PROSEMINAR IN THEORETICAL PHYSICS

# Classical String Theory

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**Abstract**

The following report is based on my talk about classical string theory given at April 15, 2013 in the course of the proseminar 'conformal field theory and string theory'.
In this report a short historical and theoretical introduction to string theory is given. This is followed by a discussion of the point particle action, leading to the postulation of the Nambu-Goto and Polyakov action for the classical bosonic string. The equations of motion from these actions are derived, simplified and generally solved. Subsequently the Virasoro modes appearing in the solution are discussed and their algebra under the Poisson bracket is examined.

# Contents

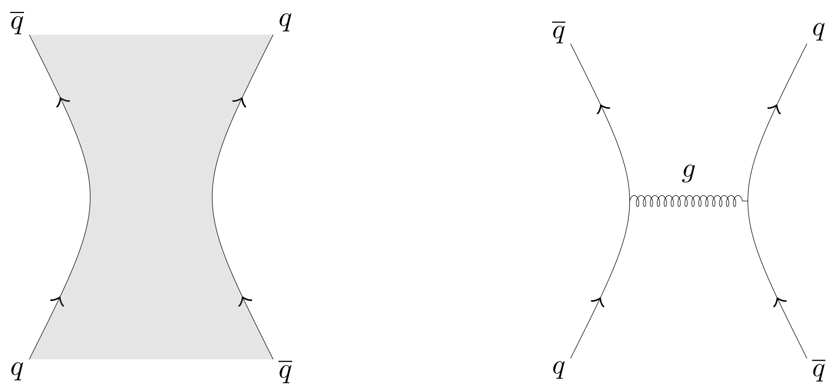
# 1 Introduction

In 1969 the physicists Yoichiro Nambu, Holger Bech Nielsen and Leonard Susskind proposed a model for the strong interaction between quarks, in which the quarks were connected by one-dimensional strings holding them together. However, this model - known as string theory - did not really succeded in describing the interaction. In the 1970's Quantum Chromodynamics was recognized as the theory of strong interactions. String theory, whose quantum theory was discovered to need 10 dimensions to work - in these times a very unlikely assumption -, went into the dustbin of histoy.

However, in this early stage there was one remarkable fact about the new theory:
From all calculations a massless particle with spin two appeared. These properties -mass zero and spin two - were later recognized as the properties of the hypothetical *graviton*, the carrier of the gravitational force.
In the 1980's, after the full discovery of the standard model, it was this fact, which brought string theory back into the minds of the physicists as a possible candidate for a theory of quantum gravitation.

By postulating one dimensional fundamental objects instead of zero dimensional particles, string theory also elludes the problem of renormalization in quantum field theory. The particles of the standard model follow in the large scale limit as the different quantized oscillation modes of the string. This is often compared with the different oscillation modes of a violin leading to different sounds.
The two major drawbacks of the theory were its postulation of additional dimensions - which can be elluded by compactification arguments - and its very small working scale.
Especially this scale - a string length is of order of the Planck length - makes string theory for many practically minded physicists a not falsifiable theory.

Despite these problems string theory develops as a major candidate for a unifying theory and brought - as a sideeffect - many interesting and important mathematical theories to life.



(a) quarks connected by a string (b) gluon exchange

Figure 1.1: String theory was introduced as a model for strong interaction.

# 2 The Relativistic String

Classical string theory is built up of one dimensional, fundamental objects moving in spacetime due to dynamical laws determined by an action principle. The goal of this chapter is to postulate this action and to find the resulting equations of motion.

To quantify the stated ideas we will work in a $D + 1$- dimensional spacetime with one timelike and $D$ spacelike directions. This spacetime is equipped with a general metric tensor $g_{\mu\nu}(x)$ with signature $(-1, 1 \ldots 1)$.

A string as a one dimensional spatial object traces out a two dimensional worldsheet in this spacetime, the same way a relativistic point particle traces out a worldline.
In the derivation of the string action we will stick to this analogy between the worldsheet of a string and the worldline of a particle.

To derive the Nambu Goto Action for string motion, we will start by deriving the action for a point particle.

## 2.1 The Action of a Classical Point Particle

### 2.1.1 The Classicle Point Particle Action

The action of a particle arises from the idea of finding a functional of the particle's path, which is reparametrization invariant (*intrinsic*). This leads directly to the action of the particle being proportional to the 'length' of the particle's worldline ('length' measured with the metric $g_{\mu\nu}$) and to the well known formula $S = -mc \int_\gamma ds$. The proportionality factor is given by $-mc$ with its interpretation as mass obtained from the classical limit.
To state this action more clearly, let $x^\mu(\tau)$ be a parametrization of the worldline of the particle.

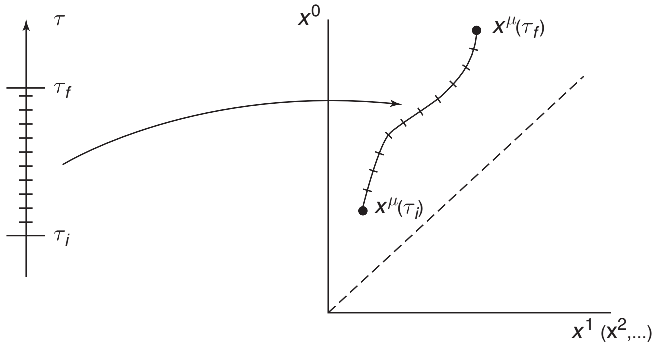



Figure 2.1: parametrization of the world line $x^\mu : [\tau_i, \tau_f] \to \mathbb{R}^D$ [6]

The $(1 \times 1)$ metric on the chart domain $\Gamma$ is simply the pullback of the ambient metric $g_{\mu\nu}$, i.e.

$$\Gamma = \frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tau} g_{\mu\nu}.$$

Therefore, one calculates the volume form as

$$ds = \sqrt{|\det(\Gamma)|}d\tau = \sqrt{-\Gamma}d\tau = \sqrt{-\dot{x}^\mu \dot{x}_\mu}d\tau.$$

Here it is used that $\Gamma < 0$, which is due to the signature of the metric (and its negative definiteness). In the end one arrives at the point particle action

$$S_{\text{point particle}} = -mc\int_{\tau_i}^{\tau_f} \sqrt{-\dot{x}^\mu \dot{x}_\mu}d\tau. \qquad \left(\dot{x}^\mu := \frac{\partial x^\mu}{\partial \tau}\right) \tag{2.1}$$

By choosing the eigentime parametrization

$$x^\mu(\tau) = (c\tau, \mathbf{x}(\tau))$$

this leads to the the well known formula

$$S = -mc^2 \int \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}dt.$$

### 2.1.2 Disadvantages of the Classical Action

The derived action (2.1) has some major disadvantages.
Firstly

- the square root in the Lagrangian makes the quantization difficult and
- the massless case is not described by this action.

