Basics of Lie Theory
(Proseminar in Theoretical Physics)

Andreas Wieser
ETH Zürich

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Tutor:
Dr. Peter Vrana

Abstract

The goal of this report is to give an insight into the theory of Lie groups and Lie algebras. After an introduction (Matrix Lie groups), the first topic is Lie groups focusing on the construction of the associated Lie algebra and on basic notions. The second part is on Lie algebras, where especially semisimple and simple Lie algebras are of interest. The Cartan decomposition is stated and elementary properties are proven. The final part is on the classification of finite-dimensional, simple Lie algebras.

Contents

1 An Intoductory Example

1.1 The Matrix Group SO(3)

Consider the Matrix group

$$\mathrm{SO}(3) = \{A \in \mathrm{Mat}(3, \mathbb{R}) \mid A^T A = \mathbb{1}, \det(A) = 1\} \tag{1.1.1}$$

One can think of SO(3) as being sort of smooth. In fact SO(3) can be viewed as a submanifold of Euclidean space given through the isomorphism $\mathrm{Mat}(n, \mathbb{R}) \simeq \mathbb{R}^{n^2}$ and the defining equations in 1.1.1. Define the **Lie algebra of SO(3)** as

$$\mathfrak{so}(3) = \{\dot\gamma(0) \mid \gamma : (-\varepsilon, \varepsilon) \to \mathrm{SO}(3) \text{ smooth}, \gamma(0) = \mathbb{1}\} \tag{1.1.2}$$

for $\varepsilon > 0$. Note that by the above discussion the Lie algebra can be defined this way.[1]

Proposition 1.1.1.

$$\mathfrak{so}(3) = \{A \in \mathrm{Mat}(3, \mathbb{R}) \mid A^T + A = 0\} \tag{1.1.3}$$

Proof. We have to show two inclusions.
"$\subset$": Let $\varepsilon > 0$. Consider $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{SO}(3)$, such that $\gamma(0) = \mathbb{1}$. In particular

$$\gamma(t)^T \gamma(t) = \mathbb{1} \quad \forall t \in (-\varepsilon, \varepsilon) \tag{1.1.4}$$

By differentiation

$$\begin{aligned} \dot\gamma(t)^T \gamma(t) + \gamma(t)^T \dot\gamma(t) = 0 \\ \overset{t=0}{\Rightarrow} \dot\gamma(0)^T + \dot\gamma(0) = 0 \end{aligned} \tag{1.1.5}$$

"$\supset$" Let $A \in \mathrm{Mat}(3, \mathbb{R})$ such that $A^T + A = 0$. In particular $\mathrm{Tr}(A) = 0$. Define using the Matrix exponential

$$\begin{aligned} \gamma : \mathbb{R} &\to \mathrm{Mat}(3, \mathbb{R}) \\ t &\mapsto \exp(At) \end{aligned} \tag{1.1.6}$$

Due to

$$\det(\gamma(t)) = \exp(t \,\mathrm{Tr}(A)) = 1 \tag{1.1.7}$$

and

$$\gamma(t)^T \gamma(t) = \exp(-At)\exp(At) = \mathbb{1} \tag{1.1.8}$$

γ maps into SO(3). Also note that $\gamma(0) = \mathbb{1}$ and $\dot\gamma(0) = A$. In other words: there is a curve in SO(3) satisfying the required properties and having A as a velocity vector at time 0. This concludes the second inclusion and the proof. □

[1] The Lie algebra of SO(3) is thus simply the tangent space of SO(3) at the identity, characterized through short paths, see [7], page 14.

The Lie algebra can be thought of "describing" the group SO(3) entirely. The following claim expresses this.

Claim 1.1.2. *The map*

$$\exp : \mathfrak{so}(3) \to \mathrm{SO}(3) \tag{1.1.9}$$

is surjective.

Proof. Use normal forms for orthogonal matrices. □

This discussion extends to a larger class of Matrix groups.

1.2 Other Matrix Groups

Let $n \in \mathbb{N}$ and

$$\begin{aligned}
\mathrm{GL}(n,\mathbb{R}) &= \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid \det(A) \neq 0\} \\
\mathrm{GL}(n,\mathbb{C}) &= \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid \det(A) \neq 0\} \\
\mathrm{O}(n) &= \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid A^T A = \mathbb{1}\} \\
\mathrm{U}(n) &= \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid A^\dagger A = \mathbb{1}\} \\
\mathrm{SO}(n) &= \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid A^T A = \mathbb{1}, \det(A) = 1\} \\
\mathrm{SU}(n) &= \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid A^\dagger A = \mathbb{1}, \det(A) = 1\} \\
\mathrm{SL}(n,\mathbb{R}) &= \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid \det(A) = 1\} \\
\mathrm{SL}(n,\mathbb{C}) &= \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid \det(A) = 1\}
\end{aligned} \tag{1.2.1}$$

As before, these groups can be considered sort of smooth viewing them as subsets of Euclidean space and endowing them with the natural smooth structure. The corresponding Lie algebras $\mathfrak{gl}(n,\mathbb{R}), \mathfrak{gl}(n,\mathbb{C}), \mathfrak{o}(n), \ldots$ can be defined as in the case of SO(3). It can be shown that

$$\begin{aligned}
\mathfrak{gl}(n,\mathbb{R}) &= \mathrm{Mat}(n,\mathbb{R}) \\
\mathfrak{gl}(n,\mathbb{C}) &= \mathrm{Mat}(n,\mathbb{C}) \\
\mathfrak{o}(n) &= \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid A^T + A = 0\} \\
\mathfrak{u}(n) &= \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid A^\dagger + A = 0\} \\
\mathfrak{so}(n) &= \mathfrak{o}(n) \\
\mathfrak{su}(n) &= \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid A^\dagger + A = 0, \mathrm{Tr}(A) = 0\} \\
\mathfrak{sl}(n,\mathbb{R}) &= \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid \mathrm{Tr}(A) = 0\} \\
\mathfrak{sl}(n,\mathbb{C}) &= \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid \mathrm{Tr}(A) = 0\}
\end{aligned} \tag{1.2.2}$$

Notice that the above sets are all real (!) vector spaces. We define the **commutator** on $\mathrm{Mat}(n,\mathbb{R})$ as

$$\begin{aligned}[\cdot,\cdot] : \mathrm{Mat}(n,\mathbb{R}) \times \mathrm{Mat}(n,\mathbb{R}) &\to \mathrm{Mat}(n,\mathbb{R}) \\ (A,B) &\mapsto [A,B] = AB - BA\end{aligned} \tag{1.2.3}$$

and on $\mathrm{Mat}(n,\mathbb{C})$ analogously. The commutator is bilinear and antisymmetric. We now claim that the commutator restricted to each of the above subspace of $\mathrm{Mat}(n,\mathbb{R})$ or $\mathrm{Mat}(n,\mathbb{C})$ is a map to the subspace itself. Compute for $A, B \in \mathrm{Mat}(n,\mathbb{R}/\mathbb{C})$

$$\mathrm{Tr}([A,B]) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(AB) = 0 \tag{1.2.4}$$

This shows the claim for $\mathfrak{sl}(n,\mathbb{R}/\mathbb{C})$. Let $A, B \in \mathfrak{o}(n)$.

$$[A,B]^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB = -[A,B] \tag{1.2.5}$$

In the complex we can proceed analogously. The above Lie algebras are thus vector spaces equipped with a bilinear, antisymmetric map to themselves. This is exactly how Lie algebras will be defined in the general case in the next chapter. Also note that the exponential map is for all the matrix groups we have considered here a map from the corresponding Lie algebra to the group. Unlike the SO(3) case it is *not* surjective in general e.g. O(n).

2 Lie Groups

2.1 Definition and Examples

The examples we considered in the last chapter are special cases of the following

Definition 2.1.1. A **Lie group** G is a set that has compatible structures of a group and a smooth manifold. Compatible means that the natural maps defined on the group are smooth i.e. the maps $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth.

Example 2.1.1 (The General Linear Group).
Consider $G = \mathrm{GL}(n,\mathbb{R})$. G can be seen as a subset of $\mathbb{R}^{n^2}$. G is in fact an open subset of $\mathbb{R}^{n^2}$ and thus a submanifold (in particular a manifold). Why are inversion and group multiplication smooth? Group multiplication is linear in the components of the matrices and thus smooth. The inverse of any $A \in G$ is a rational function in the entries of A using Cramer's rule (see e.g. [6]).

Example 2.1.2. (Matrix Lie Groups)
A **Matrix Lie group** is by definition a Lie group that is a subgroup of $\mathrm{GL}(n,\mathbb{R})$. It can be shown that all the examples of Matrix groups treated in the previous chapter are Matrix Lie groups (it is enough to show that such a Matrix group as a submanifold of Euclidean space). Other Examples of Matrix Lie group are the symplectic groups $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{C})$ and the upper/lower triangular matrices in $\mathbb{R}$ or $\mathbb{C}$.

