ETH Zurich

Theoretical Physics Proseminar

March 25, 2013

Boundary Conformal Field Theory

Author:
Zhuli He

Supervisor:
Dr. Adel Benlagra

Abstract

The aim of this report is to give a short introduction to boundary conformal field theory. Start from the free boson example, the boundary conditions that preserve the conformal symmetry will be built. The boundary states that satisfy the boundary condition will be constructed. Some partition function will be calculated in detail, and the important loop-channel – tree-channel equivalence will be shown. Cardy condition will be introduced which put strong constrains to the possible boundary condition that could exist. Finally,the ground state degeneracy, or the g-function will be introduced.

Contents

1 Introduction

In this report I will demonstrate how the presence of boundary change the picture of conformal field theory.
In part I, starting with simple example of free boson, the conditions of boundary that still preserve conformal symmetry will be shown. Then this idea will be generalized so that the boundary states that satisfied boundary conditions can be constructed. Then some cylinder partition functions will be calculated in analogy with the calculation of torus partition function. During the calculation of partition functions, the loop-channel – tree channel equivalence which is a very essential property of boundary conformal field theory will be shown.
And then, in part II, I will generalize the idea obtained in part I to Rational Conformal Field Theory. In RCFT, the Ishibashi states satisfy the gluing condition, however the Ishibashi states are not real boundary states. The real boundary states are linear combination of the Ishibashi states. And then the Cardy condition will be introduced, which put strong constrains to the possible boundary states that could exist.
Finally, in part III, I will show that how the presence of boundary change the value of partition function in the way of ground state degeneracy, or g-function. And show how its related to the Cardy condition.

2 Boundary Condition and Boundary States

2.1 Conformal Invariance and Boundary Condition

We start by the discussing of boundary conformal field theory of free boson. In order to demonstrate how the appearance change the picture of conformal field theory, we look at the two-dimensional action for free boson $X(\sigma, \tau)$. It is given by:

$$S = \frac{1}{4\pi} \int d\sigma d\tau \left((\partial_\sigma X)^2 + (\partial_\tau X)^2 \right) \tag{1}$$

2.1.1 Least Action Principal in the Presence of Boundaries

Note now we consider a two-dimensional spacetime with boundaries such that $\tau \in (-\infty, +\infty)$ denotes the two-dimensional time coordinate and $\sigma \in [0, \pi]$ is the coordinate parametrising the distance between the boundaries. The variation of

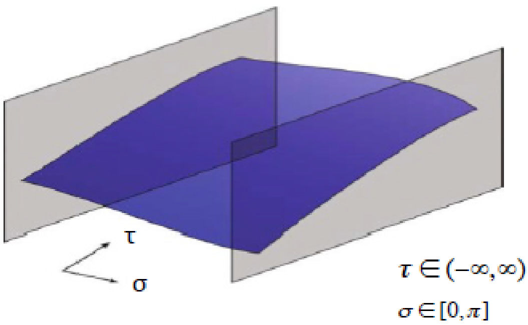

Figure 1: Two dimensional surface with boundaries

the action (1) could be easily obtained, but this time we take the boundary terms into account. More specifically, we compute the variation as follows:

$$\begin{aligned} \delta_{\mathrm{X}} S &= \frac{1}{\pi} \int d\sigma d\tau \left((\partial_\sigma X)(\partial_\sigma \delta X) + (\partial_\tau X)(\partial_\tau \delta X) \right) & (2) \\ &= \frac{1}{\pi} \int d\sigma d\tau \left(-\left(\partial_\sigma^2 + \partial_\tau^2\right) X \cdot \delta X + \partial_\tau \left(\partial_\tau X \cdot \delta X\right) + \partial_\sigma \left(\partial_\sigma X \cdot \delta X\right) \right) & (3) \end{aligned}$$

The equation of motion is obtained by requiring this expression to vanish for all variations δX. The vanishing of the first term in (3) leads to $(\partial_\sigma^2 + \partial_\tau^2) X = 0$

which is the familiar Klein-Gordon equation. The remaining two terms, which interpreted as the boundary term can be written as follows:

$$\begin{aligned} & \frac{1}{\pi}\int d\sigma d\tau\left(\partial_\tau\left(\partial_\tau X\cdot\delta X\right)+\partial_\sigma\left(\partial_\sigma X\cdot\delta X\right)\right) && (4)\\ = & \frac{1}{\pi}\int d\sigma d\tau\vec{\nabla}\cdot\left(\vec{\nabla}X\delta X\right) && (5)\\ = & \frac{1}{\pi}\int_B dl_B(\vec{\nabla}X\cdot\vec{n})\delta X && (6)\end{aligned}$$

where $\vec{\nabla}=(\partial_\tau,\partial_\sigma)^T$ and Stokes theorem was used to rewrite the integral $\int d\sigma d\tau$ as an integral over the boundary B. Furthermore, dl_B denotes the line element along the boundary and $\vec{n}$ is a unit vector normal to B. In our case, the boundary is specified by $\sigma=0$ and $\sigma=\pi$ so that $\vec{n}=(0,1)^T$ as well as $dl_B=d\tau$. The vanishing of the last two terms in (3) can therefore be expressed as:

$$0=\frac{1}{\pi}\int d\tau\left(\partial_\sigma X\right)\delta X\big|_{\sigma=0}^{\sigma=\pi} \tag{7}$$

This equation allows for two different solutions and hence for two different boundary conditions. The first possibility is a Neumann boundary condition given by $\partial_\sigma X\big|_{\sigma=0,\pi}=0$. The second possibility is a Dirichlet condition $\delta X\big|_{\sigma=0,\pi}=0$ which implies $\partial_\tau X\big|_{\sigma=0,\pi}=0$. In summary, the two different boundary conditions for the free boson theory read as follows:

$$\boxed{\begin{array}{rll} \partial_\sigma X\big|_{\sigma=0,\pi} & =0 & \textbf{Neumann Condition}\\ \delta X\big|_{\sigma=0,\pi}=\partial_\tau X\big|_{\sigma=0,\pi} & =0 & \textbf{Dirichlet Condition}\end{array}} \tag{8}$$

2.1.2 Boundary Conditions for the Laurent Modes

Above, we have considered the BCFT in terms of the real variables (τ,σ) which was convenient in order to arrive at Eq. (8). However, it is much more convenience to use a description in terms of complex variables for more advanced studies. Thus we now consider a mapping from the infinite strip described by the real variables (τ,σ) to the complex upper half-plane H^+ is achieved by $\mathrm{z}=\exp(\tau+i\sigma)$. The mapping is illustrated in Fig. 2, and the boundaries $\sigma=0,\pi$ is mapped to the real axis $z=\bar{z}$

With this map, we can express the boundary conditions (8) for the field $X(\sigma,\tau)$ in terms of the corresponding Laurent modes. Recalling that $j(z)=i\partial X(z,\bar{z})$, we