There is however one more subtle difficulty:
Define the conjugate momentum

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = mc\frac{\dot{x}_\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}}$$

where $L = -mc\sqrt{-\dot{x}^\mu \dot{x}_\mu}$ is the point particle Lagrangian. Then the matrix

$$\frac{\partial p_\mu}{\partial \dot{x}^\nu} = \frac{\partial^2 L}{\partial \dot{x}^\mu \dot{x}^\nu}$$

has vanishing eigenvalues. Explicitly, one calculates

$$\frac{\partial p_\mu}{\partial \dot{x}^\nu}\dot{x}^\nu = 0.$$

By the inverse function theorem the function $p^\mu(\dot{x}^\nu)$ is not (globally) invertible. However to calculate the Hamiltonian $H = \dot{x}^\mu(p)p_\mu - L(x^\mu, \dot{x}^\mu(p))$ one has to invert $\dot{x}^\mu(p)$. This problem is solved by not defining the Hamiltonian on the whole phasespace.
For every vanishing eigenvalue of the above matrix one finds a so called *primary constraint*. This is a constraint not following from the equations of motion but from the formulation of the action itself. In this case the primary constraint is

$$p^\mu p_\mu = -m^2c^2.$$

Hence, a suitable definition for the Hamiltonian is

$$H = H_{\text{can}} + \sum_n c_n \phi_n$$

where $\phi_n = 0$ are the primary constraints and $H_{\text{can}} = \dot{x}^\mu p_\mu - L(x^\mu, \dot{x}^\mu)$, i.e. we introduce the primary constraints via Lagrange multipliers in the Hamilton formalism. In the case of the free particle $H_{\text{can}} = 0$, the time evolution is therefore completely determined by the primary constraints represented by the Hamiltonian $H = \frac{1}{2m}\left(p^2 + m^2\right)$.
For our purpose it will be enough to recognize primary constraints and to avoid them if possible to have all relevant equations included in the equations of motion.

### 2.1.3 The Einbein Action

A possibility to avoid these difficulties is to define an alternative action such that there is neither a squareroot, nor primary constraints.
Therefore one introduces an auxiliary function $e(\tau)$, the so called *einbein*. This function can be seen as an intrinsic (and not predetermined) metric on the worldline.
The new action is defined as

$$S = \frac{1}{2}\int_{\tau_0}^{\tau_1} d\tau e(\tau) \left(e^{-2}(\tau) \left(\dot{x}^\mu(\tau)\right)^2 - m^2\right). \tag{2.2}$$

Varying the action with respect to the five quantities $x^\mu$ and $e$ leads to the equations of motion

$$\frac{\delta S}{\delta e} = 0 \quad \Rightarrow \quad \dot{x}^2 + e^2 m^2 = 0$$

$$\frac{\delta S}{\delta x^\mu} = 0 \quad \Rightarrow \quad \frac{d}{d\tau}\left(e^{-1}\dot{x}^\mu\right) = 0.$$

Solving the first equation one finds that $e = \frac{1}{m}\sqrt{-\dot{x}^2}$. Substituting this expression into the einbein action (2.2), one arrives back at the classical action, proofing the equivalence of the two actions (in the sense that they lead to the same equations of motion).
However, the einbein action has no primary constraints.

By introducing an additional degree of freedom, the primary constraints are turned into equations of motion. As a side effect one got rid of the squareroot.

## 2.2 The General p-Brane Action

One could derive this action in the much more general case of a spatial $p$-dimensional object, represented by a $(p+1)$- dimensional submanifold (which is a 'line' for $p = 0$ and a 'sheet' for $p = 1$) in a $(D+1)$ - dimensional spacetime with metric $g_{\mu\nu}$. Such an object is in general called a *p-brane*. If we want this object to be in any way 'fundamental' the best candidate for a reparametrization invariant action is in analogy to the point particle proportional to the volume of the object in spacetime.

Let $X^\mu(\tau^i)$ $(\mu = 0, \ldots D,\ i = 0, \ldots p)$ be a parametrization of the p-brane. The $(p+1 \times p+1)$ metric on the chartdomain (or parameterspace) $\Gamma^{(p)}_{ab}$ (where (p) just denotes its dimension) is again the pullback of the spacetime metric

$$\Gamma^{(p)}_{ab} = \frac{\partial X^\mu}{\partial \tau^a}\frac{\partial X^\nu}{\partial \tau^b} g_{\mu\nu}.$$

Hence, the volume form reads

$$dV = \sqrt{|\det(\Gamma_{ab}^{(p)})|}d\tau^0 \dots d\tau^p.$$

The volume of the object is

$$\mathrm{Vol} = \int \sqrt{|\det(\Gamma_{a,b}^{(p)})|}d^{p+1}\tau.$$

Because we are working with Lorentzian metrics, it holds that $\det(\Gamma_{\alpha\beta}^{(p)}) < 0$.
By defining the brane tension $T_p$ as the proportionality factor between the brane action and its *world-volume* in a very similar manner as the mass of the point particle one concludes

$$S_{\text{p-brane}} = -T_p \int \sqrt{-\det(\Gamma_{ab}^{(p)})}d^{p+1}\tau = -T_p \int \sqrt{-\det(\frac{\partial X^\mu}{\partial \tau^a}\frac{\partial X_\mu}{\partial \tau^b})}d^{p+1}\tau.$$

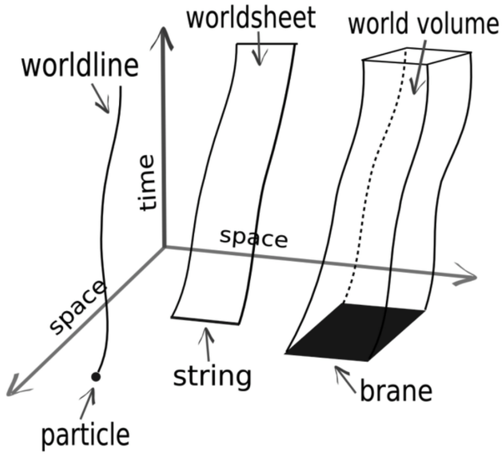



Figure 2.2: worldlines, worldsheets and branes [Wikimedia Commons]

## 2.3 The Nambu Goto Action

In the above framework a string is a 1-brane, a fundamental one (spatial-) dimensional object moving through spacetime. The *Nambu-Goto action* of the string is exactly the 1-brane action derived above.
Let

$$(\tau, \sigma) \mapsto X^\mu(\tau, \sigma)$$

be a parametrization of the string.
By introducing the proportionality factor *string tension* $T_0$ (which will be discussed later) one concludes

$$S_{\text{Nambu-Goto}} = -\frac{T_0}{c}\int \sqrt{-\det(\Gamma_{\alpha\beta})}d\tau d\sigma.$$

From now on using the determinant notation $\Gamma := \det(\Gamma_{\alpha\beta})$ and writing $T = \frac{T_0}{c}$ the most compact form of the Nambu-Goto action reads

$$S_{\text{NG}} = -T\int \sqrt{-\Gamma}\ \mathrm{d}\tau\mathrm{d}\sigma. \tag{2.3}$$

One can simplify this expression further. By writing $\dot{X}^\mu := \frac{\partial X^\mu}{\partial \tau}$ and $X'^\mu := \frac{\partial X^\mu}{\partial \sigma}$ and introducing the Minkowski-product notation $a \cdot b := a^\mu g_{\mu\nu} b^\nu$ and $a^2 := a^\mu a^\nu g_{\mu\nu}$ one finds

$$\Gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}\cdot\dot{X} & \dot{X}\cdot X' \\ X'\cdot\dot{X} & X'\cdot X' \end{pmatrix}.$$

