CFT: Basic Properties and Examples

Daniel Herr

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Abstract

This document introduces several principles known from quantum field theory and examines what conformal invariance implies for these. The main part of this report will be the calculation of the central charge for the free boson and the free fermion under conformal invariance.

1 Basic Definitions

First of all here are some definitions, which will be used later on.

Definition 1 (*Chiral Antichiral Fields*)

A chiral field is a field only dependent on z, whereas an antichiral field only depends on $\overline{z}$

As shown in the previous report a conformal transformation is for example a rescaling. In order to get a conformal invariant field one can define the conformal dimensions as follows:

Definition 2 (*Conformal Dimension*)

$$\phi(z,\overline{z}) \mapsto \lambda^h \overline{\lambda}^{\overline{h}} \phi(\lambda z, \overline{\lambda}\overline{z})$$

The quantities h and $\overline{h}$ are called the conformal dimensions.

Definition 3 (*Primary field*)

For a general conformal transformation $f(z)$ the field ϕ is called primary if it transforms like:

$$\phi(z,\overline{z}) \mapsto \phi'(z,\overline{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \phi\left(f(z), \overline{f}(\overline{z})\right) \tag{1}$$

2 Correlation Functions

In quantum field theory correlation functions describe scattering processes. Here their invariance is of huge importance because we can calculate the structure of two-point and three-point correlation functions up to a constant assuming conformal invariance. The definition for correlation functions can be derived with the means of the Feynman path integral formalism.

Definition 4 (*Correlation functions*)

$$\langle \mathcal{T}\left(\phi(t_1)\phi(t_2)\ldots\phi(t_N)\right)\rangle = \frac{\int [d\phi]\phi(t_1)\phi(t_2)\ldots\phi(t_N)\exp\left(iS_\varepsilon[\phi(t)]\right)}{\int [d\phi]\exp\left(iS_\varepsilon[\phi(t)]\right)}$$

The $\mathcal{T}$ indicates that the following operators are time ordered. This has to be included since it does not make sense to have operators of the past acting on operators of the future. Physically this means that a measurement of the future would influence the measurement in the past. This is obviously not what happens. In the following the time ordering will always be applied without explicitly mentioning it.

2.1 Two point functions

We are now going to use the invariance under conformal transformations to show how an arbitrary two point function looks like. The restrictions are sufficient enough to determine the two point function up to a constant.

$$\langle \phi_1(z)\phi_2(\omega)\rangle = g(z,\omega)$$

With translational invariance ($f(z) = z + a$) one can conclude that $g(z, w)$ only depends on the difference of the two vectors z, ω.

$$g(z,\omega) = g(z-w)$$

Let's take a look at a rescaling. From the definition of the primary fields we know that the following must hold.

$$\left\langle \lambda^{h_1}\phi_1(\lambda z)\lambda^{h_2}\phi_2(\lambda\omega)\right\rangle = \lambda^{h_1+h_2} g\left(\lambda(z-\omega)\right) \stackrel{!}{=} g(z-\omega)$$

In order to fulfill this equation the two point function has to look like:

$$g(z-\omega) = \frac{d_{12}}{(z-\omega)^{h_1+h_2}}$$

One can still restrict the two-point function further by looking at the special conformal transformation. Those transformations are compositions of inversions with translations. Since translations were already taken care of one can just take a look at a inversion $f(z) = -\frac{1}{z}$.

$$\langle \phi_1(z)\phi_2(\omega)\rangle = \left\langle \frac{1}{z^{2h_1}}\frac{1}{\omega^{2h_2}}\phi_1(-\frac{1}{z})\phi_2(-\frac{1}{\omega})\right\rangle = \frac{1}{z^{2h_1}\omega^{2h_2}}\frac{d_{12}}{(-\frac{1}{z}+\frac{1}{\omega})^{h_1+h_2}} \stackrel{!}{=} \frac{d_{12}}{(z-w)^{h_1+h_2}}$$

With this identity one needs to restrict the two-function further by demanding that $h_1 = h_2$. So in the end the conformal invariant two-point-function looks like the following.

$$\langle \phi_1(z)\phi_2(\omega)) = \frac{d_{12}\delta_{ij}}{(z-\omega)^{2h_i}} \tag{2}$$

2.2 Three point functions

With the same line of reasoning one can find again an expression for the Three-point-function up to some constant. Again this can be obtained by using the symmetries for translation, rescaling and inversion. With the definition of $z_{ij} = z_i - z_j$

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}} \tag{3}$$

For any higher number correlation function one has not enough restrictions to write it in a similar way as above.

