Basics of Lie theory
Classification of Lie Algebras

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The Matrix group $\text{SO}(3)$

Consider the Matrix group

$$\text{SO}(3) = \{ A \in \text{Mat}(3, \mathbb{R}) \mid A^T A = 1, \det(A) = 1 \}$$

Define the Lie algebra of $\text{SO}(3)$ as

$$\mathfrak{so}(3) = \{ \dot{\gamma}(0) \mid \gamma : (-\varepsilon, \varepsilon) \to \text{SO}(3), \gamma(0) = 1 \}$$

Claim

$$\mathfrak{so}(3) = \{ A \in \text{Mat}(3, \mathbb{R}) \mid A^T + A = 0 \}$$
Proof of the Claim:
"⊂" Consider $\gamma$ as in the definition of the Lie algebra. Then
\[
\gamma(t)^T \gamma(t) = 1 \quad \forall t \in [0, \varepsilon)
\]
By differentiation
\[
\dot{\gamma}(t)^T \gamma(t) + \gamma(t)^T \dot{\gamma}(t) = 0
\]
\[
t \rightarrow^0 \dot{\gamma}(0)^T + \dot{\gamma}(0) = 0
\]
"⊃" Let $A \in \text{Mat}(3, \mathbb{R})$ st. $A^T + A = 0$. In particular $\text{Tr}(A) = 0$.

Define

$$\gamma : \mathbb{R} \rightarrow \text{Mat}(3, \mathbb{R})$$

$$t \mapsto \exp(At)$$

Note that

1. $\gamma(0) = 1$
2. $\det(\gamma(t)) = \exp(t \text{Tr}(A)) = 1$
3. $\gamma(t)^T \gamma(t) = \exp(-At) \exp(At) = 1$
4. $\dot{\gamma}(0) = A$
Definition

A Lie group $G$ is a set that has compatible structures of a smooth manifold and of a group. Compatible means that group multiplication and inversion are smooth maps i.e. the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth.
A **Matrix Lie group** is a Lie group that is contained in $\text{GL}(n, K)$ for some $n$ and field $K$. Let $n \in \mathbb{N}$. Then the following groups are Lie groups:

- $\text{GL}(n, \mathbb{R})$ and $\text{GL}(n, \mathbb{C})$
- $\text{SL}(n, \mathbb{R})$ and $\text{SL}(n, \mathbb{C})$
- $\text{O}(n), \text{SO}(n), \text{U}(n), \text{SU}(n)$
- The symplectic groups $\text{Sp}(2n, \mathbb{R})$ and $\text{Sp}(2n, \mathbb{C})$
- The group $\mathbb{B}_n$ of upper-triangular matrices
Construction of the Lie algebra

Consider the action of the Lie group $G$ on itself by conjugation

$$\Psi : G \rightarrow Aut(G)$$

$$g \mapsto \psi_g$$

where

$$\psi_g(h) = ghg^{-1} \quad \forall h \in G$$

Note that the neutral element $e$ gets mapped to itself. Consider now for $g \in G$ the map

$$Ad(g) = (d\psi_g)_e : T_eG \rightarrow T_eG$$
Thus

\[ Ad : G \to Aut(T_eG) \]

Taking the differential map of Ad at the unity we get a map in the tangent spaces

\[ ad : T_eG \to End(T_eG) \]

This implies a bilinear map \( T_eG \times T_eG \to T_eG \) called the \textbf{Lie bracket} by

\[ [X, Y] := ad(X)(Y) \]
The Lie bracket fulfills

- \([X, Y] = -[Y, X]\) for all \(X, Y \in T_e G\)
- the Jacobi identity

\[
[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0
\]

for all \(X, Y, Z \in T_e G\)

The **Lie algebra associated to the Lie group** \(G\) is \(T_e G\) together with the Lie bracket on \(T_e G\), we write \(\mathfrak{g}\). A vectorspace \(\mathfrak{g}\) together with a bilinear map \([\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}\) satisfying the conditions in the theorem above is called a **Lie algebra**.
Homomorphisms of Lie groups and Lie algebras

Definition

Let $G, H$ be Lie groups and $\mathfrak{g}, \mathfrak{h}$ a Lie algebras

- A Lie group homomorphism $\rho : G \rightarrow H$ is a smooth map such that $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$.

- A Lie algebra homomorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a linear map, such that $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ for all $X, Y \in \mathfrak{g}$.