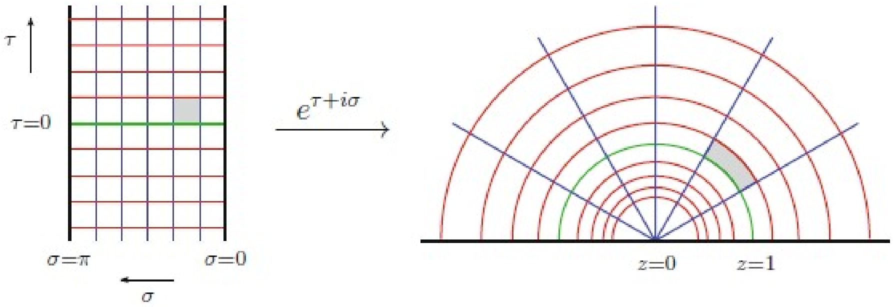

Figure 2: Mapping from the infinite strip to the complex upper half-plane

find

$$\partial_\sigma X = i(\partial - \bar\partial)X = j(z) - \bar j(\bar z) = \sum_{n\in Z}\left(j_n z^{-n-1} - \bar j_n \bar z^{-n-1}\right) \tag{9}$$

$$i\cdot\partial_\tau X = i(\partial + \bar\partial)X = j(z) + \bar j(\bar z) = \sum_{n\in Z}\left(j_n z^{-n-1} + \bar j_n \bar z^{-n-1}\right) \tag{10}$$

where we used the explicit expressions for ∂ and $\bar\partial$. For transforming the right-hand side of these equations as $z \mapsto e^w$ with $w = \sigma + i\tau$, we employ $j(z) = 1$ as a primary field of conformal dimension $h = 1$, In particular, we have $j(z)$=∂z/$\partial w^1 j(w) = zj(w)$ leading to

$$\partial_\sigma X = \sum_{n\in Z}\left(j_n e^{-n(\tau+i\sigma)} - \bar j_n e^{-n(\tau-i\sigma)}\right) \tag{11}$$

$$i\cdot\partial_\tau X = \sum_{n\in Z}\left(j_n e^{-n(\tau+i\sigma)} + \bar j_n e^{-n(\tau-i\sigma)}\right) \tag{12}$$

Thus both of the Neumann and Dirichlet boundary conditions at $\sigma = 0$ are easily obtained as

$$\partial_\sigma X|_{\sigma=0} = \sum_{n\in Z}(j_n - \bar j_n)e^{-n\tau} \tag{13}$$

$$\partial_\tau X|_{\sigma=0} = \sum_{n\in Z}(j_n + \bar j_n)e^{-n\tau} \tag{14}$$

Finally, we note that boundaries introduce relations between the chiral and the anti-chiral modes of the conformal fields which read

$$\boxed{\begin{array}{lll} j_n - \bar j_n & = 0 & \textbf{Neumann Condition} \\ j_n + \bar j_n & = 0 & \textbf{Dirichlet Condition} \end{array}} \tag{15}$$

2.1.3 Conformal Symmetry

Let us remark that Eq. (15) apply to the Laurent modes of the two $U(1)$ currents $j(z)$ and $\bar{j}(\bar{z})$ of the free boson theory leaving only a diagonal $U(1)$ symmetry. However, in addition there is always the conformal symmetry generated by the energy-momentum tensor. Since boundaries in general break certain symmetries, we expect also restrictions on the Laurent modes of energyCmomentum tensor. Recall that the energy-momentum tensor $T(z)$ and $\bar{T}(\bar{z})$ can be expressed in terms of the currents $j(z)$ and $\bar{j}(\bar{z})$ in the following way:

$$T\left(\mathrm{z}\right) = \frac{1}{2} N\left(jj\right)\left(z\right), \qquad \bar{T}\left(\bar{\mathrm{z}}\right) = \frac{1}{2} N\left(\bar{j}\bar{j}\right)\left(\bar{z}\right)$$

where N stands for normal ordering. And if we consider the Laurent mode of the energy-momentum tensor $T(z)$, which is $L(n) = N(jj)$, we find that the Neumann as well as the Dirichlet boundary conditions in Eq.(15) imply that

$$\boxed{L_n - \bar{L}_n = 0} \tag{16}$$

Note that this condition can be expressed as $T(z) = \bar{T}(\bar{z})$ which in particular means the central charges of the holomorphic and anti-holomorphic theories have to be equal, i.e. $c = \bar{c}$.

2.2 Boundary States and Gluing Condition

By far, we have described the boundaries for the free boson CFT implicitly via the boundary conditions for the fields. However, in an abstract CFT usually there is no Lagrangian formulation available and no boundary terms will arise from a variational principle. Therefore, to proceed in the more general theories, we need a more inherent formulation of a boundary.

In the following, we first take a quick look into the BCFT partition functions, and then we will illustrate the construction of the so-called boundary states for the example of the free boson.

2.2.1 Partition Function in Boundary Conformal Field Theory

For conformal field theories this is essentially the same object as in statistical mechanics where it is defined as a sum over all possible configurations weighted with the Boltzmann factor $exp(?\beta H)$. Similarly, it corresponds to the generating functional in quantum field theory.

Definition

Let us now consider the one-loop partition function for BCFTs. To do so, we first review the construction of partition functions for the boundary-free conformal field theory and then compare with the present situation.

- we defined the one-loop partition function for CFTs without boundaries as follows. We started from a theory defined on the infinite cylinder described by (τ, σ), where σ was periodic and $\tau \in (-\infty, \infty)$. Next, we imposed periodicity conditions also on the time coordinate τ yielding the topology of a torus
- In the present case, since the space coordinate σ is not periodic, we thus start from a theory defined on the infinite strip given by $\sigma = [0, \pi]$ and $\tau \in (-\infty, \infty)$. For the definition of the one-loop partition function, we again make the time coordinate τ periodic leaving us with the topology of a cylinder instead of a torus. This is illustrated in Fig.3.
- Similarly to the modular parameter of the torus, there is a modular parameter t with $0 \leq \mathrm{t} < \infty$ parametrising different cylinders. The inequivalent cylinders are described by $\{(\tau, \sigma :) \, 0 \leq \sigma \leq \pi, 0 \leq \tau \leq \mathrm{t}\}$.

For the partition function, we need to determine the operator generating translations in time circling the cylinder once along the τ direction. Because boundaries lead to an identification of the left- and right-moving sector as required by Eq. (16), we see that this operator is the Hamiltonian say in the open sector

$$H_{open} = (L_{cyl.})_0 = L_0 - \frac{c}{24} \tag{17}$$

In analogy to the case of the torus partition function, we then define the cylinder partition function as $Z = Tr \exp(-2\pi t H_{open})$ which can be brought into the following form:

$$Z^C(t) = Tr_{\mathrm{H}_B} \left(q^{L_0 - \frac{c}{24}} \right) \tag{18}$$

where $q = \exp(-2\pi t)$, and C on Z denotes the cylinder partition function, and H_B denotes the Hilbert space of all states satisfying one of the boundary conditions

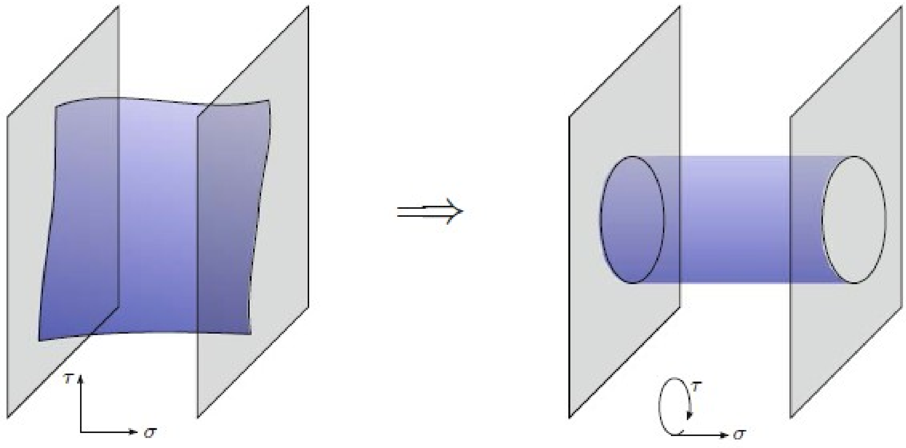

Figure 3: Illustration of obtaining the cylinder partition function

Eq.(15).Since the partition function in BCFT is defined is such way, it is necessary to understand those states satisfying the boundary conditions. We call those states boundary states.