Therefore

$$\Gamma = \det(\Gamma_{\alpha\beta}) = \dot{X}^2 X'^2 - \left(\dot{X} \cdot X'\right)^2$$

and the Nambu Goto action can be written as

$$S_{\text{NG}} = -\frac{T_0}{c} \int \sqrt{\left(\dot{X} \cdot X'\right)^2 - \dot{X}^2 X'^2} d\tau d\sigma.$$

### 2.3.1 Open and Closed Strings

At this point it is necessary to think about the form of the parametrization (and of the string itself), especially in foresight of the discussion of boundary conditions later on. A string worldsheet is conventionally parametrized as $X^\mu(\tau, \sigma)$ with

$$-\infty < \tau < \infty \qquad\qquad 0 < \sigma < \overline{\sigma}.$$

To specify the value of the endpoint of the $\sigma$ parametrization one has to think about the form of strings. There are two kind of strings:

- open strings, with loose ends and a worldsheet diffeomorphic to $\mathbb{R} \times [0, \overline{\sigma} = \pi]$ and
- closed strings, with connected ends (or better 'no ends') and a worldsheet diffeomorphic to the cylinder $\mathbb{R} \times S^1$ (and $\overline{\sigma} = 2\pi$).

The two choices for $\overline{\sigma} = \pi$ resp. $2\pi$ are purely conventional.
Closed strings have no boundary, hence there is no need for any boundary condition besides the periodicity of the function $X^\mu(\sigma)$.
Open strings do need certain boundary conditions which are discussed in chapter 3.2.

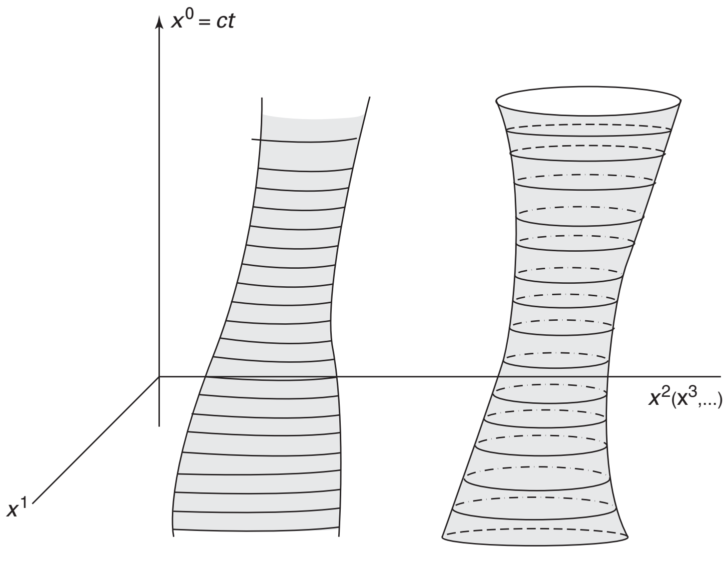



Figure 2.3: open and closed worldsheets [6]

### 2.3.2 The String Tension

To see how one can interpret the string tension $T_0$, one has to consider the (natural) eigenzeit parametrization

$$X^\mu(\tau,\sigma) = (c\tau, \mathbf{x}(\tau,\sigma)).$$

The Lagrangian for time (= $\tau$) evolution is

$$L = -\frac{T_0}{c}\int \mathrm{d}\sigma \sqrt{\left(\dot{X}\cdot X'\right)^2 - \dot{X}^2 X'^2}.$$

At a time $\tau$ for which $\frac{\partial \mathbf{x}}{\partial \tau} = 0$ (the string is at rest) the Lagrangian becomes

$$L = -T_0 \int \mathrm{d}\sigma \left|\frac{d\mathbf{x}}{d\sigma}\right| = -T_0 \text{ (length of string)}.$$

With the interpretation of the Lagrangian as the difference of kinetic and potential energy and noting that the kinetic energy vanishes for a string at rest this means

$$T_0 = \frac{E_{\text{pot}}}{\text{length of string}}.$$

From this point of view the factor $T_0$ can be interpreted as a tension.

One also sees that the string's energy is proportional to its length. This is not Hooke's law for elastic bands (where energy is proportional to length squared). Therefore, a string does not behave like a stretched elastic band.

It is also interesting to note that with this energy dependence, a string tends to collapse to a point. This problem is fixed in quantum mechanics where (roughly speaking) the Heisenberg uncertainty principle yields the opposite impetus.

## 2.4 The Polyakov Action

### 2.4.1 Disadvantages of the Nambu Goto Action

As the free particle action the Nambu Goto action has (besides its squareroot) some disadvantages as well.
With the canonical ($\tau$-) momenta

$$\Pi^{\tau\mu} = \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} = -T\frac{\left(\dot{X}\cdot X'\right)X'^\mu - \left(X'\right)^2 \dot{X}^\mu}{\left[\left(X'\cdot\dot{X}\right)^2 - \left(\dot{X}\right)^2\left(X'\right)^2\right]^{\frac{1}{2}}}$$

the matrix $\frac{\partial \Pi^{\tau\mu}}{\partial \dot{X}^\nu}$ has two zero eigenvalues corresponding to the primary constraints

$$\Pi^{\tau\mu}X'_\mu = 0$$
$$\Pi^{\tau 2} + T^2\left(X'\right)^2 = 0.$$

These constraints are called *Virasoro constraints*.
As in the case of the free particle, the canonical Hamiltonian $H_{\text{can}} = \Pi^{\sigma\mu} X'_\mu + P^{\tau\mu}\dot{X}_\mu - \mathcal{L}$ vanishes; i.e. the motion is completely determined by the constraints.

### 2.4.2 The Polyakov Action and its Equations of Motion

To avoid these difficulties one introduces an intrinsic metric tensor $h_{\alpha\beta}$ on the worldsheet (where $\alpha,\ \beta \in \{0,1\}$). This tensor can be seen as the generalization of the einbein formalism for the free particle and is as a metric tensor a symmetric $2 \times 2$ matrix.
The Polyakov action is defined via

$$S_{\text{P}} = -\frac{T}{2}\int d^2\sigma \sqrt{-h} h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} = -\frac{T}{2}\int d^2\sigma \sqrt{-h} h^{\alpha\beta}\, \Gamma_{\alpha\beta}. \tag{2.4}$$

As one could guess in analogy to the free particle the Polyakov action has no primary constraint, all constraints are contained in the equations of motion of the symmetric $2 \times 2$ tensor $h_{\alpha\beta}$.

