The conformal group in various dimensions

04 March 2013

Topics

- Repetition : Symmetries and Noethers theorem
- The structure of infinitesimal conformal transformations
- Global conformal transformations
- The Virasoro algebra in 2 dimensions

Repetition - Symmetries and Noethers theorem

- The action is defined to be

$$S = \int d^d x \, \mathcal{L}(\Phi, \partial_\mu \Phi)$$

- A general continuous transformation affects in general both the position and the fields

$$x \to x'$$

$$\Phi(x) \to \Phi'(x') =: \mathcal{F}(\Phi(x))$$

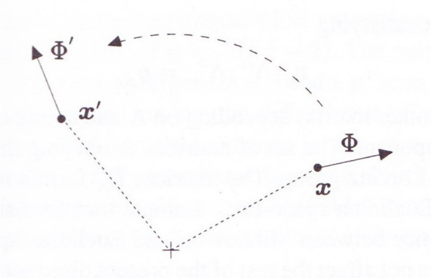

Figure: Active transformation

Repetition - Symmetries and Noethers theorem

- It changes according to

$$S' = \int d^d x \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}\left(\mathcal{F}(\Phi(x)), \left(\frac{\partial x^\nu}{\partial x'^\mu} \right) \partial_\nu \mathcal{F}(\Phi(x)) \right)$$

- Example 1 - Translation:

$$x' = x + a$$

$$\Phi'(x + a) = \Phi(x)$$

$$\rightarrow \mathcal{F} = Id$$

$$\rightarrow \frac{\partial x^\nu}{\partial x' \mu} = \delta^\nu_\mu$$

$$\rightarrow S' = S$$

$$S' = \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}(\Phi(x)), \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)\partial_\nu \mathcal{F}(\Phi(x))\right)$$

- Example 2 - Lorentz transformation:

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

$$\Phi'(\Lambda x) = L_\Lambda \Phi(x)$$

$$\rightarrow \mathcal{F} = L_\Lambda \cdot Id$$

$$\rightarrow \frac{\partial x^\nu}{\partial x' \mu} = \left(\Lambda^{-1}\right)^\nu_\mu$$

$$\rightarrow S' = \int d^d x \, \mathcal{L}\left(L_\Lambda \Phi, \Lambda^{-1}\partial\left(L_\Lambda \Phi(x)\right)\right)$$

Repetition - Symmetries and Noethers theorem

$$S' = \int d^d x \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}\left(\mathcal{F}(\Phi(x)), \left(\frac{\partial x^\nu}{\partial x'^\mu} \right) \partial_\nu \mathcal{F}(\Phi(x)) \right)$$

- Example 3 - Scale transformation:

$$x' = \lambda x$$

$$\Phi'(\lambda x) = \lambda^{-\Delta} \Phi(x)$$

$$\rightarrow \mathcal{F} = \lambda^{-\Delta} \cdot Id$$

$$\rightarrow \frac{\partial x^\nu}{\partial x' \mu} = \lambda^{-1}$$

$$\rightarrow S' = \lambda^d \int d^d x \, \mathcal{L}\left(\lambda^{-\Delta} \Phi, \lambda^{-1-\Delta} \partial_\mu \Phi \right)$$

- Let us now consider the effect of infinitesimal transformations on the action

$$x'^{\mu} = x^{\mu} + \omega_a \frac{\partial x^{\mu}}{\partial \omega_a}$$

$$\Phi'(x') = \Phi(x) + \omega_a \frac{\partial \mathcal{F}}{\omega_a}$$

- The Generator of a symmetry transformation is defined by

$$\Phi'(x) - \Phi(x) =: -i\omega_a G_a \Phi(x)$$

$$\dots = -\left(\frac{\partial \Phi}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial \omega_a} \omega_a - \frac{\partial \mathcal{F}}{\partial \omega_a} \omega_a \right)$$

$$x'^{\mu} = x^{\mu} + \omega_a \frac{\partial x^{\mu}}{\partial \omega_a}$$
$$\Phi'(x') = \Phi(x) + \omega_a \frac{\partial \mathcal{F}}{\partial \omega_a}$$

- Example 1 - Translation:

$$x'^{\mu} = x^{\mu} + a^{\mu}$$
$$\Phi'(x+a) = \Phi(x)$$
$$\to P_{\mu} = -i\partial_{\mu}$$

Repetition - Symmetries and Noethers theorem

$$x'^{\mu} = x^{\mu} + \omega_a \frac{\partial x^{\mu}}{\partial \omega_a}$$
$$\Phi'(x') = \Phi(x) + \omega_a \frac{\partial \mathcal{F}}{\partial \omega_a}$$

