $\Phi_{(r,s)}$ and its descendants. The right-hand side of equation (88) says that at most those two conformal families appear in the OPE but their coefficients could also be zero.

By generalising the same method for higher level singular vectors, the closed algebra for all conformal families in a minimal model can be found:

$$[\Phi_{r_1,s_1}]\times[\Phi_{r_2,s_2}]=\sum_{\substack{k=1+|r_1-r_2|\\ k+r_1+r_2=1 mod 2}}^{k=r_1+r_2-1}\ \sum_{\substack{l=1+|s_1-s_2|\\ l+s_1+s_2=1 mod 2}}^{l=s_1+s_2-1}[\Phi_{k,l}] \tag{89}$$

5 The Fusion Algebra

If we are interested in the fields that are possibly created by the fusion of two primary fields Φ_{h_i} and Φ_{h_j}, we can look at the OPE of these fields:

$$\Phi_{h_i}(z)\Phi_{h_j}(w) \sim \sum_h C_{h_i,h_j}^{h_k} \Phi_{h_k}(w)(z-w)^{h_k-h_i-h_j} + ... \tag{90}$$

In order to extract the relevant information we can use a notation that is similar to the one that we used to express the fusion rules for the minimal model (89). Therefore, we introduce the *fusion numbers*

$$\mathcal{N}_{ij}^k = \begin{cases} 0, & C_{h_i,h_j}^{h_k} = 0 \\ 1, & \text{otherwise} \end{cases} \tag{91}$$

which indicate if a given field Φ_{h_k} appears in the OPE. The fusion numbers $\mathcal{N}_{ij}^k$ count the number of independent possibilities to obtain a field Φ_k by fusing two fields Φ_i and Φ_j. So generally they can also take larger values than 1 but they do not do so in the case of minimal models.

The *fusion algebra* which is defined as

$$[\Phi_i] \times [\Phi_j] = \sum_k \mathcal{N}_{ij}^k [\Phi_k] \tag{92}$$

indicates, which conformal families appear in the OPE of a field of the conformal family $[\Phi_i]$ and a member of the conformal family $[\Phi_j]$ without telling the precise form of the OPE. Due to this interpretation, the fusion numbers fulfill the symmetry $\mathcal{N}_{ij}^k = \mathcal{N}_{ji}^k$. Hence the fusion algebra is commutative. The identity element of the algebra is given by Φ_0, the vacuum field, so $N_{i0}^k = \delta_{ik}$. Moreover, due to the associativity of the OPE of primary fields, the fusion algebra is associative. This can be turned into an equation containg the fusion numbers: Combining

$$\Phi_i \times (\Phi_j \times \Phi_k) = \Phi_i \times \sum_l \mathcal{N}_{jk}^l \Phi_l \tag{93}$$

$$= \sum_{l,m} \mathcal{N}_{jk}^l \mathcal{N}_{il}^m \Phi_m \tag{94}$$

and

$$(\Phi_i \times \Phi_j) \times \Phi_k) = \sum_l \mathcal{N}_{ij}^l \Phi_l \times \Phi_k \tag{95}$$

$$= \sum_{l,m} \mathcal{N}_{ij}^l \mathcal{N}_{lk}^m \Phi_m \tag{96}$$

we find:

$$\sum_l \mathcal{N}_{jk}^l \mathcal{N}_{il}^m = \sum_l \mathcal{N}_{ij}^l \mathcal{N}_{lk}^m \tag{97}$$

If we are looking at theories with a finite number of fields, it makes sense do define matrices N_i with entries given by the fusion numbers $\mathcal{N}_{ij}^k$:

$$(N_i)_{j,k} := \mathcal{N}_{ij}^k \tag{98}$$

Then the condition (97) obtained from the associativity of the fusion algebra can be rewritten as a commuting condition for the matricies:

$$N_i N_k = N_k N \tag{99}$$

5.1 The Verlinde Formula

Since the fusion matrices commute and are furthermore normal which we will not show here, we can diagonalize them simultaneously: Let us denote the diagonalizing matrix by S and the eigenvalues of N_i by $\lambda_i^{(l)}$. Then:

$$\begin{aligned} \mathcal{N}_{ij}^k &= (SDS^{-1}) && (100)\\ &= \sum_{lm} S_{jl}\lambda_i^{(l)}\delta_l^m (S^{(-1)})_{mk} && (101)\\ &= \sum_{l} S_{jl}\lambda_i^{(l)} (S^{-1})_{lk} && (102) \end{aligned}$$