Then

$$\delta S = -\frac{T}{2}\int d^2\sigma \left(\delta\sqrt{-h} h^{\alpha\beta}\Gamma_{\alpha\beta} + \sqrt{-h}\delta h^{\alpha\beta}\Gamma_{\alpha\beta} + \sqrt{-h} h^{\alpha\beta}\delta\Gamma_{\alpha\beta}\right).$$

For the variation of the determinant it holds that

$$\delta h = -h\ h_{\alpha\beta}\delta h^{\alpha\beta}$$

and therefore

$$\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h} h_{\alpha\beta}\delta h^{\alpha\beta}.$$

For the metric $\Gamma_{\alpha\beta}$

$$\delta\Gamma_{\alpha\beta} = \partial_\alpha \delta X^\mu \partial_\beta X_\mu + \partial_\alpha X_\mu \partial_\beta \delta X^\mu.$$

Using the symmetry of $h^{\alpha\beta}$ and introducing the energy momentum tensor $T_{\alpha\beta}$ the variation simplifies to

$$\delta S = -T\int d^2\sigma \left(\sqrt{-h} T_{\alpha\beta}\delta h^{\alpha\beta} + 2\partial_\alpha\left(\sqrt{-h} h^{\alpha\beta}\partial_\beta X_\mu\right)\delta X^\mu\right) + \text{boundary term} \tag{2.5}$$

where

$$T_{\alpha\beta} = -\frac{1}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}} = \frac{1}{2}\Gamma_{\alpha\beta} - \frac{1}{4}h^{\gamma\delta}\Gamma_{\gamma\delta}h_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu. \tag{2.6}$$

The equations of motion can be read off the variation:

$$T_{\alpha\beta} = 0 \tag{2.7}$$

$$\partial_\alpha\left(\sqrt{-h} h^{\alpha\beta}\partial_\beta X^\mu\right) = 0 \tag{2.8}$$

### 2.4.3 Equivalence to Nambu Goto Action

The equivalence of Polyakov and Nambu Goto follows from the vanishing energy momentum tensor. From $T_{\alpha\beta} = 0$ it follows that

$$\frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{4} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu$$

and after taking the determinant of both sides

$$\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right) = \frac{1}{4} h \left(h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu\right)^2 .$$

Now substituting the square root of the left hand side into the Polyakov action (2.4) one finds

$$S_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma \sqrt{-h} h^{\alpha\beta}\, \Gamma_{\alpha\beta} = -T \int d^2\sigma \sqrt{-\Gamma} = S_{\mathrm{NG}}.$$

Only for an $h_{\alpha\beta}$ fulfilling the equations of motion (one $h_{\alpha\beta}$ for which the Polyakov action is minimal) the two actions are the same. This is clearly not true in general. From this consideration it is very clear that the Nambu Goto action had primary constraints: They simply corresponded to the fact that the NG action is only valid if we assume that $h_{\alpha\beta}$ fulfills its equations of motion.

## 2.5 Symmetries of the Action

Besides the reparametrization invariance mentioned above, the Polyakov action carries more symmetries.

- **Global Symmetries:**
  - *Poincare Invariance*
    Let

    $$\begin{aligned} \delta X^\mu &= a^\mu{}_\nu X^\nu + b^\mu \qquad \text{where } a^\mu{}_\nu = -a_\nu{}^\mu \\ \delta h_{\alpha\beta} &= 0 \end{aligned}$$

    be a general infinitesimal Poincare transformation.
    One sees that $X^\mu$ transforms as expected as a Minkowski vector, $h_{\alpha\beta}$ as a scalar.
    *Proof*
    From above one finds that for $\delta h_{\alpha\beta} = 0$

    $$\delta S_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma \left(\sqrt{h} h^{\alpha\beta} \delta\Gamma_{\alpha\beta}\right)$$

    where

    $$\delta\Gamma_{\alpha\beta} = \partial_\alpha \delta X^\mu \partial_\beta X_\mu + \partial_\alpha X_\mu \partial_\beta \delta X^\mu = \partial_\alpha X^\nu \partial_\beta X_\mu \left(a^\mu{}_\nu + a_\nu{}^\mu\right) = 0.$$

- **Local Symmetries:**

- *Reparametrization Invariance*
  Reparametrization invariance follows directly from the choice of action. Let $\phi : \sigma^\alpha \mapsto \sigma'^\alpha$ be a reparametrization of the parametrization domain. Then by defining the (infinitesimal) function $\xi^\alpha := \delta\phi^\alpha$ one finds
  $$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu$$
  and
  $$\delta h_{\alpha\beta} = \xi^\gamma \partial_\gamma h_{\alpha\beta} + \partial_\alpha \xi^\gamma h_{\gamma\beta} + \partial_\beta \xi^\gamma h_{\alpha\gamma}.$$

- *Weyl Symmetry*
  Besides the two obvious symmetries there is a more concealed symmetry, namely the invariance of the action with respect to Weyl scaling of the worldsheet metric $h$.
  $$\delta X^\mu = 0$$
  $$\delta h_{\alpha\beta} = 2\Lambda h_{\alpha\beta}$$
  where $\Lambda$ is an arbitrary (infinitesimal) function.

The Poincare invariance is a symmetry on the target space, while the two local symmetries are symmetries on the parametrization domain. In this report the local symmetries will be more important, due to the possibility of using them to fix a convenient gauge.

# 3 Wave Equation and Solutions

In this section the string equations of motion are simplified using the symmetries of the Polyakov action. Subsequently the simplified equations are solved.

## 3.1 Conformal Gauge

### 3.1.1 Using Reparametrization Invariance

The reparametrization invariance of the parametrization domain gives us the possibility to choose 'good' coordinates and to work with them. Such a choice of 'good' coordinates is given in the following claim:

**Claim**
For any two dimensional Lorentzian metric $h_{\alpha\beta}$ (i.e. a metric with signature (-1,1) ) one can find coordinates such that in these coordinates

$$h_{\alpha\beta} = \Omega(\sigma^1, \sigma^2)\eta_{\alpha\beta}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric.

**Proof**
For any point in $\mathbb{R}^2$ choose two independent null vectors (i.e. vectors sucht that $a^\alpha b^\beta h_{\alpha\beta} = 0$). In this way one obtains two (null- ) vector fields on $\mathbb{R}^2$. This construction can be made explicit. Given a metric

$$h_{\alpha\beta} = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}$$

one can set

$$v^+ = \begin{pmatrix} h_{22} \\ -h_{12} + \sqrt{-\det(h_{\alpha\beta)})} \end{pmatrix} \qquad v^- = \begin{pmatrix} h_{22} \\ -h_{12} - \sqrt{-\det(h_{\alpha\beta})} \end{pmatrix}.$$

These (null-)vector fields exist because $\det(h_{\alpha\beta}) < 0$, they are obviously linear independent and non-vanishing. If one defines the integral curves of these vector fields as new coordinates $\sigma^\pm$, the metric tensor in these coordinates looks like $h_{\pm\pm} = v^\pm h v^\pm$. With the above choice of $v$ one finds

$$h_{++} = h_{--} = 0 \qquad h_{+-} = -\frac{1}{2}\Omega$$

where the form of $h_{+-}$ is chosen for convenience.
Written in line element form

$$ds^2 = -\Omega d\sigma^+ d\sigma^-.$$

Setting the new coordinates

$$\sigma = \frac{1}{2}\left(\sigma^+ - \sigma^-\right) \qquad \tau = \frac{1}{2}\left(\sigma^+ + \sigma^-\right)$$

yields the metric

$$h_{\alpha\beta} = \Omega\, \eta_{\alpha\beta} \qquad \text{or} \qquad ds^2 = \Omega\left(-d\tau^2 + d\sigma^2\right).$$

□

The coordinates $\sigma^\pm = \tau \pm \sigma$ introduced above are called *light cone coordinates*.

