

# Entanglement Entropy in Conformal Field Theory

Matthias Strodtkötter

*Proseminar FS13, Departement Physik, ETH Zürich*

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## Abstract

The concept of quantum mechanical entropy as a measure of entanglement of mixed states is introduced in order to investigate its behaviour in a conformal field theory considering a very simple model. The final result is nice in the sense, that it agrees with intuitive demands towards the outcome. Upon reading this report one can get an insight into the topic on a relatively basic level. The reader can use this as a short introduction to entanglement entropy and an exemplary calculation of one.

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# 1 Preliminaries

This report requires the reader to have basic knowledge about quantum mechanics. Especially the bra-ket notation will be made use of and understanding of it will be seen as a prerequisite. Chapter one serves to introduce a more general notion of physical systems, as well as to understand some mathematical tools. Only finite dimensional Hilbert spaces will be considered in this section.

## 1.1 Density operator formalism

Let us first introduce a special kind of operator, along with some fundamental properties.

### Definition 1 (*density operator*)

Let  $\rho$  be an operator on a Hilbert space  $H$ .  
If  $\rho$  is positive and  $\text{tr}(\rho) = 1$ , then  $\rho$  is a density operator.

### Definition 2 (*pure density operator*)

Let  $\rho$  be a density operator on a Hilbert space  $H$ .  
 $\rho$  is pure  $:\Leftrightarrow \exists |\phi\rangle \in H$ , such that  $\rho = |\phi\rangle\langle\phi|$ .  
Otherwise  $\rho$  is mixed.

### Definition 3 (*fully mixed density operator*)

Let  $\rho$  be a density operator on a Hilbert space  $H$ .  
 $\rho$  is fully mixed  $:\Leftrightarrow \rho = \frac{1}{\dim H} \cdot \mathbb{1}_H$

We will come to understand that “mixed” and “fully mixed” are related in the intuitive way, that is, “fully mixed” corresponds to not being pure in a maximal way. Also we want to make a lot of use of the following

### Theorem 1 (*Spectral Theorem*)

Let  $H$  be a Hilbert space and  $O \in \text{End}(H)$  normal.  
 $\Rightarrow \exists$  an orthonormal basis  $\{|i\rangle\}_i$ , such that  $O = \sum_i \lambda_i |i\rangle\langle i|$  where  $\lambda_i \in \mathbb{C}$  is the  $i$ -th eigenvalue, and  $\{|i\rangle\}_i$  is an orthonormal basis of eigenvectors of  $O$ .

Since positive operators are hermitian, and hermitian operators are normal, we find that Theorem 1 holds for density operators.

We found, that any state  $|\phi\rangle \in H$  corresponds to a pure density operator  $|\phi\rangle\langle\phi| \in \text{End}(H)$ . Hence density operators represent objects, for which states (represented by kets) are merely a special case. Let us also already note, that while states were only defined up to a phase factor, which does not change any physically relevant quantity, density operators cancel out this ambiguity by construction, as one can easily read off the definition of a pure state. Once we know how physically relevant quantities translate into density operator formalism we might be inclined to state quantum physics in terms of density operators and ask the question, if they add to the theory. That is, if there are physical systems, which can not be described by states, but rather by density operators.

In this spirit, let us introduce calculation rules.

**Definition 4 (*expectation value for a density operator*)**

Let  $A, \rho \in \text{End}(H)$ , where  $H$  is a Hilbert space and  $\rho$  is a density operator. We define the expectation value  $\langle A \rangle_\rho := \text{tr}(\rho A)$

Let us check, if we recover the definition of the expectation value for a ket in the pure limit.

$$\begin{aligned}
 \text{tr}(\rho A) &\stackrel{\text{pure}}{\underset{\text{limit}}{=}} \text{tr}(|\phi\rangle\langle\phi| A) \\
 &\equiv \sum_i \langle i|\phi\rangle\langle\phi| A|i\rangle \\
 &\stackrel{\text{lin.}}{=} \langle\phi| A \sum_i |i\rangle\langle i|\phi\rangle \\
 &\stackrel{\text{id.}}{=} \langle\phi| A|\phi\rangle. \\
 &\equiv \langle A \rangle_\phi
 \end{aligned} \tag{1}$$

To get a feeling for the correspondence between kets and density operators let us consider the following

**Example 1 (*superposition in a 2-d Hilbert-space*)**

Let  $\mathcal{B} = \{|0\rangle, |1\rangle\}$  be an orthonormal basis. Consider:

$$|\psi_\pm\rangle := \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \quad \Rightarrow \quad \rho_\psi \equiv |\psi\rangle\langle\psi| \underset{\mathcal{B}}{\cong} \begin{pmatrix} \frac{1}{2} & \pm\frac{1}{2} \\ \pm\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{2}$$

**1.2 Mixed states**

This section is about states that are not pure, that is, states that can not be written as a ket.