Example 2.1.3. (Non-Matrix Lie Groups)
The n-dimensional torus as a subgroup of $\mathbb{C}^{\times n}$ is a Lie group (e.g. S^1). Consider the group

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} \tag{2.1.1}$$

and the normal subgroup

$$H' = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a = c = 0, b \in \mathbb{Z} \right\} \tag{2.1.2}$$

Set $G = H/H'$. G is a Lie group (to show).

2.2 Construction of the Lie-Bracket

Let G a Lie group. The goal of this section is to construct a bilinear map $T_eG \times T_eG \to T_eG$ that fulfills the same properties as the commutator we have already seen. Note that T_eG is a vector space that exists for any Lie group. Being equipped with the structure of a group, we can consider the action of G onto itself by conjugation.

$$\begin{aligned} \Psi : G &\to Aut(G) \\ g &\mapsto \psi_g \end{aligned} \tag{2.2.1}$$

where

$$\psi_g(h) = ghg^{-1} \quad \forall h \in G \tag{2.2.2}$$

Note that the neutral element e gets mapped to itself and that ψ_g is a smooth map for any $g \in G$ as a composition of smooth maps. Consider now for $g \in G$ the map

$$\mathrm{Ad}(g) = (d\psi_g)_e : T_eG \to T_eG \tag{2.2.3}$$

Thus

$$\mathrm{Ad} : G \to \mathrm{Aut}(T_eG) \tag{2.2.4}$$

Taking the differential map of Ad at the unity we get a map in the tangent spaces[2]

$$\mathrm{ad} : T_eG \to \mathrm{End}(T_eG) = T_e(\mathrm{Aut}(T_eG)) \tag{2.2.5}$$

This implies a bilinear map $T_eG \times T_eG \to T_eG$ called the **Lie bracket** by

$$[X, Y] := \mathrm{ad}(X)(Y) \tag{2.2.6}$$

[2]Note that if we identify $T_eG \simeq \mathbb{R}^n$ for some n, we can view $\mathrm{Aut}(T_eG)$ as $\mathrm{GL}(n, \mathbb{R})$. The computation done in the first chapter then shows that the tangent space at the identity of the Lie group $\mathrm{Aut}(T_eG)$ is $\mathrm{End}(T_eG) \simeq \mathrm{Mat}(n, \mathbb{R})$.

Example 2.2.1 (Construction for abelian Lie groups).
If G is abelian, $\psi_g(h) = h \ \forall g, h \in G$. Thus $(d\psi_g)_e = \mathbb{1}_{T_eG}$ and Ad is a constant map. It follows that ad is identically 0 and hence $[X, Y] = 0$ for all $X, Y \in T_eG$.

Example 2.2.2 (Construction for $\mathrm{GL}(n, \mathbb{R})$).
The construction for $\mathrm{GL}(n, \mathbb{R})$ yields the usual commutator (a good motivation for the above procedure). A proof of this fact can be found in [1]. By restriction we also obtain the commutator for any Matrix Lie group being a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

2.3 Associated Lie algebra

We now claim that the bilinear map constructed above fulfills all the required properties (without proving it, see [1] for a sketch)

Proposition 2.3.1. *(On the Lie-Bracket) Let G a Lie group. The Lie-bracket* $[\cdot, \cdot] : T_eG \times T_eG \to T_eG$ *is a bilinear map satisfying*

- *The Lie-bracket is **antisymmetric**. For all* $X, Y \in T_eG$*:*

$$[X, Y] = -[Y, X] \tag{2.3.1}$$

- *The **Jacobi-identity** holds. For all* $X, Y, Z \in T_eG$*:*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \tag{2.3.2}$$

Definition 2.3.1. Let G a Lie group. The **Lie algebra associated to the Lie group** G is given by T_eG together with the Lie-bracket on T_eG. We write $\mathfrak{g}$.

Generally, we can define

Definition 2.3.2. Let $\mathbb{K}$ a field. A $\mathbb{K}$-vector space $\mathfrak{g}$ together with a bilinear, antisymmetric map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which satisfies the Jacobi-identity, is called a **Lie algebra** over $\mathbb{K}$.

Usually we will be considering $\mathbb{K} = \mathbb{R}, \mathbb{C}$. It will become clear though later on that the analysis of real Lie algebra is a lot harder than the analysis of complex Lie algebras ($\mathbb{C}$ is algebraically closed). Another remark to be made at this point: Note that by fixing the Lie bracket on elements of the basis of the Lie algebra (usually we call these generators), the Lie bracket is defined everywhere by bilinearity.

2.3.1 Examples of complex Lie algebras

Apart from the real Matrix Lie algebras considered in the first chapter, there are well-known examples of complex Matrix Lie algebras [3]. Notice first that $\mathfrak{gl}_n\mathbb{C} =$

[3] If not stated otherwise, we will always work with the usual commutator on Matrix Lie algebras

$\mathfrak{gl}(n,\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$ and $\mathfrak{sl}_n\mathbb{C}$ are also complex Lie algebras. $\mathfrak{su}(n)$ for example is not a complex, but a real Lie algebra. Define

$$\begin{aligned} \mathfrak{sp}_{2n}\mathbb{C} &= \{A \in \mathrm{Mat}(2n,\mathbb{C}) \mid M_1 A + A^T M_1 = 0\} \\ \mathfrak{so}_{2n}\mathbb{C} &= \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid M_2 A + A^T M_2 = 0\} \\ \mathfrak{so}_{2n+1}\mathbb{C} &= \{A \in \mathrm{Mat}(2n+1,\mathbb{C}) \mid M_3 A + A^T M_3 = 0\} \end{aligned} \tag{2.3.3}$$

where

$$M_1 = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}, \; M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathbb{1}_n \\ 0 & \mathbb{1}_n & 0 \end{pmatrix} \tag{2.3.4}$$

It is easy to check that the above sets are complex Lie algebras. The four different complex Lie algebras $\mathfrak{sl}_n\mathbb{C}, \mathfrak{sp}_{2n}\mathbb{C}, \mathfrak{so}_{2n}\mathbb{C}, \mathfrak{so}_{2n+1}\mathbb{C}$ are called **classical Lie algebras**. It is on first sight not so obvious, why for example $\mathfrak{so}_{2n}\mathbb{C}$ is seemingly brought into connection with the group of rotations. The Lie algebra is *not* associated to $\mathrm{SO}(2n)$, but to the complex Lie group

$$\mathrm{SO}_{2n}\,\mathbb{C} = \{A \in \mathrm{Mat}(2n,\mathbb{C}) \mid M_2 = A^T M_2 A\} \tag{2.3.5}$$

2.4 Representations of Lie Groups and Lie Algebras

Definition 2.4.1. Let G, H Lie groups and $\mathfrak{g}, \mathfrak{h}$ respectively the Lie algebras associated to them.

- A **Lie group homomorphism** $\rho : G \to H$ is a smooth map such that $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$.
- A **Lie algebra homomorphism** $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a linear map, such that $\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$ for all $X, Y \in \mathfrak{g}$.

In other words: a Lie group homomorphism is a smooth group homomorphism and a Lie algebra homomorphism is a vector space homomorphism preserving the structure of the Lie bracket. It can be shown, that the differential of a Lie group homomorphism at the identity is a Lie algebra homomorphism in the corresponding Lie algebras.

Definition 2.4.2. Let V a vector space. A **representation of a Lie group** G is a Lie group homomorphism mapping to GL(V). A **representation of a Lie algebra** $\mathfrak{g}$ is a Lie algebra homomorphism mapping to $\mathfrak{gl}(V) = \mathrm{End}(V)$ [4][5].

[4] The vector space has to be over the same field as $\mathfrak{g}$

[5] In the context of Lie algebras one usually calls a representation V of $\mathfrak{g}$ a $\mathfrak{g}$-module. See for example [2], page 66.

A natural thing to ask is whether representations of the Lie group can be obtained from the representations of the associated Lie algebra. The answer is no in general. There is a special case, where this works though.

Proposition 2.4.1. *Let G,H Lie groups and $\mathfrak{g}, \mathfrak{h}$ its Lie algebras. If G is simply connected, the Lie group homomorphisms from G to H are in one-to-one correspondence to the Lie algebra homomorphisms from $\mathfrak{g}$ to $\mathfrak{h}$.*

Proof. See [1], pages 118/119. □

Also, it would be nice to know to which extent the Lie algebra describes the Lie group (compare with discussion in the first chapter). This is the content of the following proposition:

Proposition 2.4.2. *Let G a Lie group and $\mathfrak{g}$ its Lie algebra. There is a smooth map* $\exp : \mathfrak{g} \to G$ *mapping 0 to the identity and mapping surjectively onto some neighbourhood of the identity. Any neighbourhood of the identity generates the whole Lie group, if G is connected. If G is compact and connected,* exp *is surjective onto G.*

The construction of the exponential map exp is a standard construction done in any course in differential geometry. The statement of the above proposition can be strengthened, consider [1] for example.

3 Lie Algebras

In this chapter we will develop further understanding of Lie algebras, in particular of semisimple Lie algebras, and state important results connected to the introduced notions. The main goal is to classify so-called simple Lie algebras. To start with, we shall state and explain a few elementary definitions.