Additionally, we can calculate using $N_{i0}^k = \delta_{ik}$:

$$\begin{aligned} S_{in} &= \sum kN_{i0}^k Skn && (103)\\ &= \sum_{lk} S_{0l}\lambda_i^{(l)} (S^{(-1)})_{lk} S_{kn} && (104)\\ &= \sum_{l} S_{0l}\lambda_i^{(l)} \delta_{ln} && (105)\\ &= S_{0n}\lambda_i^{(n)} && (106) \end{aligned}$$

Hence, inserting the eigenvalues

$$\lambda_i^{(l)} = \frac{S_{il}}{S_{0l}} \tag{107}$$

into (102) yields an expression for the fusion numbers in terms of the entries of the diagonalization matrix S:

$$\mathcal{N}_{ij}^k = \sum_l \frac{S_{jl}S_{il}(S^{-1})_{lk}}{S_{0l}} \tag{108}$$

How do we interpret this formula? Without knowing how S looks like, what we did is just a simple calculation. It needs great mathematicians to find a deeper meaning of a superficial accumulation of letters and indices. In this case, it was Erik Verlinde who succeeded in giving an interpretation and his name to equation (108): He stated that the *modular transformation* $\mathcal{S} : \tau \to -\frac{1}{\tau}$ diagonalizes the fusion rules. In this context, equation (108) is called *Verlinde formula* and is one of the most important results of conformal field theory.

Since proving the Verlinde formula is very demanding, we will just try to understand its deep meaning and consequences. The first question we might ask ourselves is: What is the modular transformation $\mathcal{S} : \tau \to -\frac{1}{\tau}$? This map describes an inversion in the unit circle, followed by ra reflection about $\mathrm{Re} z = 0$ and is one of the two generators of the modular group which in turn describes the equivalence classes of two-dimensional tori.

To understand the meaning of the matrix elements of this modular transformation $\mathcal{S}$ that appear in the Verlinde formula we have to be aware of a further important result of conformal field theory: Under the action of the modular transformation $\mathcal{S}$ the characters of the representations of the Virasoro algebra transform among themselves.

The *character of a Verma module* $V(c,h)$ with conformal weight h and central charge c is defined as

$$\chi_{c,h}(\tau) = \mathrm{Tr}\ q^{L_0 - \frac{c}{24}} \qquad (q := e^{2\pi i \tau}) \tag{109}$$

Since any state in the Verma module is an eigenstate of L_0 with an eigenvalue of the form $h+N$, we can also write the character as

$$\chi_{c,h}(\tau) = q^{h-\frac{c}{24}} \sum_{N=0}^{\infty} p(N) q^N \tag{110}$$

where the function $p(N)$ counts the number of states at level N.

Now it is not obvious but true that in the case of rational conformal field theories the action of the modular S-matrix can be written in terms of a linear combination of the characters themselves. The numbers appearing on the right-hand side of the Verlinde formula (108) are the entries of the S-matrix in the basis of the characters respectively the entries of its inverse S^{-1}! Especially for minimal models we can write:

$$\chi_{r,s}(-\frac{1}{\tau}) = \sum_{(\rho,\sigma)\in E_{p,p'}} \mathcal{S}_{rs,\rho\sigma} \chi_{\rho,\sigma}(\tau) \tag{111}$$

Here $E_{p,p'}$ is the set of all irreducible highest weight representations, i.e. it consists of $\frac{(p-1)(p'-1)}{2}$ elements.

The coefficients $S_{r,s}$ can be calculated to be:

$$\mathcal{S}_{rs;\rho\sigma} = 2\sqrt{\frac{2}{pp'}}(-1)^{1+s\rho+r\sigma} \sin(\pi \frac{p}{p'} r\rho) \sin(\pi \frac{p'}{p} s\rho) \tag{112}$$

A derivation of this can be found in [1], chapter 10. Note that the numbers $S_{rs,\rho\sigma}$ are real and symmetric and that the S-matrix is moreover unitary. Hence the fusion numbers $N_{rs,mn}^{kl}$ for minimal models can be computed from the S-matrix by the Verlinde formula:

$$\mathcal{N}_{rs,mn}^{kl} = \sum_{(i,j)\in E_{p,p'}} \frac{\mathcal{S}_{rs,ij}\mathcal{S}_{mn,ij}\mathcal{S}_{ij,kl}}{\mathcal{S}_{11,ij}} \tag{113}$$

In the case of minimal models the identity element of the fusion algebra is given by Φ_{11} since the vacuum state $|0\rangle$ corresponds to the highest weight $h_{1,1} = 0$ (see eq.(41)). This explains, why S_{0l} in the Verlinde formula (108) is replaced by $S_{11,ij}$ in the version for minimal models (113).