By combining Theorem 1 with the Definition 1 of a density operator  $\rho$ , we find that

$$\rho = \sum_i \lambda_i |i\rangle\langle i|, \tag{3}$$

where  $\lambda_i \geq 0$ , because  $\rho$  is positive, and  $\sum_i \lambda_i = 1$ , since  $\text{tr}(\rho) = 1$ . In other words  $\rho$  is a convex combination of  $\{|i\rangle\}_i$ . It is obvious by the definition, that a pure state corresponds to a convex combination where one  $\lambda_i = 1$  and the rest is zero. In other words a density operator is pure if and only if it is of rank one. Now we can understand why the Definition 3 of a fully mixed density operator makes sense. If we compare density operators in their diagonal form, then being of the form  $\frac{1}{\dim H} \mathbb{1}_H$  is like being the furthest away from having rank one.

Let us now answer the question whether non-pure states are of any interest with a very prominent

**Example 2 (*time independent Hamiltonian in thermal equilibrium*)**

Here we have to consider a probability distribution ( $\Rightarrow$  convex combination) of energy eigenstates. The probabilistic weights are given by  $p_n = \frac{e^{-\beta E_n}}{Z}$ , where  $H$  is the corresponding Hamiltonian,  $\beta$  is the inverse temperature and  $Z = \sum_n p_n$  is the partition function. Hence

$$\rho_{thermal} = \sum_n \frac{e^{-\beta E_n}}{Z} |n\rangle \langle n|,$$

and in general this sum will not consist of one term only.

Before we end this section let us quickly consider another deceiving

**Example 3 (*fully mixed density operator in 2-d*)**

For an orthonormal basis  $\{|i\rangle\}_{i=0,1}$  of a two dimensional Hilbert space  $H$  consider

$$\rho := \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \cong_{\mathcal{B}} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Now one is tempted to instantly read off a simple superposition and be wrong. We have seen a superposition in Example 1 and found a very different result. The meaning of the weights of the decomposition of a mixed state can differ, but the most common one is the one we witnessed in Example 2 as probabilistic weights. Hence the density operator considered here rather corresponds to probabilities  $\frac{1}{2}$  of encountering either one or the other state.

**1.3 Tensor products**

As we will have great interest in compositions of systems, let us quickly understand the mathematical terms in which these are stated. In classical mechanics the composition of two systems is given by the

**Definition 5 (*Cartesian product*)**

Let  $H_A$  &  $H_B$  be Hilbert spaces.  
Let  $\dim(H_A) = m$  &  $\dim(H_B) = n$ .

Then

$$H_A \times H_B := \{(x, y) \mid x \in H_A, y \in H_B\}, \tag{4}$$

which is the set of all ordered pairs, is called the cartesian product of  $H_A$  &  $H_B$ .

It is straight forward to write down a basis  $\mathcal{B}_{A \times B}$  of  $H_A \times H_B$ . Suppose  $\mathcal{B}_A$  is a basis of  $H_A$ , and  $\mathcal{B}_B$  is a basis of  $H_B$ , then  $\mathcal{B}_{A \times B} := (\mathcal{B}_A, 0) \cup (0, \mathcal{B}_B)$  is a basis of  $H_{A \times B}$ . Hence  $\dim(H_A \times H_B) = m + n$ .

This marks a major difference between classical mechanics and quantum mechanics, because in the latter a composition of systems is given by the

**Definition 6 (*Tensor product*)**

Let  $H_A$  &  $H_B$  be Hilbert spaces.  
 Let  $\dim(H_A) = m$  &  $\dim(H_B) = n$ .  
 Let  $\mathcal{B}_A := \{a_1, a_2, \dots, a_m\}$  be a basis of  $H_A$ .  
 Let  $\mathcal{B}_B := \{b_1, b_2, \dots, b_n\}$  be a basis of  $H_B$ .

Then

$$H_A \otimes H_B := \text{the Hilbert space, whose basis is } \{(a_i, b_j) \mid a_i \in \mathcal{B}_A, b_j \in \mathcal{B}_B\} \quad (5)$$

Since the number of basis elements defines the dimension of a vector space, the dimension of a tensor product space is the number of possible combinations of basis elements from the original vector spaces. Hence here we find  $\dim(H_A \otimes H_B) = m \cdot n$ .