A representation of a Lie group $G$ is a Lie group homomorphism mapping to $\text{GL}(V)$, where $V$ is some vector space.
A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism mapping to $\mathfrak{gl}(V) = \text{End}(V)$. 
Fact

- Let $G$ a Lie group and $\mathfrak{g}$ its Lie algebra. If $G$ is connected, it is possible to generate the whole Lie group using $\mathfrak{g}$ only.
- Let $G, H$ Lie groups and $\mathfrak{g}, \mathfrak{h}$ its Lie algebras. If $G$ is simply connected, the Lie group homomorphisms from $G$ to $H$ are in one-to-one correspondence to the Lie algebra homomorphisms from $\mathfrak{g}$ to $\mathfrak{h}$.
Examples of Lie algebras

Matrix Lie groups → Matrix Lie algebras.

Some complex Matrix Lie algebras:

- \( \mathfrak{gl}_n \mathbb{C} = \text{End}(\mathbb{C}^n) \) (or more generally \( \mathfrak{gl}(V) \) for \( V \) vector space)
- \( \mathfrak{sl}_n \mathbb{C} = \{ A \in \text{Mat}(n, \mathbb{C}) \mid \text{Tr}(A) = 0 \} \)
- \( \mathfrak{sp}_{2n} \mathbb{C} = \{ A \in \text{Mat}(2n, \mathbb{C}) \mid MA + A^T M = 0 \} \) where
  \[
  M = \begin{pmatrix}
  0 & 1_n \\
  -1_n & 0 \\
  
  \end{pmatrix}
  \]
- \( \mathfrak{so}_{2n} \mathbb{C} \). As above, but with \( M = \begin{pmatrix}
  0 & 1_n \\
  1_n & 0 \\
  
  \end{pmatrix} \)
- \( \mathfrak{so}_{2n+1} \mathbb{C} \). With \( M = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 1_n \\
  0 & 1_n & 0 \\
  
  \end{pmatrix} \)
A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, that is closed under the Lie bracket (i.e. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$) is called a **Lie subalgebra**.

**Definition**

1. A Lie subalgebra $\mathfrak{h}$ is an **ideal** if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.
2. A Lie algebra $\mathfrak{g}$ is **abelian** if $[\mathfrak{g}, \mathfrak{g}] = 0$.
3. A non-abelian Lie algebra $\mathfrak{g}$ that does not contain any non-trivial ideal, is called **simple**.
4. A Lie algebra $\mathfrak{g}$ that does not contain any abelian ideal is called **semisimple**.

**Example 1**: The center $Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \forall Y \in \mathfrak{g}\}$ is an ideal. The center of a semisimple Lie algebra contains only 0.

**Example 2**: $\mathfrak{sl}_n \mathbb{C} \subset \mathfrak{gl}_n \mathbb{C}$ is a non-abelian ideal.
The adjoint map

Let $\mathfrak{g}$ be a complex Lie algebra in what follows. The adjoint map at $X \in \mathfrak{g}$ is

$$\text{ad}_X : \mathfrak{g} \to \mathfrak{g}$$

$$Y \mapsto [X, Y]$$

One can show that

$$\text{ad}[X, Y] = [\text{ad}_X, \text{ad}_Y]$$

Thus ad is a representation of $\mathfrak{g}$ on itself → adjoint representation.
Example: a basis for $\mathfrak{sl}_2(\mathbb{C})$

We consider the following basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H$$

It can easily be shown that $\mathfrak{sl}_2(\mathbb{C})$ is simple using the relations above.
Cartan subalgebra

Let \( \mathfrak{g} \) a semisimple (finite) Lie algebra. Consider a maximal subset of \( \mathfrak{g} \) consisting of linearly independent, commuting elements, st. for each element \( H \) \( \text{ad}_H \) is diagonalizable (i.e. \( H \) is \textit{ad-diagonalizable}). The subalgebra spanned by these elements is called a \textbf{Cartan subalgebra}, denoted by \( \mathfrak{h} \). Note that

- The Cartan subalgebra is unique up to automorphisms of \( \mathfrak{g} \).
- The Cartan subalgebra is a maximal abelian subalgebra consisting of simultaneously ad-diagonalizable elements b.c.

\[
[\text{ad}_{H_1}, \text{ad}_{H_2}] = \text{ad}_{[H_1,H_2]} = 0 \quad \forall H_1, H_2 \in \mathfrak{h}
\]

- \( \mathfrak{h} \) is non trivial.
Cartan decomposition

→ action of $\mathfrak{h}$ on $\mathfrak{g}$ by adjoint representation (diagonalizable!). This yields the **Cartan decomposition**

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$$

where $\mathfrak{g}_{\alpha}$ are eigenspaces of the action of $\mathfrak{h}$. For $H \in \mathfrak{h}$, $X \in \mathfrak{g}_{\alpha}$ we have

$$\text{ad}_H(X) = [H, X] = \alpha(H)X$$

$\rightarrow \alpha \in \mathfrak{h}^*$, called **roots**. $\mathfrak{g}_{\alpha}$ are the **root spaces**
**Claim**

