CFT - Basic properties and examples

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- Structure of correlation functions
- Energy momentum tensor in conformal field theory
- The free Boson and Fermion in 2 dimensions with calculation of the central charge
2D CFT basic definitions

**Chiral and antichiral fields**
Fields depending only on $z$ are chiral fields, fields depending only on $\bar{z}$ are called antichiral fields.

**Conformal dimensions**
Property of field under scalings $z \mapsto \lambda z$

$$\phi(z, \bar{z}) \mapsto \lambda^h \overline{\lambda}^\bar{h} \phi(\lambda z, \lambda \bar{z})$$

**Primary fields**
Conformal transformation $z \mapsto f(z)$

$$\phi(z, \bar{z}) \mapsto \phi'(z, \bar{z}) = \left( \frac{\partial f}{\partial z} \right)^h \left( \frac{\partial \bar{f}}{\partial \bar{z}} \right)^\bar{h} \phi(f(z), \bar{f}(\bar{z}))$$
Structure of correlation functions

\[ \langle T(\phi(t_1)\phi(t_2)\ldots\phi(t_N)) \rangle = \frac{\int [d\phi]\phi(t_1)\phi(t_2)\ldots\phi(t_N) \exp(iS_\epsilon[\phi(t)])}{\int [d\phi] \exp(iS_\epsilon[\phi(t)])} \]

However this only makes sense if the Operators are ordered in time!
Two point function under conformal invariance

\[ \langle \phi_1(z)\phi_2(\omega) \rangle = g(z, \omega) \]

with \( \phi_1, \phi_2 \) quasi-primary fields:

- Translation invariance
- Invariance under rescalings
- Invariance under inversion

thus:

\[ \langle \phi_i(z)\phi_j(\omega) \rangle = \frac{d_{ij}\delta_{h_i,h_j}}{(z - \omega)^{2h_i}} \]
Three point function under conformal invariance

Same steps of derivation lead to this expression:

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) \rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_1+h_3-h_2}}$$
Infinitesimal conformal transformation

Consider conformal trafo $f(z) = z + \varepsilon(z)$ with $\varepsilon << 1$. Change of a primary field:

$$\delta_{\varepsilon, \bar{\varepsilon}} \phi(z, \bar{z}) = \left( h \partial \varepsilon(z) + \varepsilon(z) \partial + \overline{h \partial \bar{\varepsilon}(\bar{z})} + \bar{\varepsilon}(\bar{z}) \overline{\partial} \right) \phi(z, \bar{z})$$
Energy momentum tensor in conformal field theories

Noether

\[ \delta S = 0 \] conserved current \( j \) from infinitesimal transformation

\[ x'\mu = x + \epsilon \omega^\mu \]

\[
0 = L \left( \phi(x'), \frac{\partial \phi}{\partial x'\nu}, x' \right) d^d x' - L \left( \phi, \frac{\partial \phi}{\partial \phi}, x \right) d^d x
\]

\[
0 = \epsilon \partial_\mu \left( \eta^{\mu\nu} L \omega_\nu - \omega \frac{\partial L}{\partial \partial_\mu \phi} \right) d^d x
\]
Energy momentum tensor in conformal field theories

Conserved current

\[ j^\mu = \eta^{\mu\nu} L \omega_\nu - \omega_\nu \partial^\nu \phi \frac{L}{\partial(\partial_\mu \phi)} \]

Definition of the Energy momentum tensor \( T^{\mu\nu} \)

\[ j^\mu = T^{\mu\nu} \omega_\nu \]
Energy momentum tensor in conformal field theories

What implications does conformal invariance have on the energy momentum tensor?

- \( \partial_\mu T^{\mu\nu} = 0 \)
- \( T^{\rho\nu} = T^{\nu\rho} \)
- \( T^\mu_\mu = 0 \)
Energy momentum tensor in 2D

Transformation to complex coordinates:

\[ T_{zz} = \frac{1}{4} \left( T_{00} - 2iT_{10} - T_{11} \right) \]

\[ T_{z\bar{z}} = \frac{1}{4} \left( T_{00} + 2iT_{10} - T_{11} \right) \]

\[ T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4} T_{\mu}^{\mu} = 0 \]

Energy momentum tensor

In two dimensions one will get a chiral and an antichiral field:

\[ 2\pi T_{zz}(z, \bar{z}) = T(z), \quad 2\pi T_{z\bar{z}}(z, \bar{z}) = \bar{T}(\bar{z}) \]
Radial ordering 2D

Compactification

Two variables \(x^0\) for time and \(x^1\) for space. Mapping of space variable onto circle.

Introduction of complex variables \(\omega = x^0 + ix^1\)

Mapping from the Cylinder to the Complex Plane

Mapping function: \(z = e^\omega = e^{x^0} \cdot e^{ix^1}\)
Conserved Charges

**Definition**

\[
Q = \int dx^1 j_0 = \frac{1}{2\pi i} \oint_C \left( dz T(z) \epsilon(z) + d\bar{z} \bar{T}(\bar{z}) \bar{\epsilon}(\bar{z}) \right)
\]

at \( x^0 = \text{const} \) and \( j_\mu = T_{\mu\nu} \epsilon^\nu \)

From QFT we know that:

\[
\delta A = [Q, A]
\]
Radial ordering 2D

\[
\delta_{\varepsilon, \omega} \phi(\omega, \overline{\omega}) = \frac{1}{2\pi i} \oint_{|z|>|\omega|} dz \varepsilon(z) T(z) \phi(\omega, \overline{\omega}) - \oint_{|z|<|\omega|} dz \varepsilon(z) \phi(\omega, \overline{\omega}) T(z)
\]

\[
= \oint_{C(\omega)} dz \varepsilon(z) \mathcal{R} ([T(z), \phi])
\]
Operator Product expansion