3.1 Basic Notions

Let $\mathfrak{g}$ a complex Lie algebra[6].

Definition 3.1.1. A **Lie subalgebra** $\mathfrak{h}$ is a subspace of $\mathfrak{g}$, such that it is closed under the Lie bracket i.e.

$$\forall X, Y \in \mathfrak{h} : [X, Y] \in \mathfrak{h} \tag{3.1.1}$$

[6] The definitions in this subchapter also apply to Lie algebras over other fields

Immediate examples are $\mathfrak{g}$ itself and the trivial vector space $\{0\}$. Let $X \in \mathfrak{g}$. The set $\mathfrak{i}_X := \mathrm{span}_{\mathbb{C}}\{X\}$ is a subalgebra, since $[X, X] = 0$ by antisymmetry. Let now $\mathfrak{h}_1, \mathfrak{h}_2$ two subalgebras of $\mathfrak{g}$ and define

$$[\mathfrak{h}_1, \mathfrak{h}_2] = \mathrm{span}_{\mathbb{C}}\{[X, Y]\}_{X,Y\in\mathfrak{g}} \tag{3.1.2}$$

called the *commutator* of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ in what follows. $[\mathfrak{h}_1, \mathfrak{h}_2]$ is again a vector space. It contains all the elements of the form $[X, Y]$ for $X \in \mathfrak{h}_1, Y \in \mathfrak{h}_2$

Note that in general the subspace $[\mathfrak{h}_1, \mathfrak{h}_2]$ is not a Lie subalgebra.

Definition 3.1.2. A Lie algebra is **abelian** iff $[\mathfrak{g}, \mathfrak{g}] = \{0\}$.

In other words: In an abelian Lie algebra, $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$, we say that X, Y *commute*. Any subspace of an abelian Lie algebra is a Lie subalgebra. It is also possible to define

Definition 3.1.3. The (exterior) **direct sum** of two complex Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ with Lie brackets $[\cdot,\cdot]_1, [\cdot,\cdot]_2$ respectively is the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ together with the Lie bracket $[X_1\oplus, X_2, Y_1\oplus, Y_2] = [X_1, X_2]_1 \oplus [Y_1, Y_2]_2$.

The definitions that follow are best motivated by recalling concepts from basic group theory. Let G a group and N a subgroup. We call N a normal subgroup, if for all $g \in G$: $gN = Ng$. Note that the definiton does not require N to be abelian. Of course any subgroup constisting of elements that commute with every other element of G (i.e. a subset of the center) is a normal subgroup. A group is called simple, if the only normal divisors of G are $\{e\}$ and G, where e is the neutral element.[7]

Definition 3.1.4. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is an **ideal** if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. A Lie algebra $\mathfrak{g}$ is **simple**, if it is not abelian and does not contain any ideals other than $\{0\}, \mathfrak{g}$ [8].

Definition 3.1.5. A Lie algebra $\mathfrak{g}$ is **semisimple**, if it does not contain any abelian ideal other than $\{0\}$ (trivial ideal).

One ought to remark that this is a non-standard definition. For the usual approach please consider the appendix. Every semisimple Lie algebra can be written as direct sum of simple Lie algebras.

Example 3.1.1 (Center of a Lie algebra)**.**
Define

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \ \forall Y \in \mathfrak{g}\} \tag{3.1.3}$$

the **center** of $\mathfrak{g}$. $Z(\mathfrak{g})$ is an abelian ideal. Thus if $\mathfrak{g}$ is semisimple, the center has to be trivial.

[7] Even better is to recall ring theory. Let R a commutative ring. An additive subgroup I of R is an ideal iff for all $a \in R$ and $x \in I$ we have $ax \in I$.

[8] The condition that $\mathfrak{g}$ is not abelian can be replaced by $\dim(\mathfrak{g}) > 1$.

Example 3.1.2 (Matrix Lie algebras)**.** Let $\mathfrak{g} = \mathfrak{gl}_n\mathbb{C}$. Recall that $\mathrm{Tr}([X, Y]) = 0$ for all $X, Y \in \mathfrak{gl}_n\mathbb{C}$ i.e. $[\mathfrak{gl}_n\mathbb{C}, \mathfrak{gl}_n\mathbb{C}] \subset \mathfrak{sl}_n\mathbb{C}$. In other words: $\mathfrak{sl}_n\mathbb{C}$ is an ideal (non-abelian). The vector space spanned by the diagonal matrices is an example of an abelian subalgebra that is not an ideal.

In what follows we will be focussing on semisimple Lie algebras. This is motivated by the fact that representations of a semisimple Lie algebra are completely reducible.[9]

Definition 3.1.6. The **adjoint representation** is the map

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}) \\ X &\mapsto \mathrm{ad}_X \end{aligned} \tag{3.1.4}$$

where

$$\mathrm{ad}_X(Y) = [X, Y] \quad \forall Y \in \mathfrak{g} \tag{3.1.5}$$

Lemma 3.1.1. *The adjoint map* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *defines a representation of* $\mathfrak{g}$ *onto itself.*

Proof. The adjoint map is clearly linear by bilinearity of the Lie bracket. Let $X, Y, Z \in \mathfrak{g}$. Then

$$\mathrm{ad}_{[X,Y]} Z = [[X, Y], Z] = [X, [Y, Z]] + [Y, [X, Z]] = [\mathrm{ad}_X, \mathrm{ad}_Y]Z \tag{3.1.6}$$

by the Jacobi identity. Thus $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$ □

The adjoint representation of a semisimple Lie algebra is faithful (injective). If it were not injective, there would be $X, Y \in \mathfrak{g}$, such that

$$[X, Z] = \mathrm{ad}_X(Z) = \mathrm{ad}_Y(Z) = [Y, Z] \quad \forall Z \in \mathfrak{g} \tag{3.1.7}$$

But this implies that $X - Y \in Z(\mathfrak{g})$ and since by example 3.1.1 the center is trivial, $X = Y$.

3.2 Representation Theory of $\mathfrak{sl}_2\mathbb{C}$

In this subsection we will analyse the structure of $\mathfrak{sl}_2\mathbb{C}$. The theory of $\mathfrak{sl}_2\mathbb{C}$ is a very strong motivation of the later definition of a Cartan subalgebra and is an essential tool in later proofs. Recall:

$$\mathfrak{sl}_2\mathbb{C} = \{X \in \mathrm{Mat}(2, \mathbb{C}) \mid \mathrm{Tr}(X) = 0\} \tag{3.2.1}$$

[9] Let V a representation of $\mathfrak{g}$ and W an invariant subspace. The representation is completely reducible if there is a completely invariant complementary subspace to W, say W', such that $W \oplus W' = V$.

We choose the following basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.2.2}$$

We can easily compute that

$$[H, X] = 2X, \; [H, Y] = -2Y, \; [X, Y] = H \tag{3.2.3}$$

This means that ad_H acts "diagonally". In the basis $\{H, X, Y\}$ we can write

$$\mathrm{ad}_H = \mathrm{diag}(0, 2, -2) \tag{3.2.4}$$

It is now easy to show that $\mathfrak{sl}_2\mathbb{C}$ is simple. Suppose $\mathfrak{I}$ is a npn-trivial ideal in $\mathfrak{sl}_2\mathbb{C}$ and let $Z = a_1H + a_2X + a_3Y \in \mathfrak{I}$. Then the following brackets are contained in $\mathfrak{I}$:

$$\begin{aligned} [X, Z] &= -2a_1X + a_3H \\ [Y, Z] &= 2a_1X - a_2H \end{aligned} \tag{3.2.5}$$

Hence

$$[X + Y, Z] = (a_3 - a_2)H \in \mathfrak{I} \tag{3.2.6}$$

Thus $H \in \mathfrak{I}$ or $a_3 = a_2$. In the first case we are done by equation 3.2.3 (since $\mathfrak{I}$ is an ideal by assumption. Assume now the second case. Then $X - Y \in \mathfrak{I}$ by eq. 3.2.5 and $[X - Y, Y] = H \in \mathfrak{I}$, if $a_2 \neq 0$. If $a_2 = 0$, $a_1H \in \mathfrak{I}$ and $H \in \mathfrak{I}$, since $Z \neq 0$ and we are done. This procedure can also be applied to the more general case of $\mathfrak{sl}_n\mathbb{C}$

Generally, given a semisimple Lie algebra $\mathfrak{g}$, our goal will be to find a basis of $\mathfrak{g}$, such that some elements of the basis act diagonally just as the H above does. Let V a (finite-dimensional) representation of $\mathfrak{sl}_2\mathbb{C}$. Applying Theorem 5.2.2 to the adjoint representation, we see that the action of H onto V is diagonalizable i.e. we can write

$$V = \bigoplus_{\alpha} V_\alpha \tag{3.2.7}$$

where the V_α's are eigenspaces of the action of H. Now let $v \in V_\alpha$ i.e. $H(v) = \alpha v$. We claim that $X(v) \in V_{\alpha+2}$. Indeed

$$H(X(v)) = X(H(v)) + [H, X](v) = \alpha X(v) + 2X(v) = (\alpha + 2)X(v) \tag{3.2.8}$$

Thus we may think of X as a map $X : V_\alpha \to V_{\alpha+2}$. Analogously $Y : V_\alpha \to V_{\alpha-2}$. Now suppose that V is an irreducible representation. Then there is a number $\alpha \in \mathbb{C}$, such that

$$V = \bigoplus_{k\in\mathbb{Z}} V_{\alpha+2k} \tag{3.2.9}$$

Else the above sum would be an invariant subspace. Strings in the above sum must not be broken i.e. if the eigenspaces of α and $\alpha + 2k$ are non-trivial, then so are the eigenspaces of $\alpha, \alpha + 2, ..., \alpha + 2(k - 1), \alpha + 2k$. Since we assume that V is finite-dimensional, the above sum must stop at a certain point. To be precise: There is a number $k \in \mathbb{N}$ and $n := \alpha + 2k \in \mathbb{C}$, such that $V_{n+2} = \{0\}$ and $V_n \neq \{0\}$. Let $0 \neq v \in V_n$.