It is very astonishing that the combination of $S_{ij} \in \mathbb{R}$ on the right-hand side of (113) always sums up to either 0 or 1! So the S-matrix is greatly restricted by this formula! The outstanding importance of the Verlinde formula can be seen in the fact that it combines local as well as global properties in conformal field theory: The fusion numbers N_{ij}^k contain information about the *local* OPE of two fields whereas the modular transformation S is related to the *global* modular invariance of partition functions on the torus!

6 Example: The Ising Model

Now we want to illustrate the basic theory of minimal models, fusion rules and the Verlinde formula that we have learnt in the preceding chapters. Therefore we look at the simple example of the two-dimensional Ising model.

The conformally invariant action of the Ising model at the critical point of its second-order phase transition yields a central charge of $c = \frac{1}{2}$. In the holomorphic part of the theory we have three conformal families arising from three different primary fields: These are the vacuum field $\mathbb{1}$ ($h_{\mathbb{1}} = 0$), the spin field σ ($h_\sigma = \frac{1}{16}$) and the energy field ϵ ($h_\epsilon = \frac{1}{2}$). We can identify this model with the minimal model $M(4,3)$ characterized by $p = 4$ and $p' = 3$. Plugging in these values into the expressions for the conformal weights $h_{r,s}$, $(1 \leq r < 3; 1 \leq s < 4)$ and the central charge c

$$h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'} \tag{114}$$

$$c = 1 - \frac{6(p - p')^2}{pp'} \tag{115}$$

leads exactly to the given values for the Ising model. We can draw a conformal grid which shows the conformal weights in dependance on r and s and which is invariant by a rotation by π around its center due to the symmetry $h_{r,s} = h_{p'-r,p-s}$:

	$r = 1$	$r = 2$
$s = 1$	0	$\frac{1}{2}$
$s = 2$	$\frac{1}{16}$	$\frac{1}{16}$
$s = 3$	$\frac{1}{2}$	0

We identify the following fields:

$$\mathbb{1} \quad \Leftrightarrow \quad \Phi_1 := \Phi_{1,1} \tag{116}$$
$$\epsilon \quad \Leftrightarrow \quad \Phi_2 := \Phi_{2,1} \tag{117}$$
$$\sigma \quad \Leftrightarrow \quad \Phi_3 := \Phi_{2,2} \tag{118}$$

Since the Kac-formula predicts the first singular vector of a representation with highest weight $h_{r,s}$ at level rs, we can conclude that both the energy and the spin primary fields have singular vectors at level $N = 2$ while the vacuum field has one at level $N = 1$. The singular vectors at level $N = 2$ are exactly the ones that we calculated in section 4.1: We can certify this statement by plugging the central charge $c = \frac{1}{2}$ into equation (82) which results in $h = \frac{1}{2}$ and $h = \frac{1}{16}$.

What are the fusion rules of the primary fields? We want to apply the Verlinde formula and for this reason have to find the modular S-matrix in the basis of the characters. The characters of the three different representations are given by:

$$\chi_0(\tau) = \frac{1}{2}\left(\sqrt{\frac{\Theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\Theta_4(\tau)}{\eta(\tau)}}\right) \tag{119}$$
$$\chi_{\frac{1}{2}}(\tau) = \frac{1}{2}\left(\sqrt{\frac{\Theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\Theta_4(\tau)}{\eta(\tau)}}\right) \tag{120}$$
$$\chi_{\frac{1}{16}}(\tau) = \frac{1}{\sqrt{2}}\sqrt{\frac{\Theta_2(\tau)}{\eta(\tau)}} \tag{121}$$
$$\tag{122}$$

where the Theta-functions and the η-function are defined as ($\tau \in \mathbb{H}; q = e^{2\pi i \tau}$):

$$\Theta_2(\tau) = 2q^{\frac{1}{8}} \prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 \tag{123}$$
$$\Theta_3(\tau) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2 \tag{124}$$
$$\Theta_4(\tau) = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2 \tag{125}$$
$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \tag{126}$$

For a derivation of the characters have a look at [5] (chapter 6).