- Example 2 - Lorentz transformation:

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \omega^{\mu}_{\nu} x^{\nu} \\ &= x^{\mu} + \omega_{\rho\nu} g^{\rho\mu} x^{\nu} \end{aligned}$$

$$\begin{aligned} \Phi'(x') &= L_{\Lambda}\Phi(x) \\ &= \left(1 - \frac{1}{2} i\omega_{\rho\nu} S^{\rho\nu}\right)\Phi \end{aligned}$$

- $\omega_{\rho\nu} = -\omega_{\nu\rho}$ is antisymmetric

$$\rightarrow L^{\rho\nu} = i\left(x^{\rho}\partial^{\nu} - x^{\nu}\partial^{\rho}\right) + S^{\rho\nu}$$

Repetition - Symmetries and Noethers theorem

- The effect of an infinitesimal transformation on the action can be shown to be

$$\begin{aligned} \delta S &= S - S' \\ &= -\int d^d x \, \partial_\mu j_a^\mu \omega_a \end{aligned}$$

with the conserved current

$$j_a^\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta_\nu^\mu \mathcal{L} \right) \frac{\partial x^\nu}{\partial \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \frac{\partial \mathcal{F}}{\partial \omega_a}$$

- For a continuous symmetry transformation ($\delta S = 0$ for all ω_a) this implies

$$\partial_\mu j_a^\mu = 0$$

- This is Noether's theorem
- Every conserved current gives us an integral of motion. We can solve a system with n degrees of freedom if n currents are conserved.

Topics

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- The structure of infinitesimal conformal transformations
- Global conformal transformations
- The Virasoro algebra in 2 dimensions

The structure of infinitesimal conformal transformations - Conformal transformations

- Conformal transformations conserve angles

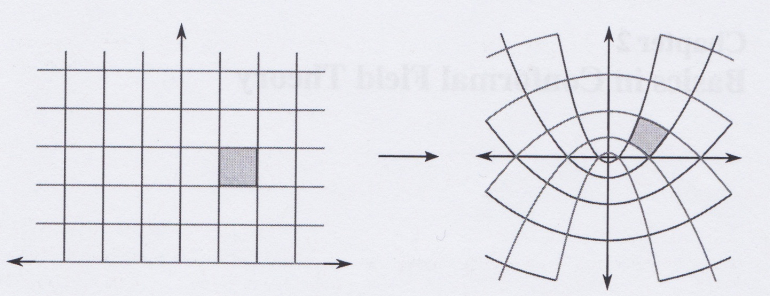

Figure: Conformal transformation

The structure of infinitesimal conformal transformations - Conformal transformations

- Conformal transformations conserve angles
- This is equivalent to leaving to leaving the metric tensor invariant up to a scaling factor

$$g_{\rho\sigma}\frac{\partial x^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)g_{\mu\nu}$$

The structure of infinitesimal conformal transformations - Conformal transformations

- Conformal maps include Lorentz transformations and translations (Poincaré group) - this corresponds to $\Lambda(x) = 1$
- But there are more! Let's discuss the conditions for conformal invariance in more detail...

The structure of infinitesimal conformal transformations - Conditions

- Consider an infinitesimal transformation

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho} + O(\epsilon^2)$$

- Now impose the constraint that the metric tensor be invariant up to a scale factor

$$\cdots \to \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}$$

- One can also derive

$$(d-1)\partial^\mu \partial_\mu (\partial \cdot \epsilon) = 0$$

and

$$2\partial_\mu \partial_\nu \epsilon_\rho = \frac{2}{d}\left[g_{\rho\mu}\partial_\nu + g_{\nu\sigma}\partial_\mu + g_{\mu\nu}\partial_\rho\right]$$

- Let us now turn to the general case $d \geq 2$

The structure of infinitesimal conformal transformations - The four types

- By equation (2) we can make the ansatz

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho$$

- We can consider all of the terms separately

The structure of infinitesimal conformal transformations - The four types

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho$$

- No constraint for a_μ
- Infinitesimal translations

$$x'^\mu = x^\mu + a^\mu$$

- Generator:

$$P_\mu = -i\partial_\mu$$

The structure of infinitesimal conformal transformations - The four types

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho$$

- Constraint for $b_{\mu\nu}$

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\lambda_\lambda g_{\mu\nu}$$