Lemma 3.2.1. *The vectors* $\{v, Y(v), Y^2(v), ...\}$ *span* V.

Proof. Denote by W the subspace spanned by the vectors $\{v, Y(v), Y^2(v), ...$ Since V is irreducible, it is enough to show that W is invariant (W is non-trivial). By definition W is carried to itself under Y. Under the action of H:

$$H(Y^m(v)) = (n-2m)Y^m(v) \tag{3.2.10}$$

by induction. For the action of X:

$$\begin{aligned} X(Y^m(v)) &= Y(X(Y^{m-1}(v))) + H(Y^{m-1}(v)) \\ &= Y(X(Y^{m-1}(v))) + (n-2(m-1))v \\ &= ... = m(n-m+1)Y^{m-1}(v) \end{aligned} \tag{3.2.11}$$

by induction. This proves the claim. □

We have also proven that the subspaces V_α are one-dimensional. Now now use the fact that there is a lower bound to α. Let m minimal s.t. $Y^m(v) = 0$. Then

$$X(Y^m(v)) = 0 = m(n-m+1)Y^{m-1}(v) \tag{3.2.12}$$

Thus $n = m-1$ is a positive integer. $Y^{m-1}(v) = Y^n(v)$ is by equation 3.2.10 has the eigenvalue -n with respect to the action of H.

To wrap it up: if V is an irreducible representation of $\mathfrak{sl}_2\mathbb{C}$, then V can be decomposed into eigenspace of the action of H. The eigenvalues are integers distributed symmetrically around the origin in $\mathbb{Z}$ and differing by multiples of 2. All the eigenspaces are one-dimensional.

3.3 Cartan Decomposition

Let $\mathfrak{g}$ a complex, semisimple Lie algebra.

Definition 3.3.1. An element $X \in \mathfrak{g}$ is called **ad-diagonalizable** or **semisimple**, if ad_X is diagonalizable.

Similar to the case of $\mathfrak{sl}_2\mathbb{C}$ we will try to find a set of such ad-diagonalizable elements, which is maximal. An explicit computation will be given later on in the case of $\mathfrak{sl}_3\mathbb{C}$. Additionally we require that these elements commute.

Definition 3.3.2. A **Cartan subalgebra** is a subalgebra spanned by a maximal set of linearly independent, commuting, ad-diagonalizable elements.

Equivalently, a Cartan subalgebra is a maximal, abelian subalgebra consisting of simultaneously ad-diagonalizable elements because

$$[\mathrm{ad}_{H_1}, \mathrm{ad}_{H_2}] = \mathrm{ad}_{[H_1,H_2]} = 0 \quad \forall H_1, H_2 \in \mathfrak{h} \tag{3.3.1}$$

First note that the Cartan subalgebra is not unique. It can be shown though that it is unique up to an automorphism. We will denote our choice of a Cartan subalgebra by $\mathfrak{h}$.

Claim 3.3.1. $\mathfrak{h}$ *is non-trivial.*

Proof. The proof uses two theorems mentioned in the appendix. Suppose that $\mathfrak{h}$ is trivial. Equivalently, $\mathfrak{g}$ does not contain any ad-diagonalizable element. By theorem 5.2.2 every element of $\mathfrak{g}$ has a nilpotent adjoint map i.e. $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ consists of nilpotent elements. By Engel's theorem there is an element $0 \neq X \in \mathfrak{g}$, such that for every $Y \in \mathfrak{g}$ we have $\mathrm{ad}_Y X = [Y, X] = 0$. Thus $X \in Z(\mathfrak{g})$, which is a contradiction to $\mathfrak{g}$ being semisimple. □

The statement of this claim is essential for everything that follows. Note that semisimplicity is needed. We now consider the action of $\mathfrak{h}$ on $\mathfrak{g}$. Since the elements of $\mathfrak{h}$ are simultaneously ad-diagonalizable, we can decompose $\mathfrak{g}$ into eigenspaces of the action of $\mathfrak{h}$ as follows:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha \tag{3.3.2}$$

Now pick $H \in \mathfrak{h}, X \in \mathfrak{g}_\alpha$ and note that

$$\mathrm{ad}_H X = [H, X] = \alpha(H)X \tag{3.3.3}$$

Thus $\alpha \in \mathfrak{h}^*$. The elements of $\mathfrak{h}^*$ appearing in equation 3.3.2 (non-trivially) are called **roots**. The vector spaces $\mathfrak{g}_\alpha$ are called **root spaces**. One usually identifies $\mathfrak{g}_0 = \mathfrak{h}$ (this would have to be proven, one inclusion is trivial), but 0 is not seen as a root. The decomposition 3.3.2 is called **Cartan decomposition**. We will denote the set of roots by R.

Claim 3.3.2. *In the adjoint representation* $\mathfrak{g}_\alpha : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$

Proof. Let $X_\alpha \in \mathfrak{g}_\alpha, X_\beta \in \mathfrak{g}_\beta$ and $H \in \mathfrak{h}$. Then

$$\begin{aligned}[H, [X_\alpha, X_\beta]] &= -[X_\beta, [H, X_\alpha]] - [X_\alpha, [X_\beta, H]] \\ &= -\alpha(H)[X_\beta, X_\alpha] + \beta(H)[X_\alpha, X_\beta] \\ &= (\alpha + \beta)(H)[X_\alpha, X_\beta] \quad \square \end{aligned} \tag{3.3.4}$$

Note the similarity of this fundamental computation to equation 3.2.8. □

Remark 3.3.1. This shows that a semisimple Lie algebra $\mathfrak{g}$ together with a Cartan subalgebra $\mathfrak{h}$ is an $\mathfrak{h}^*$-graded algebra, which means exactly that there is a decomposition of the sort $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha$, such that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

Example 3.3.1 (The semisimple Lie algebra $\mathfrak{sl}_3\mathbb{C}$)**.**
As said before, we will try to imitate the picture we have of $\mathfrak{sl}_2\mathfrak{C}$. We thus choose the following spanning set of $\mathfrak{sl}_3\mathbb{C}$:

$$\begin{gathered} H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ H_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ H_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ X_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ Y_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Y_{13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ Y_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{gathered} \tag{3.3.5}$$

This is easier written, if we denote by E_{ij} the matrix that has a 1 at (i, j) and is zero everywhere else. Then

$$H_{12} = E_{11} - E_{22}, H_{13} = E_{11} - E_{33}, ... \tag{3.3.6}$$

It is easy to see that

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \tag{3.3.7}$$

and

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj} \tag{3.3.8}$$

If D is a diagonal matrix and M some matrix, we get $[D, M]_{ij} = (d_{ii} - d_{jj})m_{ij}$. It is clear that

$$[H_{12}, H_{13}] = [H_{12}, H_{23}] = [H_{13}, H_{23}]) = 0 \tag{3.3.9}$$

(diagonal matrices commute). We set

$$\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\{H_{12}, H_{13}, H_{23}\} = \mathrm{span}_{\mathbb{C}}\{H_{12}, H_{23}\} \tag{3.3.10}$$