In the adjoint representation $g_\alpha : g_\beta \rightarrow g_{\alpha + \beta}$

**Proof:** Let $X_\alpha \in g_\alpha$, $X_\beta \in g_\beta$ and $H \in h$. Then

$$[H, [X_\alpha, X_\beta]] = -[X_\beta, [H, X_\alpha]] - [X_\alpha, [X_\beta, H]]$$

$$= -\alpha(H)[X_\beta, X_\alpha] + \beta(H)[X_\alpha, X_\beta]$$

$$= (\alpha + \beta)(H)[X_\alpha, X_\beta] \quad \square$$

We will denote the set of roots by $R$. 
Proposition

Let \( g \) a semisimple, complex, finite-dim. Lie algebra. Let \( \mathfrak{h} \) a Cartan subalgebra. Consider the Cartan-decomposition

\[ g = \mathfrak{h} \oplus \bigoplus_{\alpha} g_{\alpha} \]

Then

- The roots span the dual space \( \mathfrak{h}^* \).
- Every root space is one dimensional.
- The only multiples of a root \( \alpha \), which are roots are \( \pm \alpha \).

A basis of \( g \) consisting of a basis of \( \mathfrak{h} \) and of elements spanning \( g_{\alpha} \) is called a **Cartan-Weyl basis**.
Remark

We can show that \([g_\alpha, g_{-\alpha}] \neq 0\), \([[g_\alpha, g_{-\alpha}], g_\alpha] \neq 0\). Thus

\[ s_\alpha := g_\alpha \oplus g_{-\alpha} \oplus [g_\alpha, g_{-\alpha}] \cong \mathfrak{sl}_2 \mathbb{C} \]

We can thus choose \(X_\alpha \in g_\alpha\), \(Y_\alpha \in g_{-\alpha}\) and set \(H_\alpha = [X_\alpha, Y_\alpha] \in \mathfrak{h}\), such that the usual commutation relations of \(\mathfrak{sl}_2 \mathbb{C}\) hold i.e.

\[ [H_\alpha, X_\alpha] = 2X_\alpha, [H_\alpha, Y_\alpha] = -2Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha] \]

In particular \(\alpha(H_\alpha) = 2\).
It is possible to ”build up” the Cartan subalgebra with elements \( \{ H_\alpha \}_{\alpha \in R} \). In fact we can choose a subset of \( R \) st. the above elements form a basis.

**Proposition**

There are elements \( \{ H_\alpha \}_{\alpha \in R} \) spanning \( \mathfrak{h} \) such that \( \beta(H_\alpha) \) is an integer for every \( \alpha, \beta \in R \) and \( \alpha(H_\alpha) = 2 \).
The Killing form

For $X, Y \in \mathfrak{g}$ we define the Killing form as

$$B(X, Y) = \text{Tr}(\text{ad}_X \circ \text{ad}_Y)$$

Note that $B$ is a linear map

$$B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$$

It also clear, by definition of $B$, that $B$ is symmetric.
Nondegeneracy of the Killing form

Proposition

The Killing form is positive definite on the real subspace of \( \mathfrak{h} \) spanned by \( \{H_\alpha\}_\alpha \).

Proposition

\( \mathfrak{g} \) is semisimple iff its Killing form is nondegenerate.

Idea of the Proof: "\( \Rightarrow \)" Show that the kernel of \( B \) is an ideal. "\( \Leftarrow \)" Show that if \( \mathfrak{l} \) is an ideal, then \( \mathfrak{l}^\perp \) is also an ideal.
Killing form on $\mathfrak{h}^*$

**Remark**

The nondegeneracy of the bilinear form (on the real subspace spanned by $\{H_\alpha\}_\alpha$) supplies an isomorphism $\mathfrak{h} \rightarrow \mathfrak{h}^*$ under which

$$T_\alpha := 2H_\alpha / B(H_\alpha, H_\alpha) \mapsto \alpha$$

The Killing form on $\mathfrak{h}^*$ is defined by

$$B(\alpha, \beta) = B(T_\alpha, T_\beta)$$

for two roots $\alpha, \beta \in R$ (pos.def. on the subspace spanned by $R$).

By definition

$$\beta(H_\alpha) = \frac{2B(\beta, \alpha)}{B(\alpha, \alpha)}$$
The Weyl group

**Proposition**

For any $\alpha \in R$ the map (an involution)

$$W_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$$

$$\beta \mapsto \beta - \beta(H_\alpha)\alpha$$

leaves $R$ invariant.