3 Energy Momentum Tensor

From Noether's Theorem one knows that any symmetry can be related to a conserved current ($\partial_\nu j^\nu = 0$). This current can be calculated and looks like this.

$$j^\mu = \eta^{\mu\nu} L\omega_\nu - \omega_\nu \partial^\nu \phi \frac{L}{\partial(\partial_\mu \phi)}$$

Definition 5 (*Energy Momentum Tensor*)

The quantity $T^{\mu\nu}$ is called energy momentum tensor and is defined such that $j^\mu = \omega_\nu T^{\mu\nu}$

By assuming conformal symmetry one can find that the energy momentum tensor has some important properties.
First one can consider the simplest conformal transformation: an arbitrary translation. The current has to be conserved no matter what conformal transformation is considered, so it has to be conserved under translations.

$$0 = \partial_\mu j^\mu = \partial_\mu(\omega_\nu T^{\mu\nu}) = \omega_\nu(\partial_\mu T^{\mu\nu})$$

The last equality holds because the derivative of ω for a translation vanishes. One will thus get.

$$\partial_\mu T^{\mu\nu} = 0 \tag{4}$$

Now we will prove that the energy momentum tensor is symmetric. The conserved current has to stay the same under rotations $\omega_\nu = m_{\mu\nu} x^\nu$ where m is antisymmetric ($m_{\mu\nu} = -m_{\nu\mu}$).

$$\begin{aligned} 0 &= \partial_\mu j^\mu = \partial_\mu (T^{\mu\nu} m_{\nu\rho} x^\rho) \\ &= \frac{1}{2}\partial_\mu m_{\nu\rho}(T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu) \\ &= (\partial_\mu T^{\mu\nu}) x^\rho + (\partial_\mu T^{\mu\rho}) x^\nu + T^{\rho\nu} + T^{\rho\nu} \end{aligned}$$

The first two terms vanish because of equation 4 giving us the **symmetry** condition of the energy momentum tensor. Now consider an arbitrary conformal transformation. The invariance under this transformation will yield yet another property of the energy momentum tensor.

$$0 = (\partial_\mu \omega_\nu) T^{\mu\nu} + (\partial_\mu T^{\mu\nu})\omega_\nu$$

The second term vanishes because of the property from 4 above. With the symmetry condition of the energy momentum tensor one will get to the following.

$$0 = \frac{1}{2} T^{\mu\nu} (\partial_\mu \omega_\nu + \partial_\nu \omega_\mu) = \frac{1}{d} T^\mu_\mu \partial\omega \tag{5}$$

Since ω was arbitrary the only way the right term vanishes, is if the energy momentum tensor is **traceless**.

3.1 Energy Momentum Tensor in 2D

The derivations until now are valid for any dimension, but in the case of $d = 2$ one finds some further properties of the energy momentum tensor. With the transformation rules $z = x_0 + ix_1$ and $\overline{z} = x_0 - ix_1$ one can shift to complex coordinates. The quantities T_{ij} are the energy momentum parts in real coordinates. In the following the properties derived in the chapter above were used to simplify the equations.

$$T_{zz} = \frac{1}{4}\left(T_{00} - 2iT_{10} - T_{11}\right) = \frac{1}{2}\left(T_{00} - iT_{10}\right)$$

$$T_{\overline{zz}} = \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right) = \frac{1}{2}\left(T_{00} + iT_{10}\right)$$

$$T_{z\overline{z}} = T_{\overline{z}z} = \frac{1}{4} T^\mu_\mu = 0$$

Again with the property from above one will find the following:

$$\begin{aligned} \partial_{\overline{z}} T_{zz} &= \frac{1}{4}\left(\partial_0 + i\partial_1\right)\left(T_{00} - iT_{10}\right) \\ &= \frac{1}{4}\left(\partial_0 T_{00} + \partial_1 T_{10} - i\partial_1 T_{11} - i\partial_0 T_{01}\right) = 0 \end{aligned}$$