We see, that while in classical mechanics the composition of two systems seems to be describable by the sum of information, quantum mechanics postulates a multiplicative growth of necessary information for a complete description of a system. Even more surprising will be the upcoming finding, that possession of all information about a tensor product space can imply the absence of information about parts of the system.

**1.4 Partial trace**

**Definition 7 (*Partial trace*)**

Let  $H$  be a Hilbert space, such that  $H = H_A \otimes H_B$  for Hilbert spaces  $H_A$  &  $H_B$ .  
 The partial trace  $tr_B$  is the unique map, that for any  $F \in \text{End}(H)$  given by  $F = F_A \otimes F_B$ , where  $F_A \in \text{End}(H_A)$  &  $F_B \in \text{End}(H_B)$  maps

$$tr_B : \text{End}(H) \rightarrow \text{End}(H_A)$$

$$F = F_A \otimes F_B \mapsto tr(F_B) \cdot F_A.$$

The following properties are easy to proof and provide a better understanding of this mapping.

Let  $H_A, H_B, H_C$  be Hilbert spaces,  $S_{AB} \in H_A \otimes H_B$ ,  $S_{ABC} \in H_A \otimes H_B \otimes H_C$  and  $T_A \in \text{End}(H_A)$ . Then:

- The partial trace is linear.
- $tr_{AB}(S_{ABC}) = tr_A(tr_B(S_{ABC})) = tr_B(tr_A(S_{ABC}))$
- $tr_B(S_{AB}(T_A \otimes \mathbb{1}_B)) = tr_B(S_{AB})T_A$
- $tr_B((T_A \otimes \mathbb{1}_B)S_{AB}) = T_A tr_B(S_{AB})$

A feature of the partial trace and tensor products in general already looks through. It is, that tensor products stay tensor products, be it under mappings or taking traces. To put in symbols and make clear what was stated in words, consider the following

**Example 4**

Let  $\rho_{ABC}, \rho_A, \rho_B, \rho_C$  be density operators such that  $\rho_{ABC} = \rho_A \otimes \rho_B \otimes \rho_C$ .

Let  $T \otimes V \otimes \mathbb{1}$  be an exemplary mapping.

Then:

1.  $tr_B(\rho_{ABC}) = tr_B(\rho_A \otimes \rho_B \otimes \rho_C) = \rho_A \otimes \rho_C$
2.  $(T \otimes V \otimes \mathbb{1})(\rho_{ABC}) = (T \otimes V \otimes \mathbb{1})(\rho_A \otimes \rho_B \otimes \rho_C) = T\rho_A \otimes V\rho_B \otimes \rho_C$

Let us also find out how nice an element of a tensor product space can look in an educated choice of basis.

**Theorem 2 (Schmidt decomposition)**

Let  $H_A$  &  $H_B$  be Hilbert spaces of dimensions  $n$  and  $m$  respectively.

Then:

$\forall |\psi\rangle \in H_A \otimes H_B \exists$  orthonormal sets  $\{|i\rangle_A\}_i \subset H_A$  &  $\{|k\rangle_B\}_k \subset H_B$ , such that

$$|\psi\rangle = \sum_{i=1}^{\min\{n,m\}} \gamma_i |i\rangle_A \otimes |i\rangle_B,$$

where the  $\gamma_i$  are non-negative and, as a set, uniquely determined by  $|\psi\rangle$ .

$|i\rangle_A$  &  $|k\rangle_B$  are eigenstates of  $\rho_A := tr_B(|\psi\rangle \langle\psi|)$  &  $\rho_B := tr_A(|\psi\rangle \langle\psi|)$  respectively and  $\gamma_i^2$  are the corresponding eigenvalues.

Let us quickly reproduce the latter part of the theorem and emphasize the statement, that  $\rho_A$  &  $\rho_B$  have the same eigenvalues:

Let

$$|\psi\rangle = \sum_{i=1}^{\min\{n,m\}} \gamma_i |i\rangle_A \otimes |i\rangle_B$$

like in the theorem,

$$\Rightarrow \rho_{AB} := |\psi\rangle \langle\psi| = \sum_{i=1}^{\min\{n,m\}} \sum_{k=1}^{\min\{n,m\}} \gamma_i \gamma_k^\dagger |i\rangle_A \langle k|_A \otimes |i\rangle_B \langle k|_B$$

$$\begin{aligned}
\Rightarrow \rho_A &:= \text{tr}_B(\rho_{AB}) \\
&\equiv \sum_{j=1}^m \langle j|_B \rho_{AB} |j\rangle_B \\
&= \sum_{j=1}^m \langle j|_B \left( \sum_i \sum_k \gamma_i \gamma_k^\dagger |i\rangle_A \langle k|_A \otimes |i\rangle_B \langle k|_B \right) |j\rangle_B \\
&= \sum_{j=1}^m \sum_i \sum_k \gamma_i \gamma_k^\dagger |i\rangle_A \langle k|_A \cdot \langle j|_B |i\rangle_B \langle k|_B |j\rangle_B \\
&= \sum_{j=1}^m \sum_i \sum_k \gamma_i \gamma_k^\dagger |i\rangle_A \langle k|_A \cdot \delta_{ji} \delta_{kj} \\
&= \sum_{j=1}^{\min\{m,n\}} \gamma_j \gamma_j^\dagger |j\rangle_A \langle j|_A \\
&\stackrel{\gamma_i \in \mathbb{R}}{=} \sum_{j=1}^{\min\{m,n\}} \gamma_j^2 |j\rangle_A \langle j|_A
\end{aligned} \tag{6}$$

Remembering (3) and identifying  $\gamma_i^2 = \lambda_i$ , as well as  $\lambda_i$  as the  $i^{\text{th}}$  eigenvalue we just have to follow through almost the same calculation and find

$$\rho_B := \text{tr}_A(\rho_{AB}) = \dots = \sum_{j=1}^{\min\{m,n\}} \gamma_j^2 |j\rangle_B \langle j|_B, \tag{7}$$

which proves the claim, that  $\rho_A$  &  $\rho_B$  share the same set of eigenvalues.

A density matrix produced by tracing out one of the spaces defining the product space we call reduced density matrix. Hence we just found, that for any tensor product Hilbert space, the reduced density operators share the same set of eigenvalues. So suppose someone is interested in these eigenvalues, as we will be considering entanglement, then it does not matter the eigenvalues of which of the reduced density operators one determines.

This ends section one and hopefully set up the reader to comprehend section two and subsequently, under the supply of knowledge about conformal field theory, as well as some quantum field theory, which are seen as a premise in the context of the seminar, and can be learned about by considering previous talks of the seminar, section three.

## 2 Entanglement

This section is about the concept, or phenomenon of entanglement. Let us first understand the idea by considering an example, and then introduce the full formality.

### Example 5 (*Totally anti-correlated state in a bipartite system*)

Let  $H_A$  &  $H_B$  be 2-dimensional Hilbert spaces.

Let  $\{|0\rangle_A, |1\rangle_A\}$  &  $\{|0\rangle_B, |1\rangle_B\}$  be orthonormal bases.

Suppose  $H_A$  is located in Alice laboratory in Adelaide, and  $H_B$  is located at Bobs laboratory in Berlin.

Consider a state

$$H_A \otimes H_B \ni |\psi\rangle = \frac{1}{\sqrt{2}} (|1\rangle_A \otimes |0\rangle_B + |0\rangle_A \otimes |1\rangle_B).$$

If the state is left alone, neither Alice nor Bob can tell what she or he would measure in the given basis. But suppose Alice undertakes a measurement in the given basis. If she measures a “1”, then the state collapses to  $|1\rangle_A \otimes |0\rangle_B$  and Bob is bound to measure “0”, and vice versa in the other case! This is a prime example for what is called entanglement. Alice only has access to a reduced density operator corresponding to a mixed state. So the state Alice has access to is called entangled with the state Bob has access to.

### 2.1 Definitions and properties

Now knowing which phenomenon we want to give a name motivates the following

#### Definition 8 (*Separable / entangled*)

Let  $H$ ,  $H_A$  &  $H_B$  be Hilbert spaces, such that  $H = H_A \otimes H_B$ .

Let  $\rho$  be a density operator on  $H$ .

$\rho$  is separable  $:\Leftrightarrow \exists$  sets of density operators  $\{\rho_A^i\}_i \subset \text{End}(H_A)$  &  $\{\rho_B^i\}_i \subset \text{End}(H_B)$ , such that  $\rho$  is the convex combination

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i.$$

Otherwise  $\rho$  is entangled.

The definition provides exactly what we want. If a state is separable by the definition just stated, then any subsystem can project its reduced density operator on any state it wishes, without influencing the other reduced density operator. Once one can not separate the density operator like in the definition any more, this behaviour is lost and the density operator becomes “entangled”.