- Dilations

$$x'^\mu = \alpha x^\mu$$

  and rotations

$$x'^\mu = M^\mu_\nu x^\nu$$

- Their respective generators are:

$$D = -ix^\mu \partial_\mu$$

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

The structure of infinitesimal conformal transformations - The four types

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho$$

- Constraint for $c_{\mu\nu\rho}$

$$c_{\mu\nu\rho} = g_{\mu\rho}c^\sigma_{\sigma\nu} + g_{\mu\nu}c^\sigma_{\sigma\rho} - g_{\nu\rho}c^\sigma_{\sigma\mu}$$

- "Special Conformal Transformations":

$$x'^\mu = \frac{x^\mu - (x \cdot x)b^\mu}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)}$$

- Generator:

$$K_\mu = -i(2x_\mu x^\nu \partial_\nu - (x \cdot x)\partial_\mu)$$

The structure of infinitesimal conformal transformations - The four types

- Special Conformal Transformations consist of an inversion followed by a translation and another inversion, i.e.:

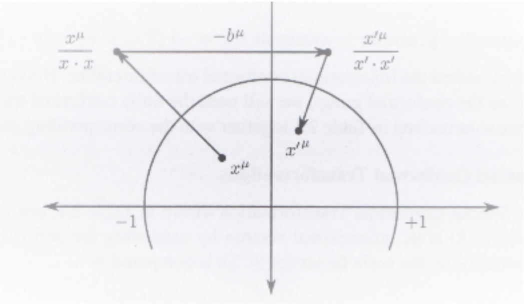

Figure: Special conformal transformation

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Global conformal transformations - The Conformal Group and Algebra

- The group $SO(p,q)$ is a generalization of the usual special orthogonal group.
- It leaves invariant the scalar product $< x, y >= x^T gy$ where g is the metric tensor with p times $+1$ on the diagonal and q times -1.
- The Lorentz group is $SO(3,1)$
- There is a basis J_{ab} in which the commutation relations read

$$[J_{ab}, J_{cd}] = i(g_{ad}J_{bc} + g_{bc}J_{ad} - g_{ac}J_{bd} + g_{bd}J_{ac})$$

Global conformal transformations - The Conformal Group and Algebra

- The Conformal Group is the group consisting of globally defined, invertible and finite conformal transformations
- The Conformal Algebra is the Lie Algebra corresponding to the Conformal Group

Global conformal transformations - The Conformal Group and Algebra

- We now seek similarities between the Conformal Group and known groups
- There were four types of conformal transformations
  1. Translations $P_\mu = -i\partial_\mu$
  2. Dilations $D = -ix^\mu \partial_\mu$
  3. Rotations $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
  4. Special Conformal Transformations s$K_\mu = -i(2x_\mu x^\nu \partial_\nu - (x \ \cdot x)\partial_\mu)$

Global conformal transformations - The Conformal Group and Algebra

- Now compute the commutation relations
  1. $[D, P_\mu] = iP_\mu$
  2. $[D, K_\mu] = -iK_\mu$
  3. $[K_\mu, P_\nu] = 2i(g_{\mu\nu} D - L_{\mu\nu})$
  4. $[K_\rho, L_{\mu\nu}] = i(g_{\rho\mu} K_\nu - g_{\rho\nu} K_\mu)$
  5. $[P_\rho, L_{\mu\nu}] = i(g_{\rho\mu} P_\nu - g_{\rho\nu} P_\mu)$
  6. $[L_{\mu\nu}, L_{\rho\sigma}] = i(g_{\nu\rho} L_{\mu\sigma} + g_{\mu\sigma} L_{\nu\rho} + g_{\mu\rho} L_{\nu\sigma} + g_{\nu\sigma} L_{\mu\rho})$
- Define
  1. $J_{\mu\nu} = L_{\mu\nu}$
  2. $J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu)$
  3. $J_{-1,0} = D$
  4. $J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu)$
- These obey

$$[J_{ab}, J_{cd}] = i(g_{ad} J_{bc} + g_{bc} J_{ad} - g_{ac} J_{bd} + g_{bd} J_{ac})$$

- Hence, the Conformal Group in d dimensions is isomorphic to the group $SO(d+1, 1)$ with $\frac{(d+2)(d+1)}{2}$ parameters

Global conformal transformations - The case $d = 2$

- Let us now turn to the case $d = 2$

Global conformal transformations - The case $d = 2$

- For $d = 2$,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}$$

  turns into the Cauchy-Riemann equations

- Infinitesimal holomorphic functions $f(z) = z + \epsilon(z)$ give rise to infinitesimal conformal transformations