The **Weyl group** is the group generated by the set of automorphisms $\{W_\alpha\}_{\alpha \in R}$. By the above the set of roots $R$ is invariant under the Weyl group.
Since

\[ W_\alpha(\beta) = \beta - \frac{2B(\beta, \alpha)}{B(\alpha, \alpha)} \alpha \]

\( W_\alpha \) corresponds to a reflection in the hyperplane

\[ \Omega_\alpha = \{ \beta \in \mathfrak{h}^* : B(\beta, \alpha) = 0 \} \]
Pick a hyperplane in $\mathfrak{h}^*$ such that no point of the lattice spanned by $R$ is contained and call by convention the points on one side the plane **positive** and on the other negative. A positive root is called **simple** if it cannot be written as a sum of two positive roots. E.g.

![Diagram](image)

**Figure**: Root system of $\mathfrak{sl}_3 \mathbb{C}$, splitting of the space by the thick line, simple roots in red.
An introductory example
Lie groups
Lie algebras
Classification of simple Lie algebras
Ordering of the roots
Root systems
Dynkin diagrams
Classification of simple Lie algebras

Angles between roots

Denote by $\mathbb{E}$ the real subspace of $\mathfrak{h}^*$ spanned by the roots together with the scalar product given by the Killing form (denoted simply by $(\cdot, \cdot)$). Recall: $\forall \alpha, \beta \in R$

$$n_{\beta \alpha} := \frac{2B(\beta, \alpha)}{B(\alpha, \alpha)} = \beta(H_\alpha) \in \mathbb{Z}$$

If $\theta$ is the angle between $\alpha$ and $\beta$, then

$$n_{\beta \alpha} = 2 \cos(\theta) \frac{||\beta||}{||\alpha||}$$

Thus

$$n_{\beta \alpha} n_{\alpha \beta} = 4 \cos^2(\theta) \leq 4$$
Hence $4\cos^2(\theta)$ is an integer. The allowed angles in $[0, \pi)$ are 
\[ \theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}. \]

**Example:** Assume $|n_{\beta\alpha}| \geq |n_{\alpha\beta}|$ and $\theta = \frac{\pi}{6}$ for instance. Then 
\[ \cos(\theta) = \frac{\sqrt{3}}{2} \] and $n_{\beta\alpha} n_{\alpha\beta} = 3$. Hence $n_{\beta\alpha} = 3$ and $n_{\alpha\beta} = 1$ 
\[ \Rightarrow \frac{||\beta||}{||\alpha||} = \sqrt{3}. \]
Examples of root systems

We call

$$r := \dim_{\mathbb{R}} E = \dim_{\mathbb{C}} \mathfrak{h}$$

the \textbf{rank} of the Lie algebra.

\textbf{rank 1} There is exactly one possible root system that can be drawn

$$(A_1) \quad \longleftrightarrow \quad \bullet$$

This is precisely the root system of $\mathfrak{sl}_2 \mathbb{C}$. 
**Examples of root systems**

**rank 2** There are 4 different root system in 2 dimensions.

\[(A_1)_x(A_1)\]  
\[\text{(A2)}\]  
\[\text{(B2)}\]  
\[\text{(G2)}\]
Further symmetries of the root system

Recall: a Lie algebra is *simple* if it is non-abelian and contains no non-trivial ideals.

**Lemma**

A semisimple Lie algebra is simple iff its root system is irreducible i.e. cannot be written as a direct sum of two root systems.

Also recall that a simple root is a root that cannot be written as a sum of two positive roots. One can show that:

- If $\alpha, \beta$ simple, then neither $\alpha - \beta$ nor $\beta - \alpha$ are roots.
- The angle between two simple roots cannot be acute.
- The simple roots are linearly independent and span $\mathbb{F}$. Every positive root can be uniquely written as a non-negative integral linear combination of simple roots.
The **Dynkin diagram** of a root system is drawn as follows.

- Every simple root is represented by a node °.
- Two simple roots are connected in the following way
  - not connected, if $\theta = \frac{\pi}{2}$
  - one line, $\theta = \frac{2\pi}{3}$
  - two lines and an arrow pointing from the longer to the shorter root, if $\theta = \frac{3\pi}{4}$.
  - three lines and an arrow pointing from the longer to the shorter root, if $\theta = \frac{5\pi}{6}$. 
The Dynkin diagrams of irreducible root systems are:

- $(A_n)$
- $(B_n)$
- $(C_n)$
- $(D_n)$
- $(E_6)$
- $(E_7)$
- $(E_8)$
- $(F_4)$
- $(G_2)$
Given any Dynkin diagram of an irreducible root system, one can prove that:

- The Dynkin diagram contains no loops/cycles and is connected (i.e. it’s a tree).
- Any node has at most three lines to it.