By the above, this is an abelian subalgebra. Its elements are ad-diagonalizable, because by eq. 3.3.8 the commutation relations for H_{12} are

$$\begin{aligned} &[H_{12}, X_{12}] = 2X_{12},\ [H_{12}, Y_{12}] = -2Y_{12}, \\ &[H_{12}, X_{13}] = X_{13},\ [H_{12}, Y_{13}] = -Y_{13}, \\ &[H_{12}, X_{23}] = -X_{23},\ [H_{12}, Y_{23}] = Y_{23} \end{aligned} \tag{3.3.11}$$

and analogously for $\{H_{12}, H_{23}\}$ (see e.g. [4]). Now suppose that $\mathfrak{h}$ is not maximal. Then there would be another element $A \in \mathfrak{sl}_3\mathbb{C}$, that commutes with all other elements of $\mathfrak{h}$. But since A certainly commutes with $\mathbb{1}$, it commutes with all diagonal matrices and is thus diagonal. Thus $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}_3\mathbb{C}$. The next thing to note is that the subalgebras spanned by $\{H_{12}, X_{12}, Y_{12}\}$, $\{H_{13}, X_{13}, Y_{13}\}$, $\{H_{23}, X_{23}, Y_{23}\}$ are isomorphic to $\mathfrak{sl}_2\mathbb{C}$. This can be seen through ignoring matrix columns/rows or through computing (for the first case)

$$[H_{12}, X_{12}] = 2X_{12},\ [H_{12}, Y_{12}] = -2Y_{12},\ [X_{12}, Y_{12}] = H_{12} \tag{3.3.12}$$

and analogously in the two other cases. The next obvious question is: what are the root spaces and the roots? We will consider the basis given by $\{H_{12}, H_{23}\}$ for $\mathfrak{h}$; note that $H_{12} + H_{23} = H_{13}$. The root spaces are each spanned by one of the elements $X_{12}, X_{13}, X_{23}, Y_{12}, Y_{13}, Y_{23}$. If α_{12} is the root corresponding to X_{12}, we have $\alpha_{12}(H_{12}) = 2$ and $\alpha_{12}(H_{23}) = -1$. Analogously, $\alpha_{13}(H_{12}) = 1$, $\alpha_{13}(H_{23}) = 1$, $\alpha_{23}(H_{12}) = -1$, $\alpha_{23}(H_{23}) = 2$. In the basis of the dual space $\mathfrak{h}^*$ corresponding to $\{H_{12}, H_{13}\}$ of $\mathfrak{sl}_3\mathbb{C}$, α_{12} looks like $(2, -1)$. Analogously, $\alpha_{13} \leftrightarrow (1, 1)$ and $\alpha_{23} \leftrightarrow (-1, 2)$. Note that the roots corresponding to Y_{12}, Y_{13}, Y_{23} are just $-\alpha_{12}, -\alpha_{13}, -\alpha_{23}$. Also observe that

$$[X_{12}, X_{23}] = X_{13} \tag{3.3.13}$$

and

$$\alpha_{12} + \alpha_{23} = \alpha_{13} \tag{3.3.14}$$

agreeing with claim 3.3.2.

3.4 The Killing Form

We are now equipped with a set of elements of the dual space $\mathfrak{h}$ that describe the Cartan subalgebra (we will see later how). To analyse this set further, we would like to have a scalar product on $\mathfrak{h}^*$. This is not achievable on the whole of $\mathfrak{h}^*$, but it is possible to do that for the real subspace spanned by the roots.

Definition 3.4.1. The **Killing form** is the bilinear, symmetric form

$$\begin{aligned} B : \mathfrak{g} \times \mathfrak{g} &\to \mathbb{C} \\ (X, Y) &\mapsto \mathrm{Tr}(\mathrm{ad}_X \, \mathrm{ad}_Y) \end{aligned} \tag{3.4.1}$$

Claim 3.4.1. *For $X, Y, Z \in \mathfrak{g}$ we have $B(X, [Y, Z]) = B([X, Y], Z)$.*

Proof. Let $X, Y, Z \in \mathfrak{g}$. Then

$$\begin{aligned} B(X, [Y, Z]) &= \mathrm{Tr}(\mathrm{ad}_X \, \mathrm{ad}_{[Y,Z]}) = \mathrm{Tr}(\mathrm{ad}_X [\mathrm{ad}_Y, \mathrm{ad}_Z]) \\ &= \mathrm{Tr}(\mathrm{ad}_X \, \mathrm{ad}_Y \, \mathrm{ad}_Z - \mathrm{ad}_X \, \mathrm{ad}_Z \, \mathrm{ad}_Y) \\ &= \mathrm{Tr}(\mathrm{ad}_X \, \mathrm{ad}_Y \, \mathrm{ad}_Z - \mathrm{ad}_Y \, \mathrm{ad}_X \, \mathrm{ad}_Z) \\ &= \mathrm{Tr}(\mathrm{ad}_{[X,Y]} \, \mathrm{ad}_Z) = B([X, Y], Z) \end{aligned} \tag{3.4.2}$$

□

Proposition 3.4.2. *Let $\mathfrak{g}$ a Lie algebra. Then $\mathfrak{g}$ is semisimple iff its Killing form is nondegenerate.*

Proof. See [1], page 480. □

3.5 Properties of the Cartan Decomposition

As before let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. We will now prove (or state) a few essential results concerning the Cartan decomposition.

Lemma 3.5.1. *The roots span the dual space $\mathfrak{h}^*$.*

Proof. Suppose the contrary. Then there would be an element $H \in \mathfrak{h}$, such that $\alpha(H) = 0 \quad \forall \alpha \in R$. Thus $[H, \mathfrak{g}_\alpha] = 0$. Since by definition of the Cartan subalgebra $[H, \mathfrak{h}] = 0$, we have $[H, \mathfrak{g}] = 0$. Thus the center is non-trivial. Contradiction. □

Proposition 3.5.2. $\alpha \in R \;\Rightarrow -\alpha \in R$

Proof. Let $\alpha \in R$. Note that by claim 3.3.2 and for $\beta \in R$

$$\mathfrak{g}_\beta \circ \mathfrak{g}_\alpha : \mathfrak{g}_\gamma \longrightarrow \mathfrak{g}_{\alpha+\gamma+\beta} \tag{3.5.1}$$

Thus for $X_\alpha \in \mathfrak{g}_\alpha, X_\beta \in \mathfrak{g}_\beta$ we have $B(X_\alpha, X_\beta) = 0$ if $\beta \neq -\alpha$. Note that in the above $\beta = 0$ is also allowed. But since the Killing form is nondegenerate, $\mathfrak{g}_{-\alpha}$ cannot be trivial. □

Proposition 3.5.3. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$

Proof. By the proof above we can choose $X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}$, such that $B(X_\alpha, Y_\alpha) \neq 0$. Let $H_\alpha = [X_\alpha, Y_\alpha]$. We claim that $H_\alpha \neq 0$. Let $H \in \mathfrak{h}$, such that $\alpha(H) \neq 0$ (possible by Lemma 3.5.1)

$$\begin{aligned} B(H_\alpha, H) &= B([X_\alpha, Y_\alpha], H) = B(X_\alpha, [Y_\alpha, H]) \\ &= \alpha(H) B(X_\alpha, Y_\alpha) \end{aligned} \tag{3.5.2}$$

In particular $H_\alpha \neq 0$. □

Proposition 3.5.4. $[\mathfrak{g}_\alpha, [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]] \neq 0$

Proof. As above. □

We now let $X_\alpha, Y_\alpha, H_\alpha$ in the proof above and define

$$\mathfrak{s}_\alpha = \mathrm{span}_\mathbb{C}\{X_\alpha, Y_\alpha, H_\alpha\} \tag{3.5.3}$$

This is by the propositions 3.5.3 and 3.5.4 a Lie subalgebra. It is easy to see that

$$\mathfrak{s}_\alpha \simeq \mathfrak{sl}_2\mathbb{C} \tag{3.5.4}$$

We can thus adjust $X_\alpha, Y_\alpha, H_\alpha$ by scalars to achieve that

$$[H_\alpha, X_\alpha] = 2X_\alpha, [H_\alpha, Y_\alpha] = -2Y_\alpha, [X_\alpha, Y_\alpha] = H_\alpha \tag{3.5.5}$$

Theorem 3.5.5. *The root spaces $\mathfrak{g}_\alpha$ for $\alpha \in R$ are one-dimensional and the only multiples of $\alpha \in R$, which are roots are $\pm\alpha$*

Proof. Consider $\alpha \in R$ and let

$$V = \mathfrak{h} \oplus \bigoplus_{k\in\mathbb{C}} \mathfrak{g}_{k\alpha} \tag{3.5.6}$$

representation of $\mathfrak{s}_\alpha$ as above. Note that

$$\mathfrak{h} = \ker(\alpha) \oplus \mathrm{span}_\mathbb{C}\{H_\alpha\} \tag{3.5.7}$$

$\mathfrak{s}_\alpha$ acts trivially on $\ker(\alpha)$ and irreducibly on itself. By the representation theory of $\mathfrak{sl}_2\mathbb{C}$ we can decompose into irreducible representations of $\mathfrak{s}_\alpha$, being direct sums of one-dimensional eigenspaces, where the string of eigenvalues is integer valued and symmetric in $\mathbb{Z}$. But since $\mathfrak{s}_\alpha$ acts trivially on $\ker(\alpha)$ and irreducibly on itself, all the other root spaces are trivial and $\mathfrak{g}_\alpha = \mathrm{span}_\mathbb{C}\{X_\alpha\}$. $\square$

Thus the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha \tag{3.5.8}$$

is a decomposition into an abelian subalgebra and one-dimensional root spaces (eigenspaces of $\mathfrak{h}$). A basis consisting of elements of $\mathfrak{h}$ and $\mathfrak{g}_\alpha$ for every $\alpha \in R$ is called a **Cartan-Weyl basis**. Due to the above construction we have found a collection of elements of $\mathfrak{h}$ being $\{H_\alpha\}_{\alpha\in R}$ that satisfy $\alpha(H_\alpha) = 2$. Note that by the theorem above

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \tag{3.5.9}$$

Representing $\mathfrak{s}_\alpha$ on the α-string through β being $\bigoplus_{n\in\mathbb{Z}} g_{\beta+n\alpha}$, we see that for all $\beta \in R$ $\beta(H_\alpha)$ is an integer.