Global conformal transformations - The Witt Algebra

- A general meromorphic function can be expanded into a Laurent series

$$f(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n (-z^{n+1})$$

Global conformal transformations - The Witt Algebra

- A general meromorphic function can be expanded into a Laurent series

$$f(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n(-z^{n+1})$$

- There is a generator for each of the terms

$$l_n = -z^{n+1}\partial_z$$

- Their commutation relation is

$$[l_m, l_n] = (m-n)l_{m+n}$$

- We call this the Witt Algebra
- It is infinite dimensional

Global conformal transformations - Local and Global Conformal Transformations

- The Witt Algebra was induced by infinitesimal conformal invariance. What about global transformations?
- We know from complex analysis that the complete set of global conformal transformations (projective or Möbius transformations) of the Riemann sphere is

$$f(z) = \frac{az+b}{cz+d} \qquad \text{with } ad - bc = 1$$

- They form a group and one can associate the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with each of them
- Therefore, the global conformal group is isomorphic to $SL(2,\mathbb{C}) \cong SO(3,1)$

Global conformal transformations - Local and Global Conformal Transformations

- Now reconsider the Witt Algebra
- Notice that the generators of the Witt Algebra generate transformations which are singular at certain points
- The principal part of the Laurent series diverges for $n < -1$
- We are actually working on the Riemann Sphere rather than the complex plane
- All transformations generated by generators with $n > 1$ are singular at ∞
- The only globally defined generators of the Witt Algebra are

$$l_{-1}, l_0, l_1$$

- This is a particuliarity of the $2d$-case

Global conformal transformations - Local and Global Conformal Transformations

- Now is the Conformal Group in two dimensions infinite dimensional or does it have dimension 6?
- Locally: Infinite dimensions
- Globally: Six dimensions

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The Virasoro Algebra in two dimensions

- A central extension of a Lie algebra of $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$ is an exact sequence of Lie algebra homomorphisms

$$0 \to \mathfrak{a} \to \mathfrak{h} \to \mathfrak{g} \to 0$$

  where $[\mathfrak{a}, \mathfrak{h}] = 0$ and the image of every homorphism is the kernel of the succeeding one
- For every such sequence there is a linear map $\beta : \mathfrak{g} \to \mathfrak{h}$
- Let $\Theta : \mathfrak{g} x \mathfrak{g} \to \mathfrak{a}$

$$\Theta(X, Y) := [\beta(X), \beta(Y)] - \beta([X, Y])$$

The Virasoro Algebra in two dimensions

$$\Theta(X, Y) := [\beta(X), \beta(Y)] - \beta([X, Y])$$

- We can check that Θ fulfills
  1. It is bilinear and alternating
  2. $\Theta(X, [Y, Z]) + \Theta(Y, [Z, X]) + \Theta(Z, [X, Y]) = 0$
- A map satisfying those requirements is a cocyle
- Every central extension has exactly one associated cocycle and it is trivial if there is a homomorphism μ such that

$$\Theta(X, Y) = \mu([X, Y])$$

The Virasoro Algebra in two dimensions

$$0 \to \mathfrak{a} \to \mathfrak{h} \to \mathfrak{g} \to 0$$

- For the central extension of the Witt Algebra $\mathfrak{W}$ we identify:
  - $\mathfrak{a} \leftrightarrow \mathbb{C}$
  - $\mathfrak{h} \leftrightarrow \mathfrak{Vir}$
  - $\mathfrak{g} \leftrightarrow \mathfrak{W}$

The Virasoro Algebra in two dimensions

- We propose that

$$\omega(L_n, L_m) := \delta_{n+m,0}\frac{n}{12}(n^2 - 1)$$

is the cocycle defining the only nontrivial central extension of $\mathfrak{W}$ by $\mathbb{C}$

  - It is bilinear and alternating
  - It fulfills $\omega(L_k, [L_m, L_n]) + \omega(L_m, [L_n, L_k]) + \omega(L_n, [L_k, L_m]) = 0$
  - If there was a homomorphism μ with $\Theta(X, Y) = \mu([X, Y])$ it would satisfy

$$\mu(L_0) = \frac{1}{24}(n^2 - 1)$$

    which cannot be true for every n
  - It can be shown that every other cocycle Θ is a multiple of ω.

The Virasoro Algebra in two dimensions

$$\Theta(X, Y) := [\beta(X), \beta(Y)] - \beta([X, Y])$$

- Hence we define the Virasoro algebra $\mathfrak{Vir}$ as the unique central extension of the Witt algebra $\mathfrak{W}$ by $\mathbb{C}$, i.e
- $\mathfrak{Vir} = \mathfrak{W} \oplus \mathbb{C}$

$$[L_m, L_n] = (m-n)L_{m+n} + c\frac{n}{12}(n^2-1)\delta_{m+n,0}$$