With the same calculation can be conducted with $T_{\overline{zz}}$ and it can be seen that there is only a chiral and an antichiral field.

$$2\pi T_{zz}(z, \overline{z}) = T(z) \quad 2\pi \overline{T}_{\overline{zz}}(z, \overline{z}) = \overline{T}(\overline{z})$$

4 Time ordering & radial ordering

As mentioned before time ordering is of major importance for the description of a physical system. A two dimensional system can be thought of as the surface of a cylinder (see fig 1). Here x_0 describes the time and x_1 is the coordinate for space.
One can map this cylinder to the complex plane with the help of the exponential function. Now the bigger the radius is, the further in the future the event takes place. Temporal ordering is now expressed by radial ordering.

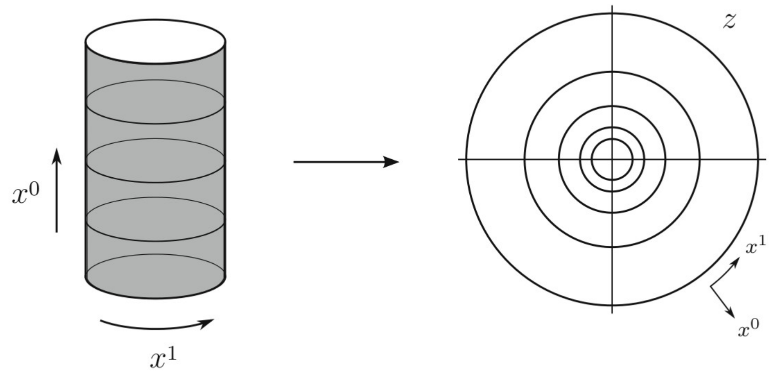

Figure 1: Mapping from the cylinder to the complex plane

5 Conserved Charge

Noethers Theorem implies a conserved current. Furthermore one can get from the current a conserved charge.

$$\partial_\mu j^\mu = \frac{\partial j^0}{\partial t} + \frac{\partial j^i}{\partial x^i}$$

Now, one can integrate over the whole space and use that the current vanishes in infinity.

$$\int dx \frac{\partial j^0}{\partial t} = 0$$

This leads to the definition of the conserved charge.

Definition 6 (*Conserved Charge*)

$$Q = \int dx^1 j_0 = \frac{1}{2\pi i} \oint_C \left(dz T(z)\epsilon(z) + d\overline{z}\overline{T}(\overline{z})\overline{\epsilon}(\overline{z}) \right)$$

From quantum field theory one can calculate that the conserved charge is a generator of the infinitesimal conformal transformations.

$$\delta\phi = [Q, \phi]$$

Plugging in the definition for the conformal charge one will get some integral description of an infinitesimal (conformal) transformation. This can be compared to the first order Taylor expansion of the primary field:

$$\delta_{\varepsilon,\overline{\varepsilon}}\phi(z,\overline{z}) = \left(h\partial\varepsilon(z) + \varepsilon(z)\partial + \overline{h\partial}\overline{\varepsilon}(\overline{z}) + \overline{\varepsilon}(\overline{z})\overline{\partial} \right) \phi(z,\overline{z})$$

The complete derivation will be done in the following paper from Pascal Debus. The result of this calculation will be the operator product expansion. With this OPE one can easily find out the value for the conformal dimension of a primary field.

$$\mathcal{R}\left(T(z)\phi(\omega,\overline{\omega})\right) = \frac{h}{(z-\omega)^2}\phi(\omega,\overline{\omega}) + \frac{\partial_\omega}{z-\omega}\phi(\omega,\overline{\omega}) + \dots \tag{6}$$

Now most of the important principles are known such that the calculation of the free boson can be understood.