For the special case of a pure density operator  $\rho = |\psi\rangle \langle\psi| \in \text{End}(H_A \otimes H_B)$  the definition reduces to the statement, that  $|\psi\rangle$  is separable  $\Leftrightarrow \exists |\psi\rangle_A \in H_A$  &  $|\psi\rangle_B \in H_B$ , such that  $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$ . Otherwise, as before,  $|\psi\rangle$  is entangled.



This allows us to understand, that for the pure density operator  $\rho_{AB} := |\psi_{AB}\rangle\langle\psi_{AB}|$  the following statements are equivalent:

1.  $|\psi_{AB}\rangle\langle\psi_{AB}|$  is separable
2.  $\exists |\psi_A\rangle, |\psi_B\rangle$  such that  $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$
3.  $\text{rank}(tr_B(\rho_{AB})) = 1$
4.  $\text{rank}(tr_A(\rho_{AB})) = 1$
5.  $\exists$  only one non-vanishing Schmidt-coefficient of either  $tr_A(\rho_{AB})$  or  $tr_B(\rho_{AB})$

## 2.2 Entropy

We now learned, that the amount of entanglement goes with the mixedness of either reduced density operator. But already then it turned out to be a rather hazy business telling how mixed a state actually is. Hence our interest lies in finding a measure of entanglement. To do so we will first introduce and then investigate the

### Definition 9 (*Von Neumann entropy*)

Let  $\rho$  be a density operator.

Then

$$S(\rho) := -tr(\rho \cdot \ln(\rho))$$

is called the von Neumann entropy of  $\rho$ .

If one writes such a  $\rho$  in terms of its eigenvectors  $|i\rangle$  and eigenvalues  $\eta_i$  according to Theorem 1 (Spectral theorem) like

$$\rho = \sum_i \eta_i |i\rangle\langle i|,$$

then the entropy turns out to be

$$S(\rho) = - \sum_i \eta_i \cdot \ln(\eta_i). \quad (8)$$

To see how the usefulness of the entropy as a measure of entanglement turns out to be, let us come back to the previously considered Example 5, of a totally anti-correlated state in a bipartite system.

### Example 6

Reconsider

$$H_A \otimes H_B \ni |\psi\rangle = \frac{1}{\sqrt{2}} (|1\rangle_A \otimes |0\rangle_B + |0\rangle_A \otimes |1\rangle_B).$$

$$\begin{aligned} \Rightarrow \rho_{AB} &:= |\psi\rangle\langle\psi| \\ &= \frac{1}{2} (|0\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|) \end{aligned}$$

where indices were dropped for visibility.

$$\Rightarrow \rho_A := \text{tr}_B(\rho_{AB}) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|),$$

which is fully mixed.

Now if the entropy is to measure entanglement, it better be big, because this is a totally anti-correlated state.

$$S(\rho_A) \equiv -\text{tr}\left(\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \cdot \ln\left(\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)\right)\right) = -2 \cdot \frac{1}{2} \ln\left(\frac{1}{2}\right) = \ln(2)$$

it turns out, that for an  $n$ -dimensional Hilbert space  $H$  and a density operator  $\rho$  on it, the maximal value the entropy can actually become, is  $\ln(n)$ .

Before moving on, lets get some exemplary insight in the other extreme.

### Example 7 (*Pure product-state*)

Consider a state

$$\begin{aligned} |\psi_{AB}\rangle &= |\psi_A\rangle \otimes |\psi_B\rangle. \\ \Rightarrow \rho_{AB} &:= |\psi_{AB}\rangle\langle\psi_{AB}| = |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|. \\ \Rightarrow \rho_A &:= \text{tr}_B(\rho_{AB}) = |\psi_A\rangle\langle\psi_A|, \text{ which is pure.} \\ \Rightarrow S(\rho_A) &= \ln(1) = 0 \end{aligned}$$

To sum up our findings:

1. Pure states  $\Rightarrow S(\rho) = 0$
2. Mixed states  $\Rightarrow S(\rho) > 0$
3. Fully mixed states  $\Rightarrow S(\rho) = \ln(\dim(H))$

Most commonly the entropy of a quantum mechanical state is interpreted as the amount of missing information. If one undertakes a measurement of a pure state in the right basis, one can beforehand know what is the outcome with one hundred percent certainty. However, for mixed states one can never choose a basis, such that the outcome is pre-decided. In other words, there is information missing about that state, meaning one can not tell how it will interact with the rest of the universe. In this context it is evident why for example a fully mixed state has to correspond to a maximum in a measure of entanglement. In a fully mixed state there is not even a tendency as to what will be the outcome of a measurement. As a consequence one could not know less about such a state, or, equivalently, there could not be more information missing.

