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Proseminar in Theoretical Physics

Gauge/Gravity Duality

The AdS_5/CFT_4 Correspondence

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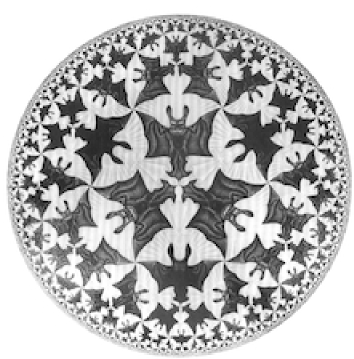


Abstract
The introductory point consisting of a description of 't Hooft's large N limit and its relation with string theory, a quick jump is then made towards the exposition of Maldacena's argument of the AdS_5/CFT_4 duality. The holographic principle is introduced and analysed in the context of the correspondence. Finally, correlation functions, in the context of the AdS_{d+1}/CFT_d, are studied for scalar fields and the simple examples are computed in detail.

Contents

1 Introduction

This document reports the presentation given on the 27^{th} of May at ETH Zürich in the frame of the *Proseminar in Theoretical Physics: Conformal Field Theory and String Theory*, coordinated by the Prof. Matthias Gaberdiel.
The objective of this talk as intended by its author was to present the basic features of the $\text{AdS}_5/\text{CFT}_4$ correspondence, namely:

- a brief description of the first ideas that lead to conjecture a gauge/string duality (the 't Hooft limit, following [1],[9])

- the emergence of the conjecture from the D3-brane dual low energy description in terms of a maximally supersymmetric gauge theory on the world-volume and its backreaction on the geometry (following mainly [1])

- Anti-de-Sitter space and its features (following [1],[9],[2])

- a brief account for the precise statement of the strong form of the duality and its formulation (following [2],[1])

- the realisation of the duality as a boundary-bulk relation and how holography may be incorporated (following [2])

- a brief description of the holographic ideas and their realisation in AdS/CFT (following [6],[4],[1])

- AdS/CFT correlation functions for scalar fields (following [2],[9]).

The conjecture of the AdS/CFT correspondence was first made by Juan Maldacena in 1997, and since then it has been subject of intense research. One of the main concerns of today 's research on the subject is integrability, which will be briefly accounted for towards the end of this report.

Firstly, a brief exposition of 't Hooft's ideas on gauge/string duality will be presented as a hint to what may such a duality look like, and how it may be formulated. This is done via an expansion in terms of Feynman diagrams on the large N limit of a $U(N)$ gauge theory, mainly following [9].

In this document the argument presented in [1] is followed intending to conjecture the duality between Type IIB String Theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ U(N) gauge theory in $\mathbb{M}^{1,3}$ Minkowski space. In this argument, the low energy description of D3-branes as classical p-dimensional supergravity solutions (black p-branes) in 10-dimensional type IIB string theory and their dual description as a $\mathcal{N} = 4$ Super-Yang Mills gauge theory in the 4-dimensional world-volume with a 10-dimensional type IIB string theory in the bulk, play a fundamental role[1]. Indeed, it is the analysis of a stack of N D3-branes in a low energy limit in both descriptions that allows the conjecture to relate a gravity-including string theory in a non-trivial background created by the stack and the gauge theory living in the stack world-volume itself.

Among these considerations, a brief presentation of the 5-dimensional Anti-de Sitter space (AdS_5) will be given. This will be done by considering the usual description of AdS_5 as a hypersurface in $\mathbb{M}^{2,4}$ 6-dimensional Minkowski space. A brief account for different patches will be given, emphasising the Poincaré patch, which is the one naturally arising when considering the near-horizon limit of the D3-branes geometry.

Supported by this, the strong form of the conjecture is made and a brief account of its consequences and first matches is done. In particular, the symmetry groups of both theories are compared (the bosonic subgroup only), and certain tractable regimes are briefly studied. Also, the correspondence is seen to be realised in terms of a boundary-bulk formulation where the bulk string fields couple as sources of boundary CFT local operators (as given in [2]), allowing for an explicit computation of correlation functions. This will be done in the final chapter of this report for the case of a massless scalar field, following [2], and the result for the massive case will be presented. Then, the general perturbative method to compute n-point correlation functions is exposed, as given in [9], and bulk-to-bulk propagators are introduced. With this the exposition of the subject ends, as well as the report.