### 3.1.2 Weyl Scaling

Having used the reparametrization freedom we arrived at the metric $h_{\alpha\beta} = \Omega\eta_{\alpha\beta}$. Using Weyl invariance and gauging away $\Omega$ (setting $\Omega = 1$) leads to the metric

$$h_{\alpha\beta} = \eta_{\alpha\beta}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric.

This choice of metric in accordance with our symmetries is called *conformal gauge*.
As always 'gauge' simply means an arbitrary choice not affecting physics (but simplifying the problem).

That one was able to gauge fix $h_{\alpha\beta}$ completely can also be seen as a consequence of the following:
As a symmetric matrix $h_{\alpha\beta}$ has three independent components. Using reparametrization invariance on the two dimensional chart domain (two degrees of freedom) and Weyl symmetry (one degree) one is able to fix all those three independent components. After having done so there is no degree of freedom left.

## 3.2 Equations of Motion and Boundary Terms

In conformal gauge the Polyakov action simplifies to

$$S_{\mathrm{P}} = -\frac{T}{2}\int_0^{\overline{\sigma}} d\sigma \int_{\tau_i}^{\tau_f} d\tau\ \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{T}{2}\int_0^{\overline{\sigma}} d\sigma \int_{\tau_i}^{\tau_f} d\tau \left(\dot{X}^2 - X'^2\right). \tag{3.1}$$

To solve the equations of motion a short discussion of the boundary terms is necessary. Varying the action (3.1) with respect to $X^\mu$ under the conditions that $\delta X^\mu(\tau_i) = 0 = \delta X^\mu(\tau_f)$ and that

$$\delta X^\mu(\sigma = 0, \overline{\sigma}) \text{ arbitrary for open strings}$$
$$\delta X^\mu(\sigma + 2\pi) = \delta X^\mu(\sigma) \text{ for closed strings}$$

leads to

$$\delta S_{\mathrm{P}} = T\int_0^{\overline{\sigma}} d\sigma \int_{\tau_i}^{\tau_f} d\tau\ \delta X^\mu \left(\partial_\sigma^2 - \partial_\tau^2\right) X_\mu - T\int_{\tau_i}^{\tau_f} d\tau\ \partial_\sigma X_\mu\ \delta X^\mu]_{\sigma=0}^{\sigma=\overline{\sigma}}.$$

The boundary term is only present for the case of the open string.

This results in the equations of motion

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0$$

$$X^\mu(\sigma + 2\pi) = X^\mu(\sigma) \qquad \text{closed string}$$
$$X'^\mu(\sigma = 0, \pi) = 0 \qquad \text{open string.}$$

The equation of motion matches exactly the one derived earlier (2.7; 2.8) for the general case. This second derivation for the conformal case was simply conducted to find the boundary conditions.

The boundary conditions for the open string are *von Neumann* conditions. This can be interpreted as an open string with free ends in spacetime.

The boundary term would have also vanished, if one had imposed the condition $\delta X^\mu(\sigma = 0, \overline{\sigma}) = 0$, corresponding to fixed string ends. This would then have been a *Dirichlet* condition. The object on which a Dirichlet string is fixed is therefore called *D-brane*. Obviously one could have also imposed mixed conditions, corresponding to one free and one fixed end. In the following a *free*, open string is assumed.

One still has to impose the equation resulting from the variation of the action with respect to the metric, which is (2.7)

$$T_{\alpha\beta} = 0,$$

the vanishing of the energy momentum tensor.

With (2.6)

$$T_{01} = T_{10} = \frac{1}{2}\dot{X} \cdot X' = 0 \qquad T_{00} = T_{11} = \frac{1}{4}\left(\dot{X}^2 + X'^2\right) = 0.$$

This condition is equivalent to

$$\frac{1}{2}\left(\dot{X} \pm X'\right)^2 = 0.$$

### 3.3 Light Cone Coordinates

Before solving the equations of motion it is advisable to revisit light cone coordinates seen in the proof of (3.1.1). These coordinates are especially useful when dealing with wave equations.

In our conformal coordinates $\sigma$, $\tau$ the worldsheet metric $h_{\alpha\beta}$ is equal to the two dimensional Minkowski metric $\eta_{\alpha\beta}$.
When changing to light cone coordinates

$$\sigma^+ = \tau + \sigma \qquad \sigma^- = \tau - \sigma$$

The metric changes to the metric given by $\mathrm{d}s^2 = -\mathrm{d}\sigma^+\mathrm{d}\sigma^-$ or better $\eta_{++} = \eta{-}{-} = 0$ and $\eta_{+-} = \eta{-}{+} = -\frac{1}{2}$. The inverse of the metric is $\eta^{++} = \eta^{--} = 0$ and $\eta^{+-} = \eta^{-+} = -2$. For the derivatives it holds that

$$\partial_\pm = \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right).$$

Indices are lowered and raised according to

$$v^+ = \eta^{+-}v_- + \eta^{++}v_+ = -2v_- \qquad v^- = -2v_+.$$

The energy momentum tensor $T_{\alpha\beta} = \frac{1}{2}\partial_a X^\mu \partial_b X_\mu - \frac{1}{4}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu$ takes the form

$$\begin{aligned} T_{++} &= \frac{1}{2}\partial_+ X \cdot \partial_+ X = \frac{1}{8}\left(\dot{X} + X'\right)^2 \\ T_{--} &= \frac{1}{2}\partial_- X \cdot \partial_- X = \frac{1}{8}\left(\dot{X} - X'\right)^2 \\ T_{+-} &= T_{-+} = 0 \end{aligned}$$

The fact that $T_{+-} = T_{-+} = 0$ is not a constraint but corresponds directly to the choice of metric (i.e. follows directly from the above formula for the energy momentum tensor).

The vanishing of the energy momentum tensor, the so called *Virasoro constraints* are in light cone coordinates

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0. \tag{3.2}$$

The (wave-) equation of motion for $X$ itself is

$$\partial_+ \partial_- X^\mu = 0.$$

## 3.4 General Solution and Oscillator Expansion

For the general solutions one has to distinguish between open and closed strings.

### 3.4.1 Closed String

The complete set of equations for the closed string is

| | |
|---|---|
| wave equation: | $\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0$ |
| periodicity condition: | $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$ |
| Virasoro constraint: | $\left(\dot{X} \pm X'\right)^2 = 0$ |

In the following passage the unconstrained equation of motion is solved with the constraints imposed on it afterwards.
The general solution to the two dimensional wave equation is

$$X^\mu(\sigma, \tau) = X_R^\mu(\tau - \sigma) + X_L^\mu(\tau + \sigma)$$

or in light cone coordinates

$$X^\mu(\sigma^+, \sigma^-) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+).$$

The functions $X_R$ and $X_L$ are completely arbitrary, subject only to periodicity or boundary conditions and correspond to the left and right moving modes on the string. Note that for the closed string without boundary conditions these functions are independent. This is not the case for the open string, where the Neumann conditions can be interpreted as reflections of the modes at the end of the string. The left and right moving modes must here add up to standing waves and are therefore not independent anymore.

To get an explicit expression for $X_R$ and $X_L$ one examines them in terms of periodicity .