Knowing the following modular properties of the Theta- and η-functions

$$\Theta_2(-\frac{1}{\tau}) = \sqrt{-i\tau}\Theta_4(\tau) \tag{127}$$
$$\Theta_3(-\frac{1}{\tau}) = \sqrt{-i\tau}\Theta_3(\tau) \tag{128}$$
$$\Theta_4(-\frac{1}{\tau}) = \sqrt{-i\tau}\Theta_2(\tau) \tag{129}$$
$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau) \tag{130}$$

it is easy to calculate the action of the modular transformation $\mathcal{S} : \tau \to -\frac{1}{\tau}$ on the characters of the primary fields:

$$\chi_0(-\frac{1}{\tau}) = \frac{1}{2}\left(\sqrt{\frac{\Theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\Theta_2(\tau)}{\eta(\tau)}}\right) \tag{131}$$

$$= \frac{1}{2}\left(\sqrt{\frac{\Theta_2(\tau)}{\eta(\tau)}} + \frac{1}{2}\left(\sqrt{\frac{\Theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\Theta_4(\tau)}{\eta(\tau)}}\right) + \frac{1}{2}\left(\sqrt{\frac{\Theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\Theta_4(\tau)}{\eta(\tau)}}\right)\right) \tag{132}$$

$$= \frac{1}{2}(\chi_0(\tau) + \chi_{\frac{1}{2}}(\tau) + \sqrt{2}\chi_{\frac{1}{16}}(\tau)) \tag{133}$$

$$\tag{134}$$

$$\chi_{\frac{1}{2}}(-\frac{1}{\tau}) = \frac{1}{2}(\chi_0(\tau) + \chi_{\frac{1}{2}}(\tau) - \sqrt{2}\chi_{\frac{1}{16}}(\tau)) \tag{135}$$

$$\tag{136}$$

$$\chi_{\frac{1}{16}}(-\frac{1}{\tau}) = \frac{1}{\sqrt{2}}(\chi_0(\tau) - \chi_{\frac{1}{2}}(\tau)) \tag{137}$$

So from equations (135) -(137) we can conclude, that in the basis of the characters, the S-matrix is given by:

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{138}$$

Note that this matrix is real, unitary and symmetric, hence $S^{-1} = S$. In the next step, we calculate the fusion matrices N_0 (corresponding to the vacuum field $\mathbb{1}$), N_1 ($\to \epsilon$) and N_2 ($\to \sigma$). To do so, we make use of the Verlinde formula:

$$\mathcal{N}_{ij}^k = \sum_{l=0}^{2} \frac{S_{jl}S_{il}(S^{-1})_{lk}}{S_{0l}} \tag{139}$$

For example, we get:

$$(N_0)_{11} = \mathcal{N}_{01}^1 \tag{140}$$

$$= \frac{S_{10}S_{00}(S^{-1})_{01}}{S_{00}} + \frac{S_{11}S_{01}(S^{-1})_{11}}{S_{01}} + \frac{S_{12}S_{02}(S^{-1})_{21}}{S_{02}} \tag{141}$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \tag{142}$$

$$= 1 \tag{143}$$

Magically, all the entries of the three fusion matrices are either 1 or 0! It is left up to the reader to check that these matrices are in deed given by

$$N_0 = N_{\mathbb{1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{144}$$

$$\tag{145}$$

$$N_1 = N_\epsilon = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{146}$$

$$\tag{147}$$

$$N_2 = N_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{148}$$

and that the S-matrix actually diagonalizes them.

Finally we can read off the fusion rules for the Ising model from the fusion matrices using the fusion algebra $[\Phi_i] \times [\Phi_j] = \sum_k \mathcal{N}_{ij}^k [\Phi_k]$:

$$\epsilon \times \sigma = \sigma \tag{149}$$

$$\sigma \times \sigma = \mathbb{1} + \epsilon \tag{150}$$

$$\epsilon \times \epsilon = \mathbb{1} \tag{151}$$

From this simple example we can conclude that the Verlinde formula is an important tool for the calculation of fusion rules!

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