6 The Free Boson in 2D

6.1 Variational Principle

The action of a free Boson looks like

$$S = \kappa \int dz d\overline{z} \partial X(z, \overline{z}) \overline{\partial} X(z, \overline{z})$$

By using variational principle one imposes that the action is extremal under small variances of the path.

$$\begin{aligned} 0 = \delta S &= \kappa \int dz d\overline{z} \left(\partial \delta X(z, \overline{z}) \overline{\partial} X(z, \overline{z}) + \partial X(z, \overline{z}) \delta \overline{\partial} X(z, \overline{z}) \right) \\ &= \kappa \int dz d\overline{z} \left[\partial \left(\delta X \overline{\partial} \right) - \delta \partial \overline{\partial} X + \overline{\partial} \left(\delta X \partial X \right) - \delta X \partial \overline{\partial} X \right] \\ &= \kappa \int dz d\overline{z} 2 \delta X \partial \overline{\partial} X \end{aligned}$$

So the field Xcan be written as a sum of a chiral and an antichiral field. $X(z, \overline{z}) = x(z) + \overline{x}(\overline{z})$.

6.2 Two point function

The two point function can be derived by looking at the action. It has to be a propagator. Here is just the sketch of the derivation.

$$S = \kappa \int d^2 x \partial_\nu \phi(x) \partial^\nu \phi(x) = \kappa \int d^2 x d^2 y \phi(x) A(x, y) \phi(y)$$

With the quantity $A = -\kappa \delta^{(2)}(x - y) \partial_x^2$. The inverse of this quantity is just the two point function. With the use of the representation of the delta distribution one will get:

$$\langle \phi(x) \phi(w) \rangle = -\frac{1}{z \kappa 4\pi} \log(x - y)^2$$

To get back to complex coordinates one just has to apply the transformations.

$$\langle X(z, \overline{z}) X(\omega, \overline{\omega}) \rangle = -\frac{1}{\kappa 2\pi} \left(\log(z - \omega) + \log(\overline{z} - \overline{\omega}) \right) \tag{7}$$

But we know that this is not a primary field, since we calculated already what the structure of a primary field for a conformal invariant theory is like. In the following the calculation will only be conducted for the chiral part of the two point function. Similarly one can perform the calculation for the antichiral field. Now what about ∂X?

$$\langle \partial_z x(z) \partial_\omega x(\omega) \rangle = \partial_z \frac{1}{4\pi\kappa} \frac{1}{z - \omega} = -\frac{1}{4\pi\kappa} \frac{1}{(z - \omega)^2}$$

This is indeed a two point function of the structure we calculated before. By comparison with 2 one will get a first guess that the conformal dimension will be 1. The prove will need the operator product expansion of $T(z)x(z)$. But first the energy momentum tensor has to be calculated.

6.3 Energy Momentum Tensor

By looking at the action the Langrange density can be obtained as $\mathcal{L} = \kappa \partial X \overline{\partial} X$. To get the energy momentum tensor from the Lagrange density by using the definition of the conserved current. One will find that the chiral and antichiral parts look like the following.

$$T(z) = 2\pi\kappa\partial X\partial X \quad \overline{T}(\overline{z}) = 2\pi\kappa\overline{\partial}X\overline{\partial}X$$

There is still a problem which arises in quantum field theory. The vacuum expectation value of the energy is singular. Thus one has to find a way to 'subtract infinity'. This is done with normal ordering.

Definition 7 (*Normal ordering*)

$$:\phi(z)\phi(z) := \lim_{\omega\to z}\left(\phi(\omega)\phi(z) - \langle\phi(\omega)\phi(z)\rangle\right)$$

6.4 Wicks theorem

With the help of Wick's theorem any time ordered product can be written as a sum of normal ordered products. The proof can be found in any standard quantum field theory book and will not be performed here. All properties which are important for the calculation are stated in this section.

Definition 8 (*Contraction*)

A contraction is defined as:

$$:\phi_1\wick{\c1\phi_2\phi_3\c1\phi_4} :=: \phi_1\phi_3 : \langle\phi_2\phi_4\rangle$$

This definition suffices to state Wick's Theorem.

Theorem 1 (*Wick's Theorem*)

A time ordered product is equal to the normal ordered product, plus all possible contractions.