2.2.2 Boundary States

Let us start with the following observation. As illustrated in Fig.4, by interchanging τ and σ, we can interpret the cylinder partition function of the boundary conformal field theory on the left-hand side as a tree-level amplitude of the underlying theory shown on the right-hand side. From a string theory point of view, the tree-level amplitude describes the emission of a closed string at boundary A which propagates to boundary B and is absorbed there. Thus, a boundary can be interpreted as an object, which couples to closed strings. Note that in order to simplify our notation, we call the sector of the BCFT open and the sector of the underlying CFT closed. The relation above then reads

$$(\sigma, \tau)_{open} \leftrightarrow (\tau, \sigma)_{closed} \tag{19}$$

which in string theory is known as the world-sheet duality between open and closed strings.

Let us now focus on the closed sector(tree level).The boundary for the closed sector can be described by a coherent state in the Hilbert space $\mathrm{H} \otimes \bar{\mathrm{H}}$ which takes

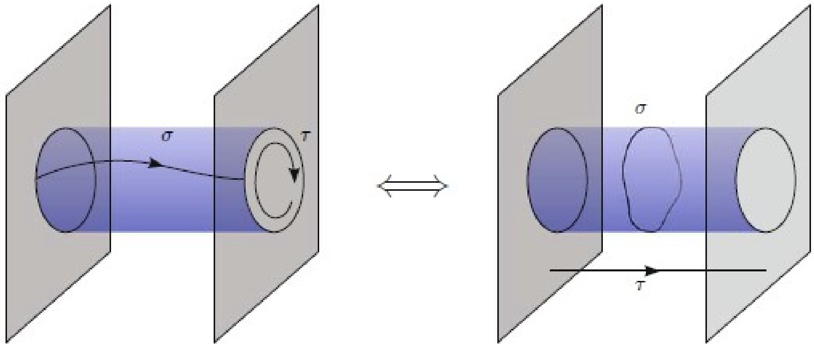

Figure 4: Illustration Loop-channel - Tree-channel Equivalence

the general form

$$|B\rangle = \sum_{i,\bar{j}\in \mathrm{H}\otimes\bar{\mathrm{H}}} \alpha_{i\bar{j}} \, |i,\bar{j}\rangle \tag{20}$$

Here $i, \bar{j}$ label the states in the holomorphic and anti-holomorphic sector of $\mathrm{H}\otimes\bar{\mathrm{H}}$, and the coefficients $\alpha_{i\bar{j}}$ encode the strength of how the closed string mode $|i,\bar{j}\rangle$ couples to the boundary $|B\rangle$. Such a coherent state is called a boundary state and provides the CFT description of a D-brane in string theory.

Boundary Conditions → Gluing Conditions

Now we can translate the boundary condition in Eq.(8) into the picture of boundary states. By using $(\sigma,\tau)_{open} \leftrightarrow (\tau,\sigma)_{closed}$, we obtain

$$\boxed{\begin{array}{lll} \partial_\tau X_{closed}|_{\tau=0}\,|B_N\rangle & = 0 & \textbf{Neumann Condition} \\ \partial_\sigma X_{closed}|_{\tau=0}\,|B_D\rangle & = 0 & \textbf{Dirichlet Condition} \end{array}} \tag{21}$$

Now for the example of free boson theory we would like to express the boundary conditions Eq.(21) of a boundary state in terms of the Laurent modes. To do so, we recall Eq.(11) and Eq.(12) and set $\tau = 0$ to obtain

$$i\cdot\partial_\tau X_{closed}|_{\tau=0} = \sum_{n\in Z}\left(j_n e^{-in\sigma} + \bar{j}_n e^{+in\sigma}\right) \tag{22}$$

$$\partial_\sigma X_{closed}|_{\tau=0} = \sum_{n\in Z}\left(j_n e^{-in\sigma} - \bar{j}_n e^{+in\sigma}\right) \tag{23}$$

We then relabel $n \to ?n$ in the second term of each line and observe again that for generic σ, the summands are linearly independent. Therefore, the boundary conditions Eq.(21) expressed in terms of the Laurent modes read

$$\boxed{\begin{array}{lll} (j_n + \bar{j}_{-n})\,|B_N\rangle & = 0 & \textbf{Neumann Condition} \\ (j_n - \bar{j}_{-n})\,|B_D\rangle & = 0 & \textbf{Dirichlet Condition} \end{array}} \tag{24}$$

for each n.
Such conditions relating the chiral and anti-chiral modes acting on the boundary state are called ***gluing conditions***.

Here, we are going to state the solutions for the gluing conditions for the example of the free boson and verify them thereafter. The boundary states for Neumann and Dirichlet conditions in terms of the Laurent modes j_n and $\bar{j}_{-n}$ read

$$\boxed{\begin{array}{lll} |B_N\rangle & = \frac{1}{N_N}\exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}j_{-k}\bar{j}_{-k}\right)|0\rangle & \textbf{Neumann Condition} \\ |B_D\rangle & = \frac{1}{N_D}\exp\left(+\sum_{k=1}^{\infty}\frac{1}{k}j_{-k}\bar{j}_{-k}\right)|0\rangle & \textbf{Dirichlet Condition} \end{array}} \tag{25}$$

where N_N and N_D are normalisation constants to be fixed later. One possibility to verify the boundary states is to straightforwardly evaluate the gluing conditions Eq.(24) for the solutions Eq.(25) explicitly. However, in order to highlight the underlying structure, we will take a slightly different approach.

Construction of Boundary States

In the following, we focus on a boundary state with Neumann conditions. First, we rewrite the Neumann boundary state in Eq.(25) as

$$|B_N\rangle = \frac{1}{N_N}\exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}j_{-k}\bar{j}_{-k}\right)|0\rangle \tag{26}$$

$$= \frac{1}{N_N}\prod_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\left(\frac{j_{-k}}{\sqrt{k}}\right)^m|0\rangle\otimes\frac{1}{\sqrt{m!}}\left(\frac{-\bar{j}_{-k}}{\sqrt{k}}\right)^m|\bar{0}\rangle \tag{27}$$

$$= \frac{1}{N_N}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\prod_{k=1}^{\infty}\frac{1}{\sqrt{m_k!}}\left(\frac{j_{-k}}{\sqrt{k}}\right)^{m_k}|0\rangle\otimes\frac{1}{\sqrt{m_k!}}\left(\frac{-\bar{j}_{-k}}{\sqrt{k}}\right)^{m_k}|\bar{0}\rangle \tag{28}$$

where we first have written the sum in the exponential as a product and then we expressed the exponential as an infinite series. Next, we note that the following states form a complete orthonormal basis for all states constructed out of the Laurent modes $j_{?k}$:

$$|\vec{m}\rangle = |m_1, m_2, m_3 \cdots\rangle = \prod_{k=1}^{\infty} \frac{1}{\sqrt{m_k!}} \left(\frac{j_{-k}}{\sqrt{k}}\right)^{m_k} |0\rangle \tag{29}$$