**Claim**
The following two statements concerning periodicity of $X^\mu$ are equivalent:

$$X^\mu(\sigma, \tau) \text{ is } 2\pi \text{ periodic in } \sigma.$$

$$\Updownarrow$$

$$\partial_- X_R^\mu(\sigma^-) \text{ and } \partial_+ X_L^\mu(\sigma^+) \text{ are } 2\pi \text{ periodic and have the same zero mode .}$$

**Proof**
Differentiating the equation (which describes the periodicity of $X^\mu(\sigma,\tau)$)

$$X_R^\mu(\sigma^- - 2\pi) + X_L^\mu(\sigma^+ + 2\pi) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+) \tag{3.3}$$

with respect to $\partial_-$, respectively $\partial_+$ yields the periodicity of $\partial_- X_R^\mu$ and $\partial_+ X_L^\mu$. The first Fourier coefficient of $\partial_- X_R^\mu(\sigma^-)$ is

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma\, \partial X_R^\mu(\sigma) = \frac{1}{2\pi}\left(X_R^\mu(2\pi) - X_R^\mu(0)\right) = \frac{1}{2\pi}\left(X_L^\mu(2\pi) - X_L^\mu(0)\right) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma\, \partial X_L^\mu(\sigma)$$

This equality follows from the above periodicity equation (3.3) with $\sigma^+ = 0, \sigma^- = 2\pi$.
The opposite direction follows by expanding $\partial_+ X_L^\mu$ and $\partial_- X_R^\mu$ in its Fourier series and integrating. This is done in detail in the following discussion. □

As proven above, equivalent conditions to the periodicity of $X^\mu$ are the periodicity of $\partial_- X_R^\mu$ and $\partial_+ X_L^\mu$ with the same first fourier coefficient. Due to their periodicity, the functions can be expanded in Fourier series

$$\partial_- X_R^\mu(\sigma^-) = \frac{1}{\sqrt{4\pi T}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\sigma^-},$$
$$\partial_+ X_L^\mu(\sigma^+) = \frac{1}{\sqrt{4\pi T}} \sum_{n=-\infty}^{\infty} \overline{\alpha}_n^\mu e^{-in\sigma^+},$$

where the constants are chosen for convenience. The Fourier coefficients $\alpha_n^\mu$ and $\overline{\alpha}_n^\mu$ are independent (except for the zero modes $\alpha_0^\mu = \overline{\alpha}_0^\mu$).

Defining the constant $p^\mu$ via $\alpha_0^\mu = \overline{\alpha}_0^\mu = \frac{1}{\sqrt{4\pi T}} p^\mu$ and integrating the above expressions yields

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu\sigma^- + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}$$
$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu\sigma^+ + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\overline{\alpha}_n^\mu e^{-in\sigma^+}.$$

Clearly, these functions are not periodic in $\sigma^+$, resp. $\sigma^-$ themself. However, the additional terms that are linear in $\sigma^\pm$ add up to a constant in $X^\mu$, leaving $X^\mu$ periodic in $\sigma$. $x^\mu, p^\mu, \alpha_n^\mu$ and $\overline{\alpha}_n^\mu$ are just (constant) Fourier coefficients.

The complete function is then written as

$$X^\mu(\sigma,\tau) = \underbrace{x^\mu + \frac{1}{2\pi T}p^\mu\tau}_{\text{center of mass motion}} + \underbrace{\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right)e^{-in\tau}}_{\text{oscillation of the string}}. \tag{3.4}$$

Since $X^\mu$ is a real function

$$x^\mu, p^\mu \text{ are both real} \qquad (\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu \qquad (\overline{\alpha}_n^\mu)^\dagger = \overline{\alpha}_{-n}^\mu.$$

Furthermore, written in the form (3.4) we see that we can interpret $x^\mu$ as the center of mass of the string at $\tau = 0$. This argument can be made more formally. It holds that

$$\begin{aligned}
&\text{center of mass at } \tau = 0 \qquad && \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma X^\mu(\sigma, 0) = x^\mu \\
&\text{center of mass at general } \tau \qquad && \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma X^\mu(\sigma, \tau) = x^\mu + \frac{\tau}{2\pi T} p^\mu .
\end{aligned}$$

The canonical $\tau$ momentum corresponding to the position $X^\mu$ is

$$\Pi^\mu = \frac{\partial \mathcal{L}_\tau}{\partial \dot{X}_\mu} = \frac{\partial}{\partial \dot{X}_\mu} \frac{T}{2}\left(\dot{X}^2 - X'^2\right) = T\dot{X}^\mu .$$

With

$$\dot{X}^\mu = \partial_- X_R^\mu + \partial_+ X_L^\mu$$

this leads to the total ($\tau$-) momentum

$$P^\mu = \int_0^{2\pi} \mathrm{d}\sigma \Pi^\mu = p^\mu .$$

The constant $p^\mu$ is therefore the preserved total momentum.

So far the equations of motion and the boundary conditions were imposed on the solution. The additional constraint resulting from the vanishing of the energy momentum tensor will give rise to an infinite number of conserved charges. Before discussing these charges in an algebraic fashion, the general solution for the open string is examined.

### 3.4.2 Open String

For the open string the complete set of equations is:

$$\begin{aligned}
&\text{wave equation:} \qquad && \left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0 \\
&\text{boundary condition:} \qquad && \partial_\sigma X^\mu |_{\sigma = 0, \pi} = 0 \\
&\text{Virasoro constraint:} \qquad && \left(\dot{X} \pm X'\right)^2 = 0
\end{aligned}$$

A similar analysis as for the closed case leads to the open string expansion

$$X^\mu(\tau, \sigma) = \underbrace{x^\mu + \frac{1}{\pi T} p^\mu \tau}_{\text{center of mass motion}} + \underbrace{\frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma)}_{\text{oscillation of the string}} . \tag{3.5}$$

Note that in this case there is only one kind of $\alpha_n^\mu$ due to the non-independence of right and left movers resulting from the van Neumann boundary condition. Physically this follows from the fact that only

standing waves are allowed on the string.
Again from the fact that $X^\mu$ is real it follows that

$$x^\mu, p^\mu \text{ are both real} \qquad (\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu.$$

For later use one can again define $\alpha_0^\mu = \frac{1}{\sqrt{\pi T}} p^\mu$ and find

$$\begin{aligned}\partial_- X^\mu &= \frac{1}{2\sqrt{\pi T}} \sum_n \alpha_n^\mu e^{-in\sigma^-} \\ \partial_+ X^\mu &= \frac{1}{2\sqrt{\pi T}} \sum_n \alpha_n^\mu e^{-in\sigma^+}.\end{aligned}$$

## 3.5 The Virasoro Constraints

Having found the general solution of the equations of motion under consideration of boundary conditions for both closed and open strings, one can now turn to the additional constraint: the vanishing of the energy momentum tensor, the so called Virasoro constraint.
As a reminder: The vanishing of the enrgy momentum tensor resulted from the equation for $h_{\alpha\beta}$ and is therefore equivalent to the primary constraints of the Nambu-Goto action. Hence the Virasoro constraints can be seen as the string analogon of the point particle constraint $p^\mu p_\mu = -m^2 c^2$.
The constraints will lead to further restricitions on the expansion coefficients $\alpha_n^\mu$ and $\overline{\alpha}_n^\mu$.