The von Neumann entropy will be what we are interested in. Nevertheless there is a more general definition for an entropy, which together with a trick can ease calculations of entropies a lot.

**Definition 10 (Rényi entropy)**

Let  $\rho$  be a density operator.

Let  $\alpha \geq 0$ .

Then

$$S_\alpha(\rho) := \frac{1}{1-\alpha} \ln(\text{tr}(\rho^\alpha))$$

is called the Rényi entropy of  $\rho$ .

For the connection between the von Neumann entropy and the Rényi entropy consider

**Claim 1**

Let  $\rho$  be a density operator.

Then

$$\lim_{\alpha \rightarrow 1} S_\alpha(\rho) = S(\rho).$$

**Proof.**

$$\begin{aligned} \lim_{\alpha \rightarrow 1} S_\alpha(\rho) &\equiv \lim_{\alpha \rightarrow 1} \frac{1}{1-\alpha} \log(\text{tr}(\rho^\alpha)) \\ &= \lim_{\alpha \rightarrow 1} \frac{1}{1-\alpha} \log\left(\sum_i \eta_i^\alpha\right) \\ &= \lim_{\alpha \rightarrow 1} \frac{\log(\sum_i \eta_i^\alpha)(1+\alpha)}{(1-\alpha)(1+\alpha)} \\ &\stackrel{vH}{=} \lim_{\alpha \rightarrow 1} \frac{\log(\sum_i \eta_i^\alpha) + (1+\alpha) \frac{\sum_i \log(\eta_i) \eta_i^\alpha}{\sum_i \eta_i^\alpha}}{-2\alpha} \\ &= \frac{2 \sum_i \log(\eta_i) \eta_i}{-2} \\ &= - \sum_i \log(\eta_i) \eta_i \\ &\equiv S(\rho) \quad \square \end{aligned} \tag{9}$$

Also there is

**Claim 2**

Let  $\rho$  be a density operator.

Then

$$-\lim_{\alpha \rightarrow 1} \frac{\partial}{\partial \alpha} \text{tr}(\rho^\alpha) = S(\rho). \tag{10}$$

**Proof.**

$$\begin{aligned} -\lim_{\alpha \rightarrow 1} \frac{\partial}{\partial \alpha} \text{tr}(\rho^\alpha) &= -\lim_{\alpha \rightarrow 1} \frac{\partial}{\partial \alpha} \left(\sum_i \eta_i^\alpha\right) \\ &= -\lim_{\alpha \rightarrow 1} \sum_i \log(\eta_i) \eta_i^\alpha \\ &= - \sum_i \log(\eta_i) \eta_i \\ &\equiv S(\rho) \quad \square \end{aligned} \tag{11}$$

### 3 Entanglement entropy in Conformal Field Theory

In this section we want to conduct a seemingly easy task, to calculate an entanglement entropy in a CFT for a very simple model.

**The task:**

- Assume the universe  $U$  is in the ground state and consists of one time dimension, and one space dimension.
- Consider an observer with limited access to the universe, that is, in general to a non-trivial density operator.
- The aim is to measure the correlation of the accessible subsystem to the rest of the universe.
- This will be done by measuring the entropy of the accessible region  $R_1$  relative to the rest of the universe  $R_2$ .

**The model:**

- The space dimension is an interval  $U = [0, \Lambda)$  with periodic boundary conditions identifying  $0 = \Lambda$ .
- $\Lambda$  is the infra red cut off.
- The accessible region for the observer in question is given by  $[0, \Sigma)$ .
- A sharp distinction between being inside or outside the accessible region is impossible and leads to infinities in the correlation. Hence we introduce a regulating parameter  $\varepsilon$  producing intervals around the edges of the system which we will not consider for the calculation. Later we will see what happens when we make  $\varepsilon$  small.
- Introduce a complex variable  $\zeta = \sigma + i\tau$ , where  $\sigma$  is the space coordinate, and  $\tau$  is the time coordinate.

Now introduce a complete set of commuting observables  $\xi_{in}$  &  $\xi_{out}$ . These are supposed to be a choice, such that the  $\xi_{in}$  are localized inside  $R_1$ , and the  $\xi_{out}$  are localized inside  $R_2$ . As a consequence the density operator of the universe  $\rho_U$  can be expressed in terms of their eigenvalues.

$$\rho_U = \rho_U(\xi_{in}^1, \xi_{out}^1; \xi_{in}^2, \xi_{out}^2). \tag{12}$$

Hence the reduced density operator for the accessible region  $\rho_{in}$  is given by

$$\rho_{in}(\xi_{in}^1; \xi_{in}^2) = \sum_{\xi_{out}} \rho_U(\xi_{in}^1, \xi_{out}; \xi_{in}^2, \xi_{out}). \tag{13}$$

The question one asks now may be if such a choice of observables exists in the first place. The answer is yes, and it is also the obvious answer to give. One just chooses the field operators  $\Phi(\sigma)$  provided by the field theory, as these are local functions only depending on position.