As an example for a time ordered product of four variables one will get this idendity:

$$\begin{aligned}\mathcal{T}(\phi_1\phi_2\phi_3\phi_4) = {} & :\phi_1\phi_2\phi_3\phi_4: + :\wick{\c1\phi_1\phi_2\c1\phi_3\phi_4}: + :\wick{\phi_1\c1\phi_2\phi_3\c1\phi_4}: + :\wick{\c1\phi_1\phi_2\phi_3\c1\phi_4}: \\ & + :\wick{\c1\phi_1\c1\phi_2\phi_3\phi_4}: + :\wick{\phi_1\phi_2\c1\phi_3\c1\phi_4}: + :\wick{\phi_1\c1\phi_2\c1\phi_3\phi_4}: + :\wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}: \\ & + :\wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4}: + :\wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4}:\end{aligned}$$

This theorem makes sense if one knows that many parts of the sum are vanishing. For example in vacuum the expectation value of any normal ordered product is zero. Thus only double contractions remain i.e. the last three contractions from above. Also from the definition of normal ordering one can see that the contraction of a already normal ordered product is 0.

$$: \phi_1 \phi_2 := \lim_{\omega \to z} \phi_1(\omega)\phi_2(z) - \langle \phi(\omega)\phi(z) \rangle = 0$$

Now all the tools are available for the calculation of the conformal dimension of the field ∂X.

$$\begin{aligned} T(z)\partial_\omega x(\omega) &= 2\pi\kappa : \partial_z x(z)\partial_z x(z) : \partial_\omega x(\omega) \\ &\approx 2\pi\kappa \left(2 \langle \partial_z x(z) \partial_\omega x(\omega) \rangle \partial x(z)\right) \\ &\approx \frac{1}{(z-\omega)^2}\partial_z x(z) \end{aligned}$$

But in comparison with the operator product expansion one needs the $\partial x(z)$ dependent on ω. Thus a Taylor expansion is necessary and the result will be

$$T(z)\partial x(\omega) = \frac{\partial x(\omega)}{(z-\omega)^2} + \frac{1}{z-\omega}\partial^2 x(\omega) + \ldots$$

This gives us proof that ∂x is indeed a primary field with conformal dimension of $h = 1$.

6.5 In- and Outstates

Consider Laurent expansion of the primary function $\phi(z, \overline{z})$:

$$\phi(z, \overline{z}) = \sum_{n, \overline{m} \in \mathbb{Z}} z^{-n-h} \overline{z}^{-\overline{m}-\overline{h}} \phi_{n, \overline{m}}$$

The states of the infinite past is defined as $|\phi\rangle = \lim_{z, \overline{z} \to 0} \phi(z, \overline{z}) |0\rangle$. A state of the infinite future is defined the same way. Now we impose a physical restriction to those states. They should not be singular. This means that the In- and Outsates should be well defined at $z = 0$ and $z \to \infty$. This implies the following properties for the Laurent coefficients.

$$\begin{aligned} \phi_{n,\overline{m}} |0\rangle = 0 \quad &\text{for} \quad n > -h \text{ or } \overline{m} > -\overline{h} \\ \langle 0| \phi_{n,\overline{m}} = 0 \quad &\text{for} \quad n < h \text{ or } \overline{m} < \overline{h} \end{aligned}$$

6.6 The Central Charge

First of all, one needs to take a look at the Laurent expansion of the energy momentum tensor.

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

The operators L_n can be calculated as:

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \qquad (8)$$

These Laurent coefficients are generators of infinitesimal transformations as can be seen in the following calculation.

$$Q_n = \oint \frac{dz}{2\pi i} T(z) \left(-\epsilon_n z^{n+1}\right) = -\epsilon \sum_{n\in\mathbb{Z}} \oint \frac{dz}{2\pi i} L_m z^{n-m-1} = -\epsilon_n L_n$$

These Laurent coefficients have the properties of the Virasoro algebra, which was discussed in the previous report in more detail.