The vanishing of the energy momentum tensor in light cone coordinates is equal to (3.2)

$$\begin{aligned}T_{++} &= \frac{1}{2} (\partial_+ X)^2 = 0 \\ T_{--} &= \frac{1}{2} (\partial_- X)^2 = 0.\end{aligned}$$

To find the restriction this imposes on the expansion modes one has to plug in the expression for $X^\mu$ in the constraint equation.

### 3.5.1 Closed String Virasoro Modes

For the closed string with (3.4)

$$\begin{aligned}\partial_- X^\mu(\sigma^+, \sigma^-) = \partial_- X_R^\mu(\sigma^-) &= \frac{1}{\sqrt{4\pi T}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\sigma^-} \\ \partial_+ X^\mu(\sigma^+, \sigma^-) = \partial_+ X_L^\mu(\sigma^+) &= \frac{1}{\sqrt{4\pi T}} \sum_{n=-\infty}^{\infty} \overline{\alpha}_n^\mu e^{-in\sigma^+}\end{aligned}$$

where $\alpha_0^\mu = \overline{\alpha}_0^\mu = \frac{1}{\sqrt{4\pi T}} p^\mu$ one finds

$$\begin{aligned}
T_{--} = \frac{1}{2}\left(\frac{1}{\sqrt{4\pi T}}\sum_{n=-\infty}^{\infty}\alpha_n^\mu e^{-in\sigma^-}\right)^2 &= \frac{1}{8\pi T}\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha_n^\mu\alpha_{k\mu}e^{-i(n+k)\sigma^-} \\
&\underbrace{=}_{m=n+k} \frac{1}{8\pi T}\sum_n\sum_m \alpha_n\cdot\alpha_{m-n}e^{-im\sigma^-} \\
&=: \frac{1}{4\pi T}\sum_m L_m e^{-im\sigma^-} \\
T_{++} &= \frac{1}{8\pi T}\sum_n\sum_m \overline{\alpha}_n\cdot\overline{\alpha}_{m-n}e^{-im\sigma^+} \\
&=: \frac{1}{4\pi T}\sum_m \overline{L}_m e^{-im\sigma^+}
\end{aligned}$$

In the last equation the Virasoro modes

$$L_m = \frac{1}{2}\sum_n \alpha_n\cdot\alpha_{m-n} \tag{3.6}$$

$$\overline{L}_m = \frac{1}{2}\sum_n \overline{\alpha}_n\cdot\overline{\alpha}_{m-n} \tag{3.7}$$

were defined. The vanishing of $T_{--}$ and $T_{++}$ implies the additional constraints on the modes given by

$$L_m = \overline{L}_m = 0 \quad \forall m\in\mathbb{Z}.$$

This way one has found an infinite number of conserved quantities $L_m, \overline{L}_m$ and by imposing the vanishing of these quantites on the modes of $X^\mu$ one has solved the classical bosonic string completely.

Especially with regard to later quantization the newfound conserved quantities shall be examined a bit further.

The conserved $L_0$ and $\overline{L}_0$ have a rather interesting interpretation.
The momentum of a particle is related to its rest mass via $p^\mu p_\mu = -M^2$. Using this equation for the total string momentum $p^\mu = \sqrt{4\pi T}\alpha_0^\mu = \sqrt{4\pi T}\overline{\alpha}_0^\mu$ one can define the string mass as

$$M^2 = -p^\mu p_\mu = -4\pi T\alpha_0^\mu\alpha_{0\mu} = -4\pi T\overline{\alpha}_0^\mu\overline{\alpha}_{0\mu}.$$

From the constraint on $L_0$ and $\overline{L}_0$

$$0 = L_0 = \frac{1}{2}\sum_n \alpha_n\cdot\alpha_{-n} = \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{n>0}\alpha_n\cdot\alpha_{-n}$$

a connection between the string's mass and its modes is given by

$$M^2 = 8\pi T\sum_{n>0}\alpha_n\alpha_{-n} = 8\pi T\sum_{n>0}\overline{\alpha}_n\overline{\alpha}_{-n}.$$

Hence, the invariant mass has two expressions, one in terms of right moving and one in terms of left moving waves.

This idea that the zero Virasoro modes correspond to energy comes from a simple calculation: The Lagrange density in conformal gauge is (3.1)

$$\mathcal{L} = \frac{T}{2}\left(\dot{X}^2 - X'^2\right),$$

leading to the $\tau$ Hamiltonian governing the $\tau$ - evolution

$$H_\tau = \int_0^{\overline{\sigma}} \mathrm{d}\sigma \ \left(\dot{X}^\mu \Pi_\mu - \mathcal{L}\right) = \frac{T}{2}\int_0^{2\pi} \mathrm{d}\sigma \ \left(\dot{X}^2 + X'^2\right).$$

This can easily be calculated using the light cone energy momentum tensor

$$\begin{aligned} \dot{X}^2 &= (\partial_+ X + \partial_- X)^2 = 2T_{++} + 2T_{--} + 2\partial_+ X \cdot \partial_- X \\ X'^2 &= (\partial_+ X - \partial_- X)^2 = 2T_{++} + 2T_{--} - 2\partial_+ X \cdot \partial_- X. \end{aligned}$$

Therefore,

$$H_\tau = 2T \int_0^{2\pi} \mathrm{d}\sigma \ \left(T_{++} + T_{--}\right).$$

The Hamiltonian is complete determined by the constraints. Using the Virasoro modes (3.7) and

$$\begin{aligned} T_{++} &= \frac{1}{4\pi T}\sum_m \overline{L}_m e^{-im\sigma^+} \\ T_{--} &= \frac{1}{4\pi T}\sum_m L_m e^{-im\sigma^-} \end{aligned}$$

leads to

$$H_\tau = \left(L_0 + \overline{L}_0\right).$$

### 3.5.2 Open String Virasoro Modes

The same analysis can be done for the open string. Plugging the open string solution

$$\partial_\pm X^\mu(\sigma^+, \sigma^-) = \frac{1}{2\sqrt{\pi T}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\sigma^\pm}$$

where $\alpha_0^\mu = \frac{1}{\sqrt{\pi T}} p^\mu$ into the constraint equation gives

$$\begin{aligned} T_{++} = \quad & \frac{1}{2}\left(\frac{1}{2\sqrt{\pi T}}\sum_{n=-\infty}^{\infty}\alpha_n^\mu e^{-in\sigma^+}\right)^2 \quad & = \frac{1}{4\pi T}\sum_{m=-\infty}^{\infty} L_m e^{-im\sigma^+} \\ T_{--} = \quad & \frac{1}{2}\left(\frac{1}{2\sqrt{\pi T}}\sum_{n=-\infty}^{\infty}\alpha_n^\mu e^{-in\sigma^-}\right)^2 \quad & = \frac{1}{4\pi T}\sum_{m=-\infty}^{\infty} L_m e^{-im\sigma^-}. \end{aligned}$$

In the last equation one defines the Virasoro modes

$$L_m = \frac{1}{2}\sum_n \alpha_n \cdot \alpha_{m-n}$$

again. The vanishing of $T_{--}$ and $T_{++}$ implies the additional constraints on the modes given by

$$L_m = 0 \quad \forall m \in \mathbb{Z}.$$

Imposing these constraints on the mode expansion leads to the complete classical open string solution. For the open string the $\tau$- Hamiltonian reads

$$H_\tau = 2T \int_0^\pi \mathrm{d}\sigma \ (T_{++} + T_{--}) = \frac{1}{\pi} \sum_m L_m \int_0^\pi \mathrm{d}\sigma \cos(m\sigma) e^{-im\tau}$$

and therefore

$$H_\tau = L_0.$$

## 3.6 The Witt Algebra in Classical String Theory

We have found the most general solutions and imposed the Virasoro constraints on them which means that we are basically done with the classical string.
However in view of later quantization it is very adviseable to discuss the algebraic relations of the introduced quantities under the Poisson bracket.