We now introduce an operator U mapping the chiral Hilbert space to its charge conjugate $U : \mathrm{H} \to \mathrm{H}^+$ and similarly for the anti-chiral sector. In particular, the action of U reads:

$$U j_k U^{-1} = -j_k = -(j_{-k})^\dagger, \qquad U \bar{j}_k U^{-1} = -\bar{j}_k = -(\bar{j}_{-k})^\dagger, \qquad U \mathrm{c} U^{-1} = \mathrm{c}^* \tag{30}$$

where c is a constant and $*$ denotes complex conjugation. In the present example, the ground state $|0\rangle$ is non-degenerate and is left invariant by U. Knowing these properties, we can show that U is anti-unitary. For this purpose, we expand a general state as $|\mathrm{a}\rangle = \sum_{\vec{m}} A_{\vec{m}} |\vec{m}\rangle$ and compute

$$U|\mathrm{a}\rangle = \sum_{\vec{m}} U A_{\vec{m}} U^{-1} \prod_{k=1}^{\infty} \frac{1}{\sqrt{m_k!}} \left(\frac{U j_{-k} U^{-1}}{\sqrt{k}}\right)^{m_k} U|0\rangle \tag{31}$$

$$= \sum_{\vec{m}} \mathrm{A}^*_{\vec{m}} \prod_{k=1}^{\infty} (-1)^{m_k} |\vec{m}\rangle \tag{32}$$

where $\vec{m}$ denotes the multi-index $m_1, m_2, \ldots$. By using that $|\vec{m}\rangle$ and $|\vec{n}\rangle$ form an orthonormal basis, we can now show that U is anti-unitary

$$\langle Ub \mid Ua\rangle = \sum_{\vec{n},\vec{m}} \langle\vec{n}| B_{\vec{n}} \prod_{k=1}^{\infty} (-1)^{n_k+m_k} \mathrm{A}^*_{\vec{m}} |\vec{m}\rangle = \sum_{\vec{m}} \mathrm{A}^*_{\vec{m}} B_{\vec{m}} = \langle a \mid b\rangle \tag{33}$$

After introducing an orthonormal basis and the anti-unitary operator U, we now express Eq.(28) in a more general way which will simplify and generalise the following calculations:

$$\boxed{|B\rangle = \frac{1}{N} \sum_{\vec{m}} |\vec{m}\rangle \otimes |\bar{U}\vec{m}\rangle} \tag{34}$$

Verification of the Gluing Conditions

Now we can verify the gluing conditions Eq(24) for Neumann boundary states, we note that these have to be satisfied also when an arbitrary state $\langle\bar{a}| \otimes \langle b|$ is multiplied from the left. We then calculate

$$\langle \bar{a}| \otimes \langle b| \left(j_n + \bar{j}_{-n} \right) |B\rangle = \frac{1}{N} \sum_{\vec{m}} \langle \bar{a}| \otimes \langle b| \left(j_n + \bar{j}_{-n} \right) |\vec{m}\rangle \otimes |\bar{U m}\rangle \tag{35}$$

$$= \frac{1}{N} \left\{ \sum_{\vec{m}} \langle b| j_n |\vec{m}\rangle \langle \bar{a}| |\bar{U m}\rangle + \langle b \mid \vec{m}\rangle \langle \bar{a}| \bar{j}_{-n} |\bar{U m}\rangle \right\} \tag{36}$$

Next, due to the identifications on the boundary, the holomorphic and the anti-holomorphic algebra are identical. We can therefore replace matrix elements in the anti-holomorphic sector by those in the holomorphic sector. Using finally the anti-unitarity of U and that $\sum_{\vec{m}} |\vec{m}\rangle \langle \vec{m}| = I$, we find

$$\langle \bar{a}| \otimes \langle b| \left(j_n + \bar{j}_{-n} \right) |B\rangle = \frac{1}{N} \sum_{\vec{m}} \langle b| j_n |\vec{m}\rangle \langle a| |U\vec{m}\rangle + \langle b \mid \vec{m}\rangle \langle a| j_{-n} |U\vec{m}\rangle \tag{37}$$

$$= \frac{1}{N} \sum_{\vec{m}} \langle b| j_n |\vec{m}\rangle \left\langle \vec{m} \mid U^{-1} a \right\rangle + \langle b \mid \vec{m}\rangle \langle \vec{m}| j_{-n} \left| U^{-1} a \right\rangle \tag{38}$$

$$= \frac{1}{N} \left(\langle b| j_n \left| U^{-1} a \right\rangle - \langle b| j_n \left| U^{-1} a \right\rangle \right) = 0 \tag{39}$$

Therefore, we have verified that the Neumann boundary state in Eq.(25) is indeed a solution to the corresponding gluing condition in Eq.(24).
Note furthermore, the construction of boundary states and the verification of the gluing conditions are also applicable for more general CFTs. for instance Rational CFTs, which we will consider later.

Conformal Symmetry

In studying the example of the free boson, we have expressed all important quantities in terms of the $U(1)$ current modes j_n and $\bar{j}_n$ However, in more general CFTs such additional symmetries may not be present but the conformal symmetry generated by the energyCmomentum tensors always is. In view of generalisations of our present example, let us therefore determine the boundary conditions of the boundary states in terms of the Laurent modes L_n and $\bar{L}_n$. Mainly guided by the final result, let us compute the following expression by employing that $T(z) = \frac{1}{2} N(jj)(z)$ which implies $L_n = \frac{1}{2} \sum_{k>-1} j_{n-k} j_k + \frac{1}{2} \sum_{k\leq -1} j_k j_{n-k}$:

$$\left(L_n - \bar{L}_{-n} \right) |B_{N,\mathrm{D}}\rangle \tag{40}$$

$$= \frac{1}{2} \left(\sum_{k>-1} \left(j_{n-k} j_k - \bar{j}_{-n-k} \bar{j}_k \right) + \sum_{k\leq -1} \left(j_k j_{n-k} - \bar{j}_k \bar{j}_{-n-k} \right) \right) |B_{N,\mathrm{D}}\rangle \tag{41}$$

If we apply the the gluing condition in Eq.(24) and continue the calculation, we will find

$$\left(L_n - \bar{L}_{-n}\right)|B_{N,\mathrm{D}}\rangle = 0 \tag{42}$$

In summary, we have shown that the boundary states are conformally symmetric.

2.3 Partition Function and Loop-Channel – Tree-Channel Equivalence

In this subsection, we will calculate the partition function of the free boson in the Neumann-Neumann boundary condition case with detail. We will calculate the partition functions for both the open(loop) and closed(tree) sector, then we will show that the partition functions in the open and closed sector are identical, which leads to a very important result of BCFT, the loop-channel - tree-channel equivalence.

2.3.1 Loop-Channel Partition Function (Open Sector)

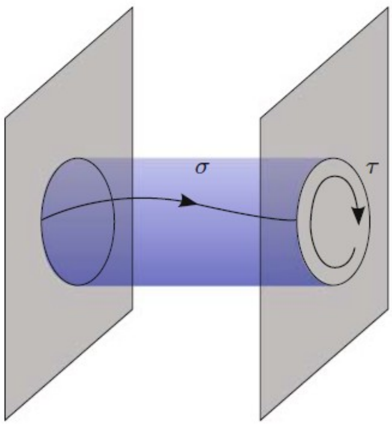

Figure 5: Illustration Loop-channel partition function

We know that we can consider the cylinder as a portion of torus, let us first recall the calculation of how the torus partition function was calculated.The partition function for a conformal field theory defined on a torus with modular parameter

τ is given by

$$Z(\tau,\bar{\tau}) = Tr_{\mathrm{H}}(q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}) \qquad \text{where} \qquad q = \exp(2\pi \mathrm{i}\tau) \tag{43}$$

For the free boson, the Laurent modes of the energy-momentum tensor are written using the modes of the current $j(z) = i\partial X(z)$. In particular, we have

$$L_0 = \frac{1}{2}j_0 j_0 + \sum_{k\geq 1} j_{-k}j_k \tag{44}$$

Since the current $j(z)$ is a field of conformal dimension one, we find $j_n|0\rangle = 0$ for $n > -1$ and that states in the Hilbert space have the following form:

$$|n_1,n_2,n_3,...\rangle = j_{-1}^{n_1}j_{-2}^{n_2}j_{-3}^{n_3}...|0\rangle \qquad \text{with} \qquad \mathrm{n_i} \geq 0 \tag{45}$$