Definition 9 (*Virasoro Algebra*)

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

The central charge c can be calculated by looking at the commutation relation of L_2 and L_{-2}.

$$\langle 0|L_2 L_{-2}|0\rangle = -4\langle 0|L_0|0\rangle + \frac{c}{2} = \frac{c}{2}$$

Here the L_0 term drops because of the definition of the in state ($0 > -2$). On the other hand a simple calculation of the commutator and applying the knowledge of in and out states the left term yields:

$$\langle 0|L_2 L_{-2} - L_{-2}L_2|0\rangle = \langle 0|L_2 L_{-2}|0\rangle$$

The central charge can now be calculated in a rather lengthy calculation:

$$\begin{aligned}
\frac{c}{2} &= \langle 0|L_2 L_{-2}|0\rangle = \oint \frac{dz}{2\pi i} \oint \frac{d\omega}{2\pi i} \frac{z^3}{\omega} \langle 0|T(z)T\omega|0\rangle \\
&= (2\pi\kappa)^2 \oint \frac{dz}{2\pi i} \oint \frac{d\omega}{2\pi i} \frac{z^3}{\omega} \langle 0| : \partial_z x(z)\partial_z x(z) :: \partial_\omega x(\omega)\partial_\omega x(\omega) : |0\rangle \\
&= (2\pi\kappa)^2 \oint \frac{dz}{2\pi i} \oint \frac{d\omega}{2\pi i} \frac{z^3}{\omega} 2 \langle \partial x(z)\partial x(\omega)\rangle \langle \partial x(z)\partial x(\omega)\rangle \\
&= \frac{1}{2} \oint \frac{dz}{2\pi i} \oint \frac{d\omega}{2\pi i} \frac{z^3}{\omega} \frac{1}{(z-\omega)^4} \\
&= \frac{1}{2} \oint \frac{dz}{2\pi i} \frac{z^3}{(z-0)^4} = \frac{1}{2}
\end{aligned}$$

The first step in this calculation was to insert equation 8. Then inserting the definition of the energy momentum tensor. Afterwards Wick's theorem was applied. Here almost every term is zero except for the double contractions. This is because any normal ordered product applied to the vacuum state will give 0 and the two double contractions are the only terms which did not have any normal ordered operators. Then the previously calculated two point functions were inserted. The rest was just applying residue theorem several times.
All in all the result for the central charge is

$$c = 1$$

7 The free fermion

The calculation for the free fermion goes along the same lines as the calculation of the free boson. The difference lies in the action of the free fermion.

$$S = \frac{\kappa}{2} \int dz d\overline{z} \left(\overline{\psi} \partial \overline{\psi} + \psi \overline{\partial} \psi \right)$$

From variational principle follows that

$$\partial \overline{\psi} = \overline{\partial} \psi = 0$$

Again the two point function can be calculated with the means of a Green's function.

$$\langle \psi(z, \overline{z}) \psi(\omega, \overline{\omega}) \rangle = \frac{1}{2\pi\kappa} \frac{1}{z - \omega} \tag{9}$$

This suggests that ψ is a conformal field of dimension $\frac{1}{2}$. This makes sense because if one exchanges two particles one will get the same term with a minus sign, this is good since fermions do have this property.
The energy momentum tensor can again be derived from its definition and the knowledge of the Lagrange density.

$$T(z) = -\pi : \psi(z) \partial \psi(z) :$$

7.1 Central charge for the free fermion

The discussion of in and out states can be applied here in exactly the same way than for the free boson. Thus the actual calculation is just the last part:

$$\begin{aligned}
\frac{c}{2} = \langle 0 | L_2 L_{-2} | 0 \rangle &= \frac{1}{(2\pi i)^2} \oint dz \oint d\omega \frac{z^3}{\omega} \langle 0 | T(z) T(\omega) | 0 \rangle \\
&= (\pi\kappa)^2 \oint \oint \frac{dz d\omega z^3}{(2\pi i)^2 \omega} \left(\langle \psi(z) \partial \psi(\omega) \rangle \langle \partial \psi(z) \psi(\omega) \rangle + \langle \psi(z) \psi(\omega) \rangle \langle \partial \psi(z) \psi(\omega) \rangle \right) \\
&= \frac{1}{4} \frac{1}{(2\pi i)^2} \oint dz \oint d\omega \frac{z^3}{\omega} \frac{1}{(z - \omega)^4} = \frac{1}{4}
\end{aligned}$$

Again first the energy momentum tensor and the equation 8 was substituted into the equation. Afterwards Wick's theorem was applied and in the end residue theorem was applied. Thus the central charge of the free boson is

$$c = \frac{1}{2}$$

References

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