Next we introduce two conformal coordinate transformations. These have unitary representations and hence leave traces alone. As a consequence they will not influence the result, as we will consider a trace in the end.

$$\omega := -\frac{\sin\left(\frac{\pi}{\Lambda}(\zeta - \Sigma)\right)}{\sin\left(\frac{\pi}{\Lambda}\zeta\right)} \quad (14)$$

$$z := \frac{1}{\kappa} \ln(\omega) \quad (15)$$

To learn how the domains transform under these mappings just consider Figure 1. Also  $L := \frac{2}{\kappa} \ln\left(\frac{\Sigma}{\varepsilon}\right)$ .  $\kappa$  is an auxiliary parameter. We also impose periodic boundary conditions in the  $L$ -direction, making the universe a cylinder.

Now that we consider the domain of the  $z$ -coordinate as in the bottom of Figure 1, let us call the field at  $R_1$   $X$ , and the field at  $R_2$   $Y$ . As the ground state is in thermal equilibrium, an entry of the density operator of the accessible part of the universe takes the following form:

$$\Psi_{XY} \propto \langle Y | e^{-\beta H} | X \rangle \quad (16)$$

Path integral formalism in imaginary times provides

$$\Psi_{XY} \propto \int D\phi e^{-S(\phi)} \quad (17)$$

The density matrix describing the subsystem on  $R_1$  after tracing out variables on  $R_2$  is

$$\rho_{XX'} \propto \int DY \Psi_{XY} \Psi_{YX'} \quad (18)$$

But in the light of the path integrals of the  $\Psi_{AB}$ , this is just integrating over all paths going from  $X$  to  $Y$ , all paths going from  $Y$  to  $X'$ , while making sure they meet at  $Y$  by integrating it out. In other words this is like gluing two of the strips like in the bottom of Figure 1 together and making it one strip of twice the length  $\frac{\pi}{\kappa} \rightarrow \frac{2\pi}{\kappa}$ . Let us introduce normalization.

$$\rho_{XX'} = \frac{1}{Z(1)} \int_{\frac{2\pi}{\kappa} \text{ Strip}} D\phi e^{-S(\phi)} \quad (19)$$

$$\Rightarrow \rho_{XX'}^a = \frac{1}{Z(1)^a} \int_{\frac{2\pi a}{\kappa} \text{ Strip}} D\phi e^{-S(\phi)} \quad (20)$$

Taking the trace now corresponds to identifying  $X' = X$ , that is, gluing the ends of the cylinder together and making it a torus. To calculate the entropy we want to use Claim 2. We will call  $\text{tr}(\rho_{XX'}^a) = \int_{\frac{2\pi a}{\kappa} \text{ Strip}} DX \rho_{XX}^a =: Z(a)$ . That also explains why the normalizer is  $Z(1)$ . Then

$$S = \left( -\frac{d}{da} \frac{Z(a)}{Z(1)^a} \right)_{a=1} = \left( 1 - a \frac{d}{da} \right)_{a=1} \ln(Z(a)). \quad (21)$$

How to calculate partition functions on a torus in a CFT has been explained in this seminar before. In analogy to Stefan Hubers talk we now introduce  $q := e^{2\pi i \tau}$ , where  $\tau = i \frac{2\pi a}{\kappa L}$ . We then perform a modular transformation  $\tau \mapsto -\frac{1}{\tau}$ , which leaves the partition function invariant. This then implies  $q = e^{-\kappa L} = q^\dagger$ . Applying these transformations to (19) we find

$$S = \left( 1 + \ln q \frac{\partial}{\partial \ln q} + \ln q^\dagger \frac{\partial}{\partial \ln q^\dagger} \right) \ln(Z(1)) \quad (22)$$

For a CFT with central extensions  $c$  &  $c^\dagger$  on a torus we know

$$Z(\tau, \tau^\dagger) = -\frac{c + c^\dagger}{24} \ln q + \ln(\text{tr}(q^{L_0 + L_0^\dagger})) \quad (23)$$

The argument of the logarithm of the second term can be expanded in powers of  $q$  around 1, but if we plug in for  $L$  we find that  $q$  will be small, when  $\varepsilon$  is small. On the other hand the first term grows, as  $\varepsilon$  becomes small. Hence we will neglect the second term to end up with a rather beautiful result:

$$S = \frac{c + c^\dagger}{6} \ln \left( \frac{\Sigma}{\varepsilon} \right) \quad (24)$$

The equation tells us, that as the “walls”, embodied by  $\varepsilon$ , get thinner, and the system, embodied by  $\Sigma$ , gets bigger, the accessible region of the universe gets correlated to the rest of the universe stronger.