And the current algebra for the Laurent modes reads

$$[j_m, j_n] = m\delta_{m,-n} \tag{46}$$

Next, let us compute the action of L_0 on a state Eq(42), Clearly, j_0 commutes with all $j_{?k}$ and annihilates the vacuum. For the other terms we calculate

$$\left[j_{-k}j_k, j_{-k}^{n_k}\right] = n_k k j_{-k}^{n_k} \tag{47}$$

and so we find for the zero Laurent mode of the energy-momentum tensor that

$$L_0|n_1,n_2,n_3,...\rangle = \sum_{k\geq 1} j_{-1}^{n_1}j_{-2}^{n_2}...(j_{-k}j_k)j_{-k}^{n_k}...|0\rangle = \sum_{k\geq 1} kn_k|n_1,n_2,n_3,...\rangle \tag{48}$$

We will utilise this last result in the calculation of the partition function where for simplicity we only focus on the holomorphic part. We compute

$$\begin{aligned}
& Tr(q^{L_0-c/24}) && (49)\\
= \;& q^{-1/24}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}...\langle n_1,n_2,n_3,...|\sum_{p=0}^{\infty}\frac{1}{p!}(2\pi\tau)^p\,(L_0)^p\,|n_1,n_2,n_3,...\rangle && (50)\\
= \;& q^{-1/24}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}...\langle n_1,n_2,n_3,...|\sum_{p=0}^{\infty}\frac{1}{p!}(2\pi\tau)^p\left(\sum_{k=1}^{\infty}kn_k\right)^p|n_1,n_2,n_3,...\rangle && (51)\\
= \;& q^{-1/24}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}...\left(q^{1\cdot n_1}\cdot q^{2\cdot n_2}\cdot q^{3\cdot n_3}\cdot ...\right) && (52)\\
= \;& q^{-1/24}\left(\sum_{n_1=0}^{\infty}q^{1\cdot n_1}\right)\cdot\left(\sum_{n_2=0}^{\infty}q^{2\cdot n_2}\right)\cdot\left(\sum_{n_3=0}^{\infty}q^{3\cdot n_3}\right)\cdot ... && (53)\\
= \;& q^{-1/24}\prod_{k=1}^{\infty}\sum_{n_k=0}^{\infty}q^{k\cdot n_k} && (54)\\
= \;& q^{-1/24}\prod_{k=1}^{\infty}\frac{1}{1-q^k} && (55)
\end{aligned}$$

where in the last step we employed the result for the infinite geometric series and the ellipses indicate that the structure extends to infinity. We then define the Dedekind η-function as

$$\boxed{\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} 1 - q^n} \tag{56}$$

so that, including also the anti-holomorphic part, the partition function of a single free boson reads

$$Z_{bos.}(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \tag{57}$$

Using this result of the torus partition function we can determine the cylinder partition function for the free boson. Recalling our calculation above and setting $\tau = it$, we obtain

$$Tr_{\mathrm{H}_B}(q^{L_0 - c/24}) = \frac{1}{|\eta(it)|^2} \qquad \text{without} \qquad j_0 \tag{58}$$

However, there we have assumed the action of j_0 on the vacuum to vanish, which in the case of string theory is in general not applicable. Taking into account the effect of j_0, we now study the three different cases of boundary conditions in turn.

- For the case of **Neumann-Neumann boundary conditions**, the momentum mode $\pi_0 = \frac{1}{2} j_0$ is unconstrained and in principle contributes to the trace. Since it is a continuous variable, the sum is replaced by an integral

$$Tr_{\mathrm{H}_B}(q^{\frac{1}{2}j_0^2}) = \sum_{n_0} \langle n_0 | e^{-\pi t \cdot j_0^2} | n_0 \rangle = \sum_{n_0} e^{-\pi t \cdot n_0^2} \rightarrow \int_{-\infty}^{\infty} d\pi_0 e^{-4\pi t \cdot \pi_0^2} \tag{59}$$

  where we utilised $n_0 = 2\pi_0$. Evaluating this Gaussian integral leads to the following additional factor for the partition function:

$$\frac{1}{2\sqrt{t}} \tag{60}$$

- For the **DirichletCDirichlet boundary conditions**, we have seen that j_0 is related to the positions of the string endpoints. Therefore, we have a contribution to the partition function of the form

$$q^{\frac{1}{2}j_0^2} = \exp\left(-2\pi t \frac{1}{2}\left(\frac{x_0^b - x_0^a}{2\pi}\right)^2\right) = \exp\left(-\frac{t}{4\pi}\left(x_0^b - x_0^a\right)^2\right) \tag{61}$$

- For the **Mixed boundary conditions**, we will not present the detail calculation here, but only the result.

In summary, the cylinder partition functions(open sector) for the example of the free boson read

$$\boxed{Z_{bos.}^{C(\mathrm{N,N})}(t) = \frac{1}{2\sqrt{t}}\frac{1}{\eta(it)}} \tag{62}$$

$$\boxed{Z_{bos.}^{C(\mathrm{D,D})}(t) = \exp\left(-\frac{t}{4\pi}\left(x_0^b - x_0^a\right)^2\right)\frac{1}{\eta(it)}} \tag{63}$$

$$\boxed{Z_{bos.}^{C(\mathrm{mixed})}(t) = \sqrt{\frac{\eta(it)}{\vartheta_4(it)}}} \tag{64}$$

2.3.2 Tree-Channel Partition Function (Closed Sector)

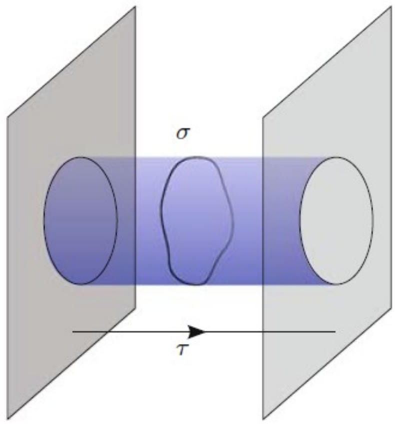

Figure 6: Illustration Tree-channel partition function

Now we compute the partition function(tree-channel). Referring again to Fig.6, in string theory, we can interpret this diagram as a closed string which is emitted at the boundary A, propagating via the closed sector Hamiltonian $H_{closed} = L_0 - \bar{L}_0 - (c+\bar{c})/24$ for a time $\tau = l$ until it reaches the boundary B where it gets absorbed. In analogy to Quantum Mechanics, such an amplitude is given by the overlap

$$\tilde{Z}^C(l) = \langle \Theta B| e^{-2\pi l\left(L_0 - \bar{L}_0 - (c+\bar{c})/24\right)} |B\rangle \tag{65}$$

where the tilde indicates that the computation is performed in the closed sector (or at tree-level) and l is the length of the cylinder connecting the two boundaries. Furthermore, we have introduced the CPT operator Θ which acts as charge conjugation **(C)**, parity transformation **(P)** and time reversal **(T)** for the two-dimensional CFT. The reason for considering this operator can roughly be explained by the fact that the orientation of the boundary a closed string is emitted at is opposite to the orientation of the boundary where the closed string gets absorbed. Without a detailed derivation, we finally note that the theory of the free boson is CPT invariant and so the action of Θ on the boundary states Eq.(25) of the free boson theory reads

$$\Theta \left|B\right\rangle = \frac{1}{N^*} \left|B\right\rangle , \qquad \Theta c \Theta^{-1} = c^* \tag{66}$$