Comparison to direct calculation of primary field yields this:

\[ R(T(z)\phi(\omega, \overline{\omega})) = \frac{h}{(z - \omega)^2} \phi(\omega, \overline{\omega}) + \frac{\partial_{\omega}}{z - \omega} \phi(\omega, \overline{\omega}) \]

**Definition**

A field is called primary if the operator product expansion between \( T(z) \) and \( \phi(z, \overline{z}) \) is of the above form.
The free boson

**Action of a free boson**

\[ S = \kappa \int dzd\bar{z} \partial X(z, \bar{z}) \bar{\partial} X(z, \bar{z}) \]

**Variation**

\[ \partial \bar{\partial} X(z, \bar{z}) = 0 \]

**Two point function**

\[ \langle x(z, \bar{z}) x(\omega, \bar{\omega}) \rangle = -\frac{1}{4\pi\kappa} \ln(z - \omega) \]
normal ordering of energy momentum tensor

\[ T(z) = 2\pi \kappa : \partial X \partial X : = 2\pi \kappa \lim_{z \to w} \left( \partial X(z) \partial X(w) - \langle \partial X(z) \partial X(w) \rangle \right) \]

conformal dimension of \( \partial x \)

\[ T(z) \partial x(\omega) = \frac{\partial x}{(z - \omega)^2} + \frac{1}{z - \omega} \partial^2 x \]

Thus the conformal dimension is \( h = 1 \)
Wick’s Theorem

From Quantum field theory:

**Contraction**

\[ \phi_1 \phi_2 \phi_3 \phi_4 \equiv \phi_1 \phi_3 : \langle \phi_2 \phi_4 \rangle \]

**Wick’s Theorem**

A time ordered product is equal to the normal ordered product, plus all possible contractions.
Asymptotic states
Consider Laurenţ expansion of the primary function $\phi(z, \bar{z})$:

$$
\phi(z, \bar{z}) = \sum_{n, \overline{m} \in \mathbb{Z}} z^{-n-h} \bar{z}^{-\overline{m}-\overline{h}} \phi_{n, \overline{m}}
$$

Definition
Take a look at the infinite past: $|\phi\rangle = \lim_{z, \bar{z} \to 0} \phi(z, \bar{z}) |0\rangle$

Singularity
We want the equation to be non singular i.e. well defined at $z = 0$ thus $\phi_{n, \overline{m}} |0\rangle = 0$ for $n > -h$ or $\overline{m} > -\overline{h}$

Out state
The same can be done with the out state. For $n < h$ or $\overline{m} < \overline{h}$

$$
\langle 0 | \phi_{n, \overline{m}} = 0
$$
Central charge of the boson

Virasoro Algebra

Extension of the Witt Algebra with following commutation relation:

\[ [L_n, L_m] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \]

\[ \frac{c}{2} = \langle L_2 L_{-2} \rangle \]

Central charge of free boson

This calculation gives us a central charge of \( c = 1 \)
The free fermion

Action of a free fermion

\[ S = \kappa \int dzd\bar{z}(\psi \overline{\partial}\psi + \overline{\psi} \partial\overline{\psi}) \]

Variation

\[ \partial\overline{\psi} = \overline{\partial}\psi = 0 \]
Two point function

\[ S = \frac{1}{2} \int dx^2 dy^2 \psi_i(x) A_{ij}(x, y) \psi_j(y) \]

With \( A_{ij} = \kappa 2\pi \delta(x - y)(\gamma^0 \gamma^\mu)_{ij} \partial_\mu \)

\[ \langle \psi_i(x) \psi_j(y) \rangle = A_{ij}^{-1} \]

two point function

\[ \langle \psi(z) \psi(\omega) \rangle = -\frac{1}{2\pi\kappa} \frac{1}{z - \omega} \]
Energy momentum tensor

\[ T(z) = -\pi \kappa : \psi(z) \partial \psi(z) : \]

Conformal charge of \( \psi \)

\[ T(z) \psi(\omega) = \frac{1}{2(z-\omega)^2} \psi(\omega) + \frac{1}{z-\omega} \partial_\omega \psi(\omega) \]

Thus \( \psi(w) \) is a field of conformal dimension: \( h = \frac{1}{2} \)
Central charge of a free fermion

Same calculation as for the boson gives the central charge for the free fermion.

\[
\frac{c}{2} = \langle 0 \vert L_2 L_{-2} \vert 0 \rangle = \frac{1}{(2\pi i)^2} \int d\omega \int dz \frac{z^3}{\omega} \langle 0 \vert T(z) T(\omega) \vert 0 \rangle = \\
(\pi \kappa)^2 \int d\omega \int dz \frac{z^3}{(2\pi i)^2 \omega}.
\]

\[
\cdot \left( \langle \psi(z) \partial \psi(\omega) \rangle \langle \partial \psi(z) \psi(\omega) \rangle + \langle \psi(z) \psi(\omega) \rangle \langle \partial \psi(z) \psi(\omega) \rangle \right) = \\
\frac{1}{4} \frac{1}{(2\pi i)^2} \int d\omega \int dz \frac{z^3}{\omega} \frac{1}{(z - \omega)^4} = \frac{1}{4}
\]

Central charge of free fermion

The central charge is \( c = \frac{1}{2} \)