Example 3.5.1. ($\mathfrak{sl}_3\mathbb{C}$) The subalgebras s_α are exactly the subalgebras spanned by $\{H_{12}, X_{12}, Y_{12}\}$, $\{H_{13}, X_{13}, Y_{13}\}$, $\{H_{23}, X_{23}, Y_{23}\}$.

Proposition 3.5.6. *The Killing form restricted to the real subspace of $\mathfrak{h}$ spanned by $\{H_\alpha\}_{\alpha\in R}$ is positive definite.*

Proof. Consider first the restriction of the Killing form B to $\mathfrak{h}$ and let $X_\alpha \in \mathfrak{g}_\alpha$. Let $H_1, H_2 \in \mathfrak{h}$. Then

$$\operatorname{ad}_{H_1} \operatorname{ad}_{H_2} X_\alpha = \alpha(H_1)\alpha(H_2) \tag{3.5.10}$$

and of course for all $H \in \mathfrak{h}$

$$\operatorname{ad}_{H_1} \operatorname{ad}_{H_2} H = 0 \tag{3.5.11}$$

Thus

$$B(H_1, H_2) = \operatorname{Tr}(\operatorname{ad}_{H_1} \operatorname{ad}_{H_2}) = \sum_{\alpha\in R} \alpha(H_1)\alpha(H_2) \tag{3.5.12}$$

Therefore B is positive definite on the set $\{H_\alpha\}_{\alpha\in R}$, since integer-valued and hence also on the real span of $\{H_\alpha\}_{\alpha\in R}$. $\square$

Example 3.5.2. ($\mathfrak{sl}_3\mathbb{C}$)

$$\begin{aligned}
B(H_{12}, H_{12}) &= \sum_{\alpha\in R} \alpha(H_{12})\alpha(H_{12}) = 2(2^2 + 1^2 + 1^2) = 12 \\
B(H_{23}, H_{23}) &= 12 \\
B(H_{12}, H_{23}) &= 2(-2 + 1 - 2) = -6 \\
B(H_{13}, H_{13}) &= 12 \\
B(H_{13}, H_{12}) &= B(H_{13}, H_{23}) = 6
\end{aligned} \tag{3.5.13}$$

We would now like to define the Killing form on the real subspace spanned by the roots. Consider first H_α for some root α and let $X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}$, such that $[X_\alpha, Y_\alpha] = H_\alpha$. We have already done the following computation:

$$\begin{aligned}
B(H_\alpha, H) = B([X_\alpha, Y_\alpha], H) &= B(X_\alpha, [Y_\alpha, H]) \\
&= \alpha(H)B(X_\alpha, Y_\alpha)
\end{aligned} \tag{3.5.14}$$

Setting $H = H_\alpha$, we get

$$B(X_\alpha, Y_\alpha) = \frac{1}{2}B(H_\alpha, H_\alpha) \tag{3.5.15}$$

and setting

$$T_\alpha = \frac{2H_\alpha}{B(H_\alpha, H_\alpha)} \tag{3.5.16}$$

we see that

$$B(T_\alpha, H) = \alpha(H) \tag{3.5.17}$$

We have proven the following statement:

Lemma 3.5.7. *The nondegeneracy of the bilinear form on the real subspace spanned by $\{H_\alpha\}_\alpha$ supplies a natural isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ under which*

$$T_\alpha := 2H_\alpha/B(H_\alpha, H_\alpha) \mapsto \alpha \tag{3.5.18}$$

(by linear extension)

The isomorphism is of course $H \mapsto B(H, \cdot)$.

Definition 3.5.1. The **Killing form on $\mathfrak{h}^*$** is given by $B(\alpha, \beta) = B(T_\alpha, T_\beta)$.

This is well defined by linear extension and using lemma 3.5.1. By what has been done earlier, the Killing form on the real subspace spanned by the roots is positive definite, symmetric and bilinear. Thus denote by $\mathbb{E}$ the real subspace of $\mathfrak{h}^*$ spanned by the roots together with the scalar product given by the Killing form. Compute

$$B(\alpha, \alpha) = B(T_\alpha, T_\alpha) = \alpha(T_\alpha) = \frac{2\alpha(H_\alpha)}{B(H_\alpha, H_\alpha)} = \frac{4}{B(H_\alpha, H_\alpha)} \tag{3.5.19}$$

and

$$\beta(H_\alpha) = B(T_\beta, H_\alpha) = \frac{B(H_\alpha, H_\alpha)}{2} B(\beta, \alpha) = \frac{2B(\beta, \alpha)}{B(\alpha, \alpha)} \tag{3.5.20}$$

Recall that this is an integer.

Example 3.5.3 ($\mathfrak{sl}_3\mathbb{C}$).

$$\begin{aligned}
B(\alpha_{12}, \alpha_{12}) &= \frac{4B(H_{12}, H_{12})}{B(H_{12}, H_{12})^2} = \frac{4}{B(H_{12}, H_{12})} = \frac{4}{12} = \frac{1}{3} \\
B(\alpha_{13}, \alpha_{13}) &= B(\alpha_{23}, \alpha_{23}) = \frac{1}{3} \\
B(\alpha_{12}, \alpha_{23}) &= \frac{4B(H_{12}, H_{23})}{B(H_{12}, H_{12})B(H_{23}, H_{23})} = -\frac{1}{6} \\
B(\alpha_{13}, \alpha_{12}) &= B(\alpha_{13}, \alpha_{23}) = \frac{1}{6}
\end{aligned} \tag{3.5.21}$$

3.6 The Weyl Group

Intuitively speaking, the Weyl group of a semisimple Lie algebra $\mathfrak{g}$ is the symmetry group of the roots (or rather of the **root system**).

Definition 3.6.1. The **Weyl group $\mathfrak{W}$** is the group generated by the linear involutions $\{W_\alpha\}_{\alpha \in R}$ given by

$$\begin{aligned}
W_\alpha : \mathfrak{h}^* &\to \mathfrak{h}^* \\
\beta &\mapsto \beta - \beta(H_\alpha)\alpha
\end{aligned} \tag{3.6.1}$$

We have already shown that for $\alpha, \beta \in R$

$$W_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha = \beta - \frac{2B(\beta, \alpha)}{B(\alpha, \alpha)}\alpha \tag{3.6.2}$$

W_α thus corresponds to a reflection in the hyperplane

$$\Omega_\alpha = \{\beta \in \mathfrak{h}^* : B(\beta, \alpha) = 0\} \tag{3.6.3}$$

Proposition 3.6.1. *R is mapped to itself under* $\mathfrak{W}$.

The proof of this is exactly the same as the proof of the fact that $\beta(H_\alpha)$ is an integer. The Weyl group shows many symmetries of the root system. For an outlook please see e.g. [1], Appendix D.4 .

4 Classification of Simple Lie Algebras

4.1 Ordering of the Roots

Let $\mathfrak{g}$ a complex, semisimple Lie algebra. The set of roots R is a finite set in $\mathfrak{h}^*$. We can thus choose a hyperplane in $\mathfrak{h}^*$ that does not contain any element of the lattice spanned by R (except 0). By convention the roots on side of the hyperplane are called positive and on the other negative (ordering of the roots). To be precise:

Definition 4.1.1. Let Λ_R be the lattice spanned by R and let $l : \mathfrak{h}^* \to \mathbb{C}$ a functional irrational with respect to Λ_R. A root α is called **positive** (negative), if $l(\alpha) > 0$ ($l(\alpha < 0)$.

Definition 4.1.2. A positive root is called **simple**, if it cannot be written as a sum of two other positive roots.

Example 4.1.1. ($\mathfrak{sl}_3\mathbb{C}$) Consider the real space $\mathbb{E}$ in the case of $\mathfrak{sl}_3\mathbb{C}$. Note that $\dim(\mathbb{E}) = 2$ (rank). The root system of $\mathfrak{sl}_3\mathbb{C}$ is

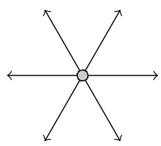


Figure 1: Root system of $\mathfrak{sl}_3\mathbb{C}$

We choose a plane (in the restriction here just a line) splitting the root system.

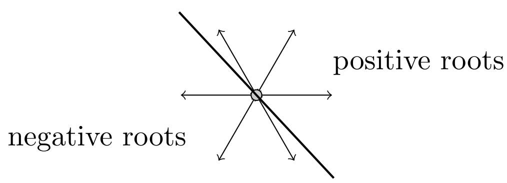


Figure 2: Root system of $\mathfrak{sl}_3\mathbb{C}$; ordering of the roots by the thick line

The simple roots are the positive roots that cannot be written as a sum of two other positive roots (here in red).