Let us now compute the overlap of two boundary states Eq.(65) for the example of the free boson. To do so, we note that for the free boson CFT we have $c = \bar{c} = 1$ and we recall that

$$L_0 = \frac{1}{2} j_0 j_0 + \sum_{k \geq 1} j_{-k} j_k \tag{67}$$

and similarly for $\bar{L}_0$. Next, we perform the following calculation in order to evaluate Eq.(65). In particular, we use $j_{-k} j_k j_k^{m_k} \left|0\right\rangle = m_k k j_k^{m_k} \left|0\right\rangle$ to find

$$\begin{aligned} q^{\sum_{k\geq 1} j_{-k} j_k \left|\vec{m}\right\rangle} &= \prod_{k=1}^{\infty} \sum_{p=0}^{\infty} \frac{(-2\pi i \tau)^p}{p!} (j_{-k} j_k)^p \prod_{l=1}^{\infty} \frac{1}{\sqrt{m_l!}} \left(\frac{j_{-l}}{\sqrt{l}} \right)^{m_l} \left|0\right\rangle & (68) \\ &= \prod_{k=1}^{\infty} \sum_{p=0}^{\infty} \frac{(-2\pi i \tau)^p}{p!} (m_k k)^p \prod_{l=1}^{\infty} \frac{1}{\sqrt{m_l!}} \left(\frac{j_{-l}}{\sqrt{l}} \right)^{m_l} \left|0\right\rangle & (69) \\ &= \prod_{k=1}^{\infty} q^{m_k k} \left|\vec{m}\right\rangle & (70) \end{aligned}$$

The cylinder diagram for the three possible combinations of boundary conditions is then computed as follows.

- For the case of **NeumannCNeumann boundary conditions**, we have $j_0 \left|B_N\right\rangle = \bar{j}_0 \left|B_N\right\rangle = 0$ and so the momentum contribution vanishes. For the remaining part, we calculate using Eq.(70) and Eq.(32)

$$\tilde{Z}^{C(\mathrm{N,N})}_{bos.}(l) = \frac{e^{-2\pi l(-\frac{2}{24})}}{N_N^2}\sum_{\vec{m}}\langle\vec{m}|\,e^{-2\pi l\sum_{k\geq1}j_{-k}j_k}\,|\vec{m}\rangle\times\times\langle U\vec{m}|\,e^{-2\pi l\sum_{k\geq1}\bar{j}_{-k}\bar{j}_k}\,|U\vec{m}\rangle \tag{71}$$

$$= \frac{e^{-2\pi l(-\frac{2}{24})}}{N_N^2}\sum_{\vec{m}}\prod_{k=1}^{\infty}e^{-2\pi l m_k k}(-1)^{\sum_{l=1}^{\infty}m_l}e^{-2\pi l m_k k}(-1)^{\sum_{l=1}^{\infty}m_l} \tag{72}$$

$$= \frac{e^{\frac{\pi l}{6}}}{N_N^2}\prod_{k=1}^{\infty}\sum_{m_k=0}^{\infty}\left(e^{-4\pi lk}\right)^{m_k} \tag{73}$$

$$= \frac{1}{N_N^2}e^{\frac{\pi l}{6}}\prod_{k=1}^{\infty}\frac{1}{1-e^{-4\pi lk}} \tag{74}$$

where in the last step, we performed a summation of the geometric series. Let us emphasise that due to the action of the CPT operator Θ, N^2 is just the square of N and not the absolute value squared. Then, with $q = exp(2\pi i\tau)$, $\tau = 2il$ and $\eta(\tau)$ the Dedekind η-function defined before, we find that the cylinder diagram for NeumannCNeumann boundary conditions is expressed as

$$\boxed{\tilde{Z}^{C(\mathrm{N,N})}_{bos.}(l) = \frac{1}{N_N^2}\frac{1}{\eta(2il)}} \tag{75}$$

- Next, we consider the case of **DirichletCDirichlet boundary conditions**. Noting that U now acts trivially on the basis states, we see that apart from the momentum contribution the calculation is similar to the case with NeumannCNeumann conditions. Here we just state the result without detail calculation:

$$\boxed{\tilde{Z}^{C(\mathrm{D,D})}_{bos.}(l) = \frac{1}{N_{\mathrm{D}}^2}\exp\left(-\frac{(x_0^b-x_0^a)^2}{8\pi l}\right)\frac{1}{\sqrt{2l}}\frac{1}{\eta(2il)}} \tag{76}$$

- Finally, we can express the cylinder diagram for **mixed boundary conditions** as

$$\boxed{\tilde{Z}^{C(mixed)}_{bos.}(l) = \frac{\sqrt{2}}{N_D N_N}\sqrt{\frac{\eta(2il)}{\vartheta_2(2il)}}} \tag{77}$$

2.3.3 Loop-Channel - Tree-Channel Equivalence

Let us come back to Fig.4. As it is illustrated there and motivated at the beginning of this section, we expect the cylinder diagram in the closed and open sectors to be

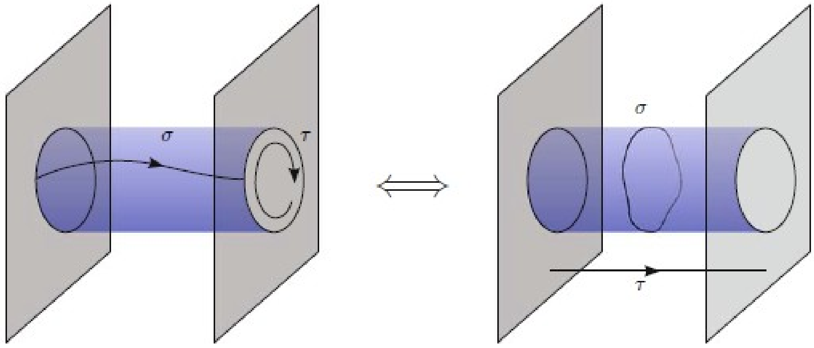

Figure 7: Illustration Loop-channel - Tree-channel Equivalence

related. More specifically, this relation is established by $(\sigma,\tau)_{open} \Leftrightarrow (\tau,\sigma)_{closed}$, where σ is the world-sheet space coordinate and τ is world-sheet time. However, this mapping does not change the cylinder, in particular, it does not change the modular parameter τ . In the open sector, the cylinder has length 1/2 and circumference t when measured in units of 2π, while in the closed sector we have length l and circumference 1. Referring then to the discussion of modular transformation, we find for the modular parameter in the open and closed sectors that

$$\tau_{open} = \frac{it}{1/2} = 2it, \qquad \tau_{closed} = \frac{i}{l} \tag{78}$$

As we have emphasised, the modular parameters in the open and closed sectors have to be equal which leads us to the relation

$$\boxed{t = 1/2l} \tag{79}$$

This is the formal expression for the pictorial ***loop-channelCtree-channel equivalence*** of the cylinder diagram illustrated in Fig.4. We now verify this relation for the example of the free boson explicitly which will allow us to fix the normalisation constants N_D and N_N of the boundary states. Recalling the cylinder partition function Eq.(75) in the open sector, we compute

$$\mathrm{Z}_{bos.}^{C(\mathrm{N,N})}(t) = \frac{1}{2\sqrt{t}}\frac{1}{\eta(it)} \to (t = 1/2l) \to \frac{1}{2\eta(2il)} = \frac{N_N^2}{2}\tilde{Z}_{bos.}^{C(\mathrm{N,N})}(l) \tag{80}$$

where we used the modular properties of the Dedekind function. Therefore, requiring the results in the loop- and tree-channels to be related, we can fix

$$\boxed{N_N = \sqrt{2}} \tag{81}$$

Next, for DirichletCDirichlet boundary conditions, we find

$$\boxed{N_D = 1} \tag{82}$$

Finally, the loop-channelCtree-channel equivalence for mixed NeumannCDirichlet boundary conditions can be verified along similar lines. This discussion shows that indeed the cylinder partition function for the free boson in the open and closed sectors is related via a modular transformation, more concretely via a modular S-transformation.