The Poisson bracket with respect to a coordinate field $X^\mu(\sigma)$ and its conjugate momentum field $\Pi^\mu(\sigma)$ of two functionals $f$ and $g$ is defined using the functional derivative as

$$\{f, g\} := \int \mathrm{d}\sigma \ \frac{\delta f}{\delta X^\mu(\sigma)} \frac{\delta g}{\delta \Pi_\mu(\sigma)} - \frac{\delta f}{\delta \Pi_\mu(\sigma)} \frac{\delta g}{\delta X^\mu(\sigma)}.$$

This definition directly leads to the canonical 'commutation' relations:

$$\{X^\mu(\sigma), X^\nu(\sigma')\} = 0 \ \ \{\Pi^\mu(\sigma), \Pi^\nu(\sigma')\} = 0$$
$$\{X^\mu(\sigma), \Pi^\nu(\sigma')\} = g^{\mu\nu} \delta(\sigma - \sigma').$$

### 3.6.1 The Closed String Witt Algebra

In the following, the algebra of the closed string is examined. We are interested in $\tau$-propagation, therefore we are interpreting $X^\mu(\sigma, \tau)$ for fixed $\tau$ as a field in $\sigma$ (This is actually the way we imagine strings at 'fixed times'). The canonical ($\tau$-) momentum is $\Pi^\mu = T\dot{X}^\mu$.
In this framework the canonical commutation relations are

$$\{X^\mu(\sigma, \tau), X^\nu(\sigma', \tau)\} = 0 \ \ \{\Pi^\mu(\sigma, \tau), \Pi^\nu(\sigma', \tau)\} = 0$$
$$\{X^\mu(\sigma, \tau), \Pi^\nu(\sigma', \tau)\} = g^{\mu\nu} \delta(\sigma - \sigma').$$

To calculate the commutation relations of the constants $\alpha_n^\mu$, $\overline{\alpha}_n^\mu$ one has to express them in terms of $X^\mu$ and $\Pi^\mu$. From the full solution

$$X^\mu(\sigma,\tau) = x^\mu + \frac{1}{2\pi T}p^\mu\tau + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right)e^{-in\tau}$$
$$\Pi^\mu(\sigma,\tau) = T\dot{X}^\mu = \frac{1}{2\pi}p^\mu + \sqrt{\frac{T}{4\pi}}\sum_{n\neq 0}\left(\alpha_n^\mu e^{in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right)e^{-in\tau}$$

one can easily verify the equations

$$\frac{1}{2\pi}\int_0^{2\pi} X^\mu(\sigma,0)\mathrm{d}\sigma = x^\mu$$
$$\int_0^{2\pi} \Pi^\mu(\sigma,0)\mathrm{d}\sigma = p^\mu$$
$$-i\frac{\sqrt{4\pi T}}{2\pi}\int_0^{2\pi} X^\mu(\sigma,0)e^{-in\sigma}\mathrm{d}\sigma = \frac{1}{n}\left(\alpha_n^\mu - \overline{\alpha}_{-n}^\mu\right)$$
$$\frac{1}{2\pi}\sqrt{\frac{4\pi}{T}}\int_0^{2\pi} \Pi^\mu(\sigma,0)e^{-in\sigma}\mathrm{d}\sigma = \alpha_n^\mu + \overline{\alpha}_n^\mu.$$

Using these expressions and the linearity of the Poisson brackets the relations for the modes can be derived. One calculates for example

$$\begin{aligned}\{x^\mu, p^\nu\} &= \frac{1}{2\pi}\{\int_0^{2\pi} X^\mu(\sigma,0)\,\mathrm{d}\sigma, \int_0^{2\pi}\Pi^\nu(\sigma',0)\,\mathrm{d}\sigma'\}\\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\int_0^{2\pi}\mathrm{d}\sigma'\{X^\mu(\sigma,0),\Pi^\nu(\sigma',0)\}\\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\int_0^{2\pi}\mathrm{d}\sigma' g^{\mu\nu}\delta(\sigma-\sigma') = g^{\mu\nu}.\end{aligned}$$

In exactly this way the relations

$$\begin{aligned}\{\alpha_m^\mu, \alpha_n^\nu\} &= \{\overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu\} = -im\delta_{m+n}g^{\mu\nu}\\ \{\overline{\alpha}_m^\mu, \alpha_n^\nu\} &= 0\\ \{x^\mu, p^\nu\} &= g^{\mu\nu}\end{aligned}$$

follow, where $\delta_{m+n} := \delta_{m+n,0}$.

With $L_m = \frac{1}{2}\sum_n \alpha_n \cdot \alpha_{m-n}, \;\; \overline{L}_m = \frac{1}{2}\sum_n \overline{\alpha}_n \cdot \overline{\alpha}_{m-n}$ and linearity, we finally arrive at the commutation relations of the Virasoro modes

$$\{L_m, L_n\} = -i(m-n)L_{m+n} \tag{3.8}$$
$$\{\overline{L}_m, \overline{L}_n\} = -i(m-n)\overline{L}_{m+n} \tag{3.9}$$
$$\{L_m, \overline{L}_n\} = 0. \tag{3.10}$$

Therefore, the Virasoro modes generate the infinite dimensional, so called *Witt-Algebra* (or better two copies of the Witt algebra).

### 3.6.2 The Open String Witt Algebra

For the open string the calculation is completely analog and will not be discussed here in detail.
The Virasoro modes of the open string $L_m = \frac{1}{2}\sum_n \alpha_n\alpha_{m-n}$ generate only one copy of the Witt algebra:

$$\{L_m, L_n\} = -i(m-n)L_{m+n}$$

### 3.6.3 Why did the Witt Algebra appear?

Finally one is left with one question: Why did the (on first sight not trivial) Witt algebra appear in a completely classical calculation. To shed some light on a possible answer to this question, consider the circle $\mathbb{S}^1$ and the group of diffeomorphisms on it.
The generator of an (infinitesimal) diffeomorphism $\theta \to \theta + a(\theta)$ looks like $ia(\theta)\frac{d}{d\theta}$. Because $a(\theta)$ is a function on the circle we can expand it into a Fourier series. This leads to the fact that the Lie algebra is generated by the operators

$$D_n = ie^{in\theta}\frac{d}{d\theta}.$$

Using the usual commutator one finds

$$[D_n, D_m] = -i(m-n)D_{m+n}.$$

Hence, the Witt algebra is simply the Lie algebra of the group of diffeomorphisms on the circle.
Due to Noether's theorem it is therefore natural that the Witt algebra appears in a theory invariant under reparametrization of a domain (at least partially) diffeomorphic to the circle.

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