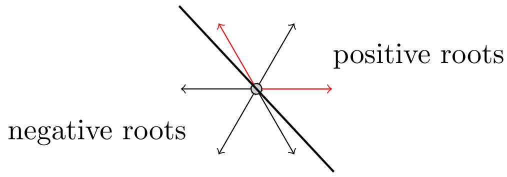


Figure 3: Root system of $\mathfrak{sl}_3\mathbb{C}$; splitting of the space by the thick line, simple roots in red.

4.2 Angles between the Roots

Recall: $\forall \alpha, \beta \in R$

$$n_{\beta\alpha} := \frac{2B(\beta,\alpha)}{B(\alpha,\alpha)} = \beta(H_\alpha) \in \mathbb{Z} \tag{4.2.1}$$

If θ is the angle between α and β, then

$$n_{\beta\alpha} = 2\cos(\theta)\frac{||\beta||}{||\alpha||} \tag{4.2.2}$$

Thus

$$n_{\beta\alpha}n_{\alpha\beta} = 4\cos^2(\theta) \leq 4 \tag{4.2.3}$$

Hence $4\cos^2(\theta)$ is an integer. The allowed angles in $[0,\pi)$ are

$$\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6} \tag{4.2.4}$$

Example 4.2.1. Assume $|n_{\beta\alpha}| \geq |n_{\alpha\beta}|$ and $\theta = \frac{\pi}{6}$ for instance. Then $\cos(\theta) = \frac{\sqrt{3}}{2}$ and $n_{\beta\alpha}n_{\alpha\beta} = 3$. Hence $n_{\beta\alpha} = 3$ and $n_{\alpha\beta} = 1 \Rightarrow \frac{||\beta||}{||\alpha||} = \sqrt{3}$.

Repeating the above procedure, we can write down the following table (assuming $|n_{\beta\alpha}| \geq |n_{\alpha\beta}|$) :

θ	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
$\cos(\theta)$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$
$n_{\beta\alpha}$	3	2	1	0	-1	-2	-3
$n_{\alpha\beta}$	1	1	1	0	-1	-1	-1
$\|\|\beta\|\|/\|\|\alpha\|\|$	$\sqrt{3}$	$\sqrt{2}$	1	*	1	$\sqrt{2}$	$\sqrt{3}$

Table 1: Allowed angles between (positive) roots and associated values

4.3 Examples of Root Systems

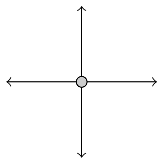
Definition 4.3.1. The **rank** of $\mathfrak{g}$ is $r = \dim_{\mathbb{R}} \mathbb{E} = \dim_{\mathbb{C}} \mathfrak{h}$.

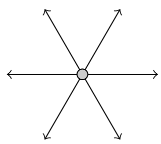
Example 4.3.1 (Rank 1). Recall: Theorem 3.5.5 states that the only multiples of a root α that are roots, are $\pm\alpha$. If $\mathfrak{g}$ has rank 1, it must thus have the root system

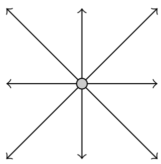
(A_1)

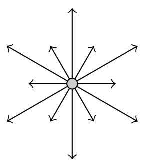
This is precisely the root system of $\mathfrak{sl}_2\mathbb{C}$.

Example 4.3.2 (Rank 2). There are 4 different root system in 2 dimensions.

$(A_1)\times(A_1)$ 

(A_2) 

(B_2) 

(G_2) 

The first root system corresponds to $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C} \simeq \mathfrak{so}_4\mathbb{C}$, the second to $\mathfrak{sl}_3\mathbb{C}$, the third to $\mathfrak{so}_5\mathbb{C} \simeq \mathfrak{sp}_4\mathbb{C}$. The forth is a Lie algebra that has not been treated so far.

4.4 Further Properties of the Root System

Recall: a Lie algebra is simple if it is non-abelian and contains no ideals other than itself and the trivial ideal.

Proposition 4.4.1. *A semisimple Lie algebra is simple iff its root system is **irreducible** i.e. cannot be written as a direct sum of two root systems.*

Proof. See [4], pages 244-246. □

Proposition 4.4.2. *Let $\alpha, \beta \in R$ such that β is not a multiple of α. If the angle between β and α is acute, $\alpha - \beta$ is a root and if it is obtuse, $\alpha + \beta$ is a root.*

Proof. See [4], pages 247-248. □

Corollary 4.4.3. *If α, β are simple, $\alpha - \beta$ and $\beta - \alpha$ are not roots and the angle between α and β is obtuse.*

Proof. Suppose $\alpha - \beta$ were a root. Then either $\alpha - \beta$ or $\beta - \alpha$ would be positive roots. If $\alpha - \beta$ were a positive root, $\alpha = \beta + (\alpha - \beta)$ would be non simple. Else $\beta = \alpha + (\beta - \alpha)$. We thus get a contradiction. By the previous proposition, the second claim follows. □

The allowed angles between two simple roots are hence $\pi/2, 2\pi/3, 3\pi/4, 5\pi/6$. The simple roots describe the whole of $\mathfrak{h}^*$:

Proposition 4.4.4. *The simple roots in R are a basis of $\mathbb{E}$. Every positive root can be uniquely written as a non-negative integral linear combination of simple roots.*

Proof. See [4], pages 251-252. □

4.5 Dynkin Diagrams

A **Dynkin diagram** of a root system is a diagram describing the distribution the simple roots in $\mathbb{E}$. It is built as follows:

- Every simple root is represented by a node $\circ$.
- Two simple roots are connected in the following way
  - not connected, if $\theta = \frac{\pi}{2}$
  - one line, $\theta = \frac{2\pi}{3}$
  - two lines and an arrow pointing from the longer to the shorter root, if $\theta = \frac{3\pi}{4}$.
  - three lines and an arrow pointing from the longer to the shorter root, if $\theta = \frac{5\pi}{6}$.

The fact that two simple roots satisfy one point of the above follows from table 4.2. We illustrate this with the examples of chapter 4.3.

Example 4.5.1 (Rank 1). There is exactly one positive root, which is automatically simple. The Dynkin diagram is thus just one node:

$$(A_1) \qquad \circ$$

Example 4.5.2 (Rank 2). The Dynkin diagrams of the rank 2 root systems are drawn as follows:

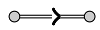
- In the first case $A_1 \times A_1$ the angle between the two simple roots is exactly $\pi/2$. We get

$(A_1 \times A_1)$

- In the case of A_2 the angle between the two simple roots is $2\pi/3$.

(A_2)

- B_2: The two simple roots span an angle of $3\pi/4$, where one root is longer that the other (easy to verify drawing a hyperplane).

(B_2) 

- G_2: As before

(G_2)

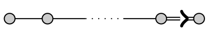
The last three Dynkin diagrams correspond to simple Lie algebras (using proposition 4.4.1).

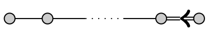
4.6 Classification of Simple Lie Algebras

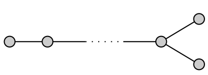
The following theorem is the generalization of the examples in the last chapter. For every rank, there is a finite number of possible Dynkin diagrams a simple Lie algebra can have.

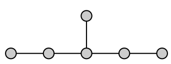
Theorem 4.6.1. *The Dynkin diagrams of irreducible root systems are:*

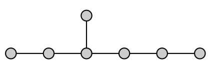
(A_n)

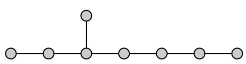
(B_n) 

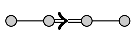
(C_n) 

(D_n) 

(E_6) 

(E_7) 

(E_8) 

(F_4) 

(G_2)

where for D_n $n \geq 3$ is required.

The Dynkin diagrams in left column are the Dynkin diagrams corresponding to the **classical Lie algebras**.

- A_n is the Dynkin diagram of $\mathfrak{sl}_{n+1}\mathbb{C}$.
- $B_n \to \mathfrak{so}_{2n+1}\mathbb{C}$
- $C_n \to \mathfrak{sp}_{2n}\mathbb{C}$
- $D_n \to \mathfrak{so}_{2n}\mathbb{C}$

The five exceptions in the right column are the Dynkin diagrams corresponding to the so-called **exceptional Lie algebras**. The structure of these Lie algebras can be deduced from the Dynkin diagram, see for example [1], chapter 22. A Lie algebra is uniquely determined through its Dynkin diagram. Two Lie algebras having the same Dynkin diagram are isomorphic.

Remark 4.6.1 (On the Proof of Theorem 4.6.1). A complete proof can be found in [1]. I would like to remark two things about the Dynkin diagrams of simple Lie algebras that are proven when proving the theorem.

- The Dynkin diagram contains no loops/cycles and is connected (i.e. it's a tree).
- Any node has at most three lines to it.