3 Cardy Condition

3.1 Boundary states for Rational CFT

Similar to free boson case in BCFT, we now generalize our finding in rational CFT, where Rational CFT is CFT with finite number of primary fields. And we will formulate the corresponding Boundary RCFT just in terms of gluing conditions for the theory on the sphere.

Boundary Conditions

We consider Rational conformal field theories with chiral and anti-chiral symmetry algebras A and $\bar{A}$, respectively. For the theory on the sphere the Hilbert space splits into irreducible representations of $A \otimes \bar{A}$ as

$$\mathrm{H} = \bigoplus_{i,\bar{j}} M_{i,\bar{j}} \mathrm{H}_i \otimes \bar{\mathrm{H}}_j \tag{83}$$

where $M_{i,\bar{j}}$ are the same multiplicities of the highest weight representation appearing in the modular invariant torus partition function. Note that for the case of RCFTs we are considering, there is only a finite number of irreducible representations and that the modular invariant torus partition function is given by a combination of chiral and anti-chiral characters as follows:

$$Z(\tau, \bar{\tau}) = \sum_{i,\bar{j}} M_{i,\bar{j}} \chi_i(\tau) \bar{\chi}_{\bar{j}}(\bar{\tau}) \tag{84}$$

Generalising the results from the free boson theory, we state without derivation that a boundary state $|B\rangle$ in the RCFT preserving the symmetry algebra $A = \bar{A}$ has to satisfy the following gluing conditions:

$$\boxed{\left(L_n - \bar{L}_n\right)|B\rangle = 0 \qquad \textbf{Conformal Symmetry}} \tag{85}$$

Ishibashi States

Next, let us recall that the charge conjugation matrix C maps highest weight representations i to their charge conjugate i^+. Denoting then the Hilbert space built upon the charge conjugate representation by $\mathrm{H}_{\mathrm{i}}^{+}$, we can state the important result of Ishibashi:

- For $A = \bar{A}$ and $\bar{\mathrm{H}}_{\mathrm{i}} = \mathrm{H}_{\mathrm{i}}^{+}$, to each highest weight representation ϕ_i of A one can associate an up to a constant unique state $|\beta_{\mathrm{i}}\rangle\rangle$ such that the gluing conditions are satisfied.

Note that since the CFTs we are considering are rational, there is only a finite number of highest weight states and thus only a finite number of such so-called Ishibashi states $|\beta_{\mathrm{i}}\rangle\rangle$ We now construct the Ishibashi states in analogy to the boundary states of the free boson. Denoting by $|\phi_{\mathrm{i}}, \mathrm{m}\rangle$ an orthonormal basis for H_i , the Ishibashi states are written as

$$|\beta_{\mathrm{i}}\rangle\rangle = \sum_{\mathrm{m}} |\phi_{\mathrm{i}}, \mathrm{m}\rangle \otimes U \left|\bar{\phi}_{\mathrm{i}}, \bar{\mathrm{m}}\right\rangle \tag{86}$$

where U is once again the anti-unitary operator.
We can easily prove that the Ishibashi states are solutions to the gluing condition Eq.(85). Since it is completely analogous to the example of the free boson and so we will not present it here.

Now, also the Ishibashi states satisfy the gluing condition, are they really the boundary states? In order to answer this question, we need to consider the following overlap:

$$\langle\langle\beta_j|\, e^{-2\pi l(L_0+\bar{L}_0-\frac{c+\bar{c}}{24})}\, |\beta_{\mathrm{i}}\rangle\rangle \tag{87}$$

Utilising the gluing conditions for the conformal symmetry generator Eq.(85), we see that we can replace L_0 by $\bar{L}_0$ and c by $\bar{c}$. Next, because the Hilbert spaces of two different HWRs ψ_i and ψ_j are independent of each other, the overlap above is only nonzero for $i = j^+$. Note that here we have written the charge conjugate j^+ of the highest weight ψ_j because the hermitian conjugation also acts as charge conjugation. We then obtain

$$\begin{aligned}
\langle\langle\beta_j|\, e^{-2\pi l(L_0+\bar{L}_0-\frac{c+\bar{c}}{24})}\, |\beta_{\mathrm{i}}\rangle\rangle &= \delta_{ij^+} \langle\langle\beta_j|\, e^{2\pi l(2il)(L_0-\frac{c}{24})}\, |\beta_{\mathrm{i}}\rangle\rangle && (88)\\
&= \delta_{ij^+} Tr_{\mathrm{H}_i}\left(q^{L_0-\frac{c}{24}}\right) && (89)\\
&= \delta_{ij^+} \chi_i\,(2il) && (90)
\end{aligned}$$

with χ_i the character of the highest weight ψ_i defined as

$$\chi_i\,(\tau) := Tr_{\mathrm{H}_i}\left(q^{L_0-\frac{c}{24}}\right) \tag{91}$$

Performing a modular S-transformation for this overlap, by the same reasoning as for the free boson, we expect to obtain a partition function in the boundary sector. However, because the S-transform of a character $\chi_i\,(2il)$ in general does not give non-negative integer coefficients in the loop-channel, it is not clear whether to interpret such a quantity as a partition function counting states of a given excitation level.
Note, S-transform of $\chi(2il)$ should give non-negative integer coefficients as required by Verline formula:

$$Z_{\alpha\beta}\,(t) = \sum_j n^j_{\alpha\beta}\chi_j\,(it) \tag{92}$$

As it turns out, the Ishibashi states are not the boundary states itself but only building blocks guaranteed to satisfy the gluing conditions. A true boundary state in general can be expressed as a linear combination of Ishibashi states in the following way:

$$|B_\alpha\rangle = \sum_i B^i_\alpha\,|\beta_{\mathrm{i}}\rangle\rangle \tag{93}$$

3.2 Cardy Condition

We have just stated that the real boundary states in RCFT are not Ishibashi states but linear combination of them:

$$|B_\alpha\rangle = \sum_i B^i_\alpha\,|\beta_{\mathrm{i}}\rangle\rangle \tag{94}$$

The complex coefficients B^i_α in Eq.(93) are called reflection coefficients and are very constrained by the so-called Cardy condition.

Cardy Condition

This condition essentially ensures the ***loop-channelCtree-channel equivalence***. Indeed, using relation Eq(89) and choosing normalisations such that the action of the CPT operator Θ introduced before reads:

$$\Theta\,|B_\alpha\rangle = \sum_i \left(B^i_\alpha\right)^*\,|\beta_{\mathrm{i}^+}\rangle\rangle \tag{95}$$

the cylinder amplitude between two boundary states of the form (93) can be expressed as follows:

$$\tilde{Z}_{\alpha\beta}(t) = \langle\Theta B_\alpha| e^{-2\pi l(L_0+\bar{L}_0-\frac{c+\bar{c}}{24})} |B_\beta\rangle \tag{96}$$

$$= \sum_{i,j} B^j_\alpha B^i_\beta \langle\langle\beta_j| e^{-2\pi l(L_0+\bar{L}_0-\frac{c+\bar{c}}{24})} |\beta_{\mathrm{i}}\rangle\rangle \tag{97}$$

$$= \sum_i B^i_\alpha B^i_\beta \chi_i(2il) \tag{98}$$

Performing a modular S-transformation $l \mapsto \frac{1}{2t}$ on the characters χ_i , this closed sector cylinder diagram is transformed to the following expression in the open sector:

$$\tilde{Z}_{\alpha\beta}(l) \to \tilde{Z}_{\alpha\beta}\left(\frac{1}{2t}\right) = \sum_{i,j} B^i_\alpha B^i_\beta S_{ij}\chi_j(2il) = \sum_j n^j_{\alpha\beta}\chi_j(it) = Z_{\alpha\beta}(t) \tag{99}$$

where S_{ij} is the modular S-matrix and where we introduced the new coefficients $n^j_{\alpha\beta}$. Now, the Cardy condition is the requirement that this expression can be interpreted as a partition function in the open sector. That is, for all pairs of boundary states $|B_\alpha\rangle$ and $|B_\beta\rangle$ in a RCFT, the following combinations have to be non-negative integers:

$$\boxed{n^j_{\alpha\beta} = \sum_{i,j} B^i_\alpha B^i_\beta S_{ij} \in Z^+_0} \tag{100}$$