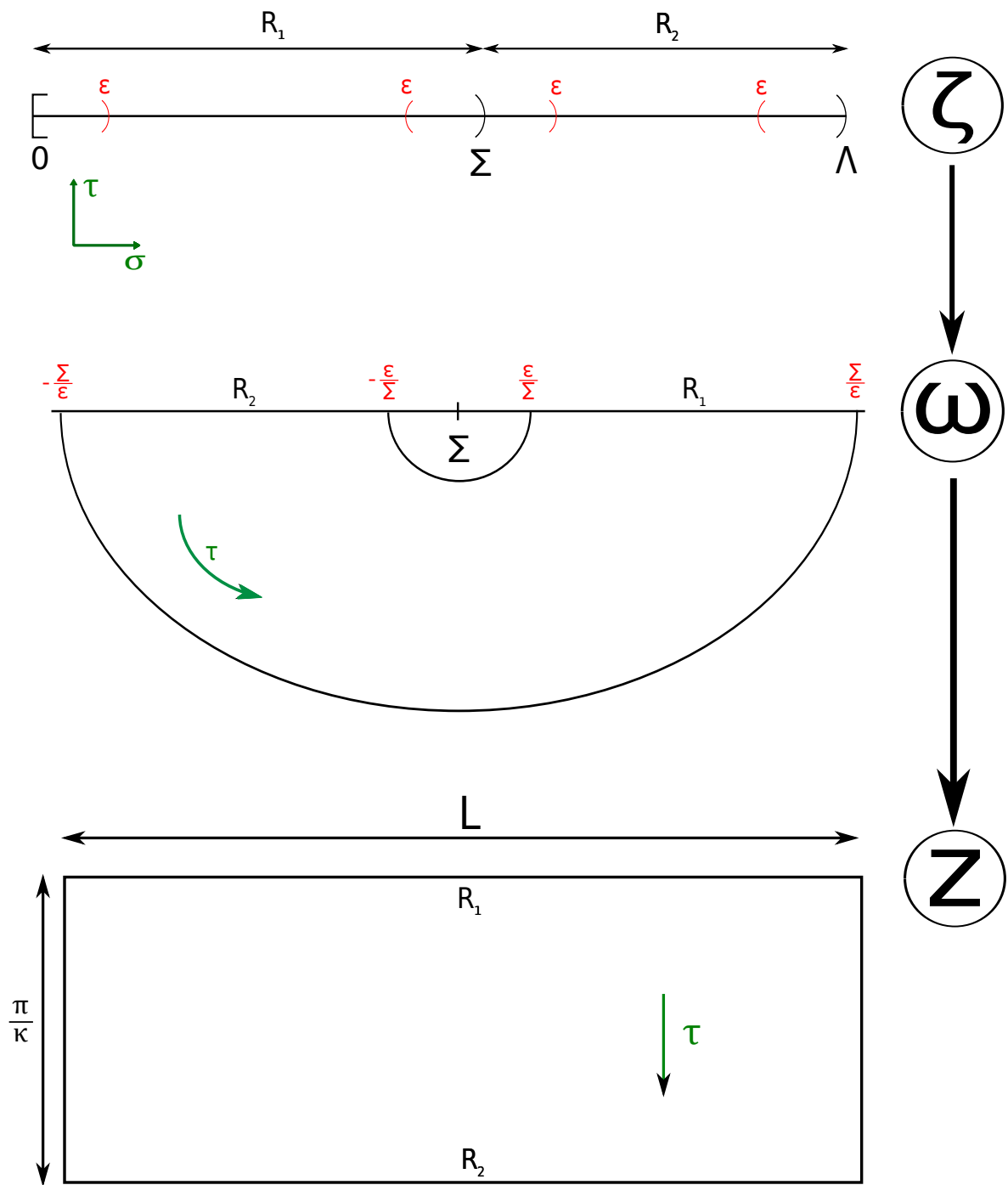


Figure 1: From the top the the bottom: The universe in the original complex coordinates and after each of the coordinate transformations.

## References

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