Example 4.6.1. Since $(A_1) = (B_1) = (C_1)$, $\mathfrak{sl}_2\mathbb{C} \simeq \mathfrak{so}_3\mathbb{C} \simeq \mathfrak{sp}_2\mathbb{C}$

Example 4.6.2. $(B_2) = (C_2) \Rightarrow \mathfrak{so}_5\mathbb{C} \simeq \mathfrak{sp}_4\mathbb{C}$

Example 4.6.3. $(A_3) = (D_3) \Rightarrow \mathfrak{so}_6\mathbb{C} \simeq \mathfrak{sl}_4\mathbb{C}$

Remark 4.6.2. To rewrite the above theorem such that the different cases are mutually exclusive, one only needs to require for (B_n) that $n \geq 2$ and for (D_n) that $n \geq 4$.

5 Appendix

5.1 On Semisimple Lie Algebras

Let $\mathfrak{g}$ a Lie algebra.

Claim 5.1.1. *If $\mathfrak{h}$ in $\mathfrak{g}$ is an ideal, than $[\mathfrak{g},\mathfrak{h}]$ and $[\mathfrak{h},\mathfrak{h}]$ are ideals.*

Proof. We first have to show that they are Lie subalgebras. By definition $[\mathfrak{g},\mathfrak{h}]$ and $[\mathfrak{h},\mathfrak{h}]$ are vector spaces. Consider $[g_1,h_1],[g_2,h_2]\in[\mathfrak{g},\mathfrak{h}]$. But since $[g_1,h_1]\in\mathfrak{g},[g_2,h_2]\in\mathfrak{h}$, we have $[[g_1,h_1],[g_2,h_2]]\in[\mathfrak{g},\mathfrak{h}]$ and thus $[\mathfrak{g},\mathfrak{h}]$ is a Lie subalgebra. Analogously for $[\mathfrak{h},\mathfrak{h}]$.

Let now $g_1,g_2\in\mathfrak{g},\ h_1,h_2\in\mathfrak{h}$. Then $[g_1,[g_2,h_2]]\in[\mathfrak{g},\mathfrak{h}]$ by definition and

$$[g_1,[h_1,h_2]] = -[h_2,[g_1,h_1]] - [h_1,[h_2,g_1]] \in [\mathfrak{h},\mathfrak{h}] \tag{5.1.1}$$

since $\mathfrak{h}$ is an ideal. □

In particular $[\mathfrak{g},\mathfrak{g}]$ is an ideal, called the **commutator subalgebra**.

Definition 5.1.1. The **lower central series** is defined by

$$\begin{aligned}\mathfrak{D}_1\mathfrak{g} &= [\mathfrak{g},\mathfrak{g}]\\ \mathfrak{D}_k\mathfrak{g} &= [\mathfrak{g},\mathfrak{D}_{k-1}\mathfrak{g}] \quad \forall k\geq 2\end{aligned} \tag{5.1.2}$$

The **derived series** is defined by

$$\begin{aligned}\mathfrak{D}^1\mathfrak{g} &= [\mathfrak{g},\mathfrak{g}]\\ \mathfrak{D}^k\mathfrak{g} &= [\mathfrak{D}^{k-1}\mathfrak{g},\mathfrak{D}^{k-1}\mathfrak{g}] \quad \forall k\geq 2\end{aligned} \tag{5.1.3}$$

By claim 5.1.1 $\mathfrak{D}^k\mathfrak{g}$ and $\mathfrak{D}_k\mathfrak{g}$ are ideals. We hence have $\mathfrak{D}^k\mathfrak{g}\subset\mathfrak{D}^{k-1}\mathfrak{g}$ and $\mathfrak{D}_k\mathfrak{g}\subset\mathfrak{D}_{k-1}\mathfrak{g}$ for all $k\geq 2$. Also $\mathfrak{D}_k\mathfrak{g}\supset\mathfrak{D}^k\mathfrak{g}$. We are only going to use the derived series in what follows, but note the following definition nonetheless.

Definition 5.1.2.

- $\mathfrak{g}$ is **nilpotent** if $\mathfrak{D}_k\mathfrak{g}=0$ for some $k\in\mathbb{N}$.
- $\mathfrak{g}$ is **solvable** if $\mathfrak{D}^k\mathfrak{g}=0$ for some $k\in\mathbb{N}$.

Every nilpotent Lie algebra is solvable.

Definition 5.1.3. $\mathfrak{g}$ is **semisimple** if it does not contain any solvable ideal.

This is the standard definition. If $\mathfrak{g}$ is simple, $\mathfrak{g}=\mathfrak{D}^1\mathfrak{g}=...=\mathfrak{D}^k\mathfrak{g}=...$, thus any simple Lie algebra is semisimple.

Claim 5.1.2. $\mathfrak{g}$ *is semisimple iff it does not contain any non-trivial, abelian ideal.*

Proof. "$\Rightarrow$" Suppose by contradiction that $\mathfrak{i} \subset \mathfrak{g}$ a non-trivial, abelian ideal. But since $[\mathfrak{i}, \mathfrak{i}] = 0$, $\mathfrak{i}$ is solvable and we have a contradiction. "$\Leftarrow$" Suppose by contradiction that $\mathfrak{i} \subset \mathfrak{g}$ is a non-trivial, solvable ideal $\Rightarrow \exists k \in \mathbb{N}$, such that $\mathfrak{D}^k\mathfrak{i} = 0 \Rightarrow \mathfrak{D}^{k-1}\mathfrak{i}$ is an abelian ideal, that is non-trivial, if k is chosen minimal $\rightarrow$ contradiction. $\square$

There is a third characterization of semisimple Lie algebras. We need the following definition first:

Definition 5.1.4. Let $\mathfrak{h} \subset \mathfrak{g}$ an ideal. The **quotient** $\mathfrak{g}/\mathfrak{h}$ is a vector space as a quotient of vector spaces and a Lie algebra together with the Lie bracket $[a+\mathfrak{h}, b+\mathfrak{h}] = [a,b] + \mathfrak{h}$ for $a, b \in \mathfrak{g}$.

Proposition 5.1.3. $\mathfrak{g}$ *is solvable iff there is a sequence of ideals in* $\mathfrak{g}$*, say* $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_2 \supset ... \supset \mathfrak{g}_k = 0$ *for some* $k \in \mathbb{N}$*, such that* $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ *for* $i = 0, 1, ..., k-1$*.*

Proof. "$\Rightarrow$" Obvious, considering the sequence $\mathfrak{D}^i\mathfrak{g}$ and the fact that by $\mathfrak{D}^i\mathfrak{g}/\mathfrak{D}^{i+1}\mathfrak{g}$ is abelian, because $[\mathfrak{D}^i\mathfrak{g}, \mathfrak{D}^i\mathfrak{g}] = \mathfrak{D}^{i+1}\mathfrak{g}$.

"$\Leftarrow$" Note that $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian iff $[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$. It can be shown by induction that $\mathfrak{D}^i\mathfrak{g} \subset \mathfrak{g}_i$. Thus in particular $\mathfrak{D}^k\mathfrak{g} = 0$. $\square$

Having shown that the sum of two solvable ideals is solvable, we can consider the sum of all solvable ideals being again a solvable ideal[10]. The so-obtained unique ideal is called **radical** of $\mathfrak{g}$, we write $\mathrm{Rad}(\mathfrak{g})$. The quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple (Proof by contradiction, using claim 5.1.2).

Proposition 5.1.4. *Any semisimple Lie algebra is a direct sum of simple Lie algebras.*

Proof. See [1], Appendix C. $\square$

5.2 On Representations of Semisimple Lie Algebras

We list here a few results concerning the representation theory of semisimple Lie algebras. Proofs to all the statement can be found in [1], Appendix C.

Proposition 5.2.1. *(Complete reducibility) Let V a representation of a semisimple Lie algebra and W an invariant subspace. Then there exits a subspace W', such that* $W' \cap W = \{0\}$ *and W' is invariant.*

[10] Using Zorn's lemma

Thus finite-dimensional representations of semisimple Lie algebras are completely reducible. Recall from linear algebra that any linear map can be decomposed into a nilpotent and a diagonalizable part.

Theorem 5.2.2 (Preservation of the Jordan decomposition)**.** *Let $\mathfrak{g}$ a semisimple Lie algebra and let $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$. Let $X \in \mathfrak{g}$. Then there is a decomposition $X = X_d + X_n$, such that*

$$\varphi(X) = \varphi(X_d) + \varphi(X_n) \tag{5.2.1}$$

where $\varphi(X_d)$ is the diagonalizable part of $\varphi(X)$ and $\varphi(X_n)$ is the nilpotent part. The decomposition of X is independent of the representation.

The following theorem describes a characteristic property of a Lie subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent elements.

Theorem 5.2.3 (Engel's Theorem)**.** *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie subalgebra, such that $\forall X \in \mathfrak{g}$ X is nilpotent. Then there exists a non-zero vector $v \in V$, such that $X(v) = 0 \quad \forall X \in \mathfrak{g}$.*

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