Construction of Boundary States

The Cardy condition just illustrated is very reminiscent of the Verlinde formula, where a similar combination of complex numbers leads to non-negative fusion rule coefficients. For the case of a charge conjugate modular invariant partition function, that is, when the characters $\chi_i(\tau)$ are combined with $\bar{\chi}_{i^+}(\bar{\tau})$ as $\mathrm{Z} = \sum_{\mathrm{i}} \chi_i(\tau)\bar{\chi}_{i^+}(\bar{\tau})$, we can construct a generic solution to the Cardy condition by choosing the reflection coefficients in the following way:

$$\boxed{B^i_\alpha = \frac{S_{\alpha i}}{\sqrt{S_{0i}}}} \tag{101}$$

Note, for each highest weight representation ψ_i in the RCFT, there exists not only an Ishibashi state but also a boundary state, i.e. the index α in $|B_\alpha\rangle$ also runs from one to the number of HWRs. Employing then the Verlinde formula:

$N_{ij}^k = \sum_{m=0}^{N-1} \frac{S_{im}S_{jm}S^*_{mk}}{S_{0m}}$,and denoting the non-negative integer fusion coefficients by $N_{j\beta}^{\alpha}$, we find that the Cardy condition for the coefficients $n_{\alpha\beta}^j$ is always satisfied:

$$n_{\alpha\beta}^j = \sum_i \frac{S_{\alpha i}S_{\beta i}S_{ij}}{S_{0i}} = \sum_i \frac{S_{\alpha i}S_{\beta i}S^*_{ij^+}}{S_{0i}} = N_{\alpha\beta}^{j^+} \in Z_0^+ \tag{102}$$

Note that here we employed $S^*_{ij} = S_{ij^+}$ which is verified by noting that $S^{-1} = S^*$.

4 G-function

Now we are going to discuss how the presence of boundary change the value of partition function. A good indicator is the so-called g-function. It was introduced by Affleck and Ludwig as a measure of the ground state degeneracy of a conformal boundary condition.

Ground State Degeneracy

Consider the partition function of a classical statistical-mechanical system de?ned on a cylinder of length l. Among the characteristics of the model might be boundary scales depending on the boundary conditions α and β imposed at the two ends of the cylinder; we will highlight the role of these quantities by denoting the partition function: $Z_{\alpha\beta}(l)$ If l is taken to in?nity , then

$$Z_{\alpha\beta}(l) \sim A_{\alpha\beta} e^{-lE_0} \tag{103}$$

where E_0 is the ground state energy of the model. To derive this asymptotic l-

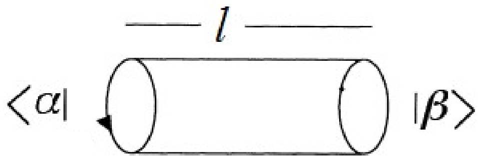

Figure 8: The space-time is periodic and has boundary states at the ends

dependence, it is sufficient to treat the boundary conditions as boundary states $|\alpha\rangle$ in a formalism where time runs along the length of the cylinder, and states are propagated by a bulk Hamiltonian $H_{circ.}$:

$$Z_{\alpha\beta}(l) = \langle\alpha| \exp(-lH_{circ.}) |\beta\rangle \tag{104}$$

At large l the contribution of the ground state $|0\rangle$ dominates, establishing Eq.(103) and also giving

$$A_{\alpha\beta} = \frac{\langle\alpha \mid 0\rangle \langle 0 \mid \beta\rangle}{\langle 0 \mid 0\rangle} \tag{105}$$

The inner products appearing in Eq.(105) should in general contain a term corresponding to a free-energy per unit length f_0, i.e.

$$\log \frac{\langle \alpha \mid 0 \rangle}{\langle 0 \mid 0 \rangle^{1/2}} = -f_0 + \log g_\alpha \tag{106}$$

Where g_α does not depend on length or mass or any energy scale, *only depends on boundary.*

This is to say, if we consider the logarithm of the partition function $Z_{\alpha\beta}$, the effect of boundaries leads to one extra term $logg$, i.e.

$$\log Z_{\alpha\beta} \to \log Z_{\alpha\beta} + \log g, \qquad Z_{\alpha\beta} \to g \cdot Z_{\alpha\beta} \tag{107}$$

For g is defined as

$$g = \langle 0 \mid \alpha \rangle \langle \beta \mid 0 \rangle, \qquad g = g_\alpha g_\beta, \qquad g_\alpha = \langle 0 \mid \alpha \rangle \tag{108}$$

the g-function is called the *ground state degeneracy.* For the example of free boson, the torus partition function is given by:

$$Z^T_{bos.}(t) = \frac{1}{\eta(it)} \tag{109}$$

and the cylinder partition function for Neumann-Neumann boundary conditions is given by:

$$Z^{C(\mathrm{N,N})}_{bos.}(t) = \frac{1}{2\sqrt{t}} \frac{1}{\eta(it)} \tag{110}$$

Using the method above, the g-function could be calculated: $g = \frac{1}{\sqrt{2}}$, which means the effect of boundaries change the value of partition function by $\frac{1}{\sqrt{2}}$.

Cardy Condition Revisited

Once again, let us consider the cylinder partition function with Verlinde formula:

$$Z^0_{\alpha\beta} = Tr(e^{\frac{\pi}{l}(L_0 - c/24)}) = \sum_i n^i_{\alpha\beta} \chi_i \tag{111}$$

Now, the infinite limit of l could be expressed as modular transformation for the character:

$$\chi_i = \sum_j S_{ij} \chi_j \tag{112}$$

at infinite limit of l, only ground state contributes to the expression, thus:

$$Z_{\alpha\beta} \to Tr(e^{\frac{\pi}{l}(L_0 - c/24)}) \sum_i n^i_{\alpha\beta} S_{i0} \to g \cdot Z^0_{\alpha\beta} \tag{113}$$

with

$$\boxed{g = g_\alpha g_\beta = \sum_i n^i_{\alpha\beta} S_{i0}} \tag{114}$$

We can interpret this result as: the consistency of this formula with $g_\alpha = \langle 0 \mid \alpha \rangle$ puts certain constrain on the possible boundary states and the coefficients n. This constraint is nothing but ***Cardy Condition***.

References

- V. Schomerus, *Lectures on branes in curved backgrounds*, Class. Quant. Grav. 19, 5781 (2002) [arXiv:hep-th/0209241].

- J.L. Cardy, *Effect of boundary conditions on the operator content of two-dimensional conformally invariant theories*, Nucl. Phys. B 275, 200 (1986).

- J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys. B 324, 581 (1989).

- I. Affleck, A.W.W. Ludwig, *Universal noninteger 'ground state degeneracy' in critical quantum systems*, Phys. Rev. Lett. 67, 161 (1991).

-R. Blumenhagen, E. Plaushinn, *Introduction to conformal field theory with application to string theory*,Springer,(2009)