Before the final chapter, a small detour on the holographic principle and its basic arguments is done; then, this principle is seen to be realised explicitly in the context of the AdS_5/CFT_4 correspondence, as in [1] and [2], with the usual regularised-boundary techniques.

I would like to thank Cristian Vergu for all his help as a supervisor of this work and good hints for the presentation, as well as Prof. Gaberdiel for the opportunity of working on this subject.

[1]This duality of descriptions was first conjectured in Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, hep-th/9510017v3.

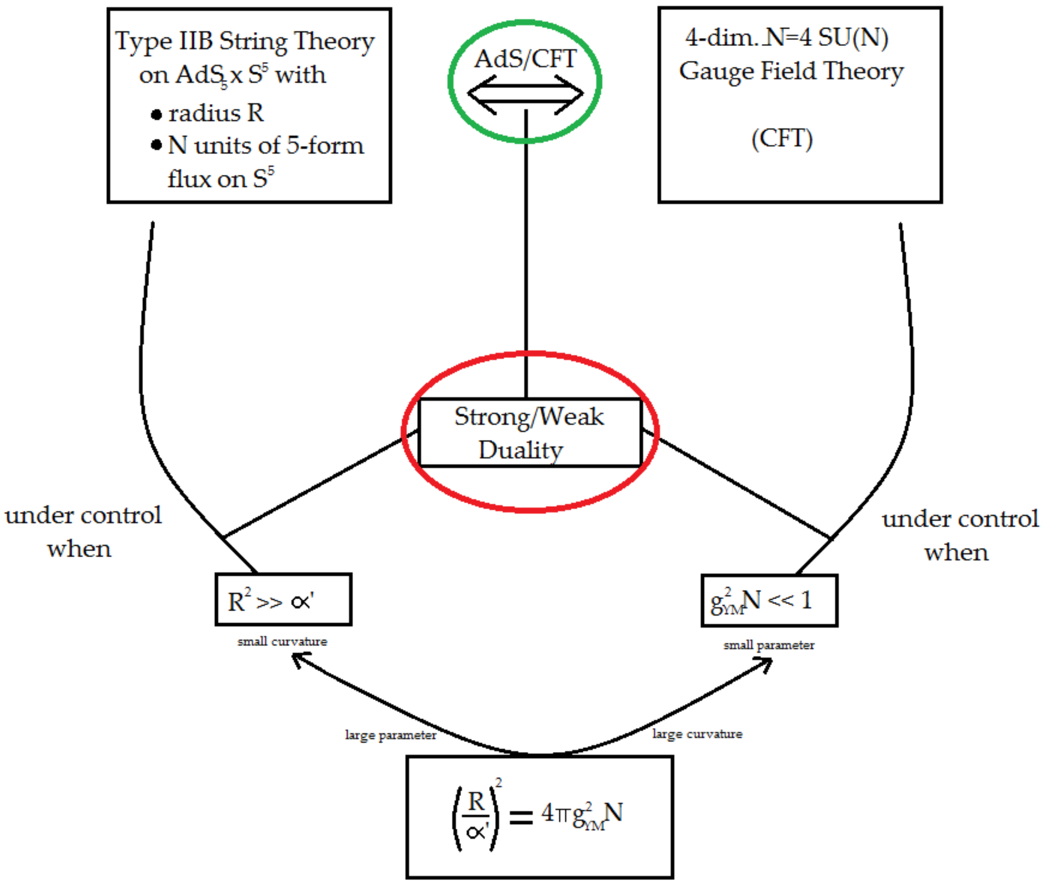


Figure 1: The AdS_5/CFT_4 correspondence. The duality allows in principle to solve theories in hard regimes using the dual theory in a soft regime. The link between the different regimes between the theories is made by the use of the relation on the bottom of the figure, issued from the black p-brane calculations.

2 Large N Gauge Theories as String Theories

In this section, we follow [9] on the analysis first made by 't Hooft of large N gauge theories with gauge group U(N). This analysis proceeds by considering a perturbative $1/N$ expansion in terms of Feynman diagrams with a slight change in notation with respect to the usual Yang-Mills notation. It will be seen explicitly that for vacuum bubbles made up of adjoint fields this leads to an expansion in topological triangulations where topologies are suppressed by N^{-2g} factors, where g is the genus. The main contributions are therefore given by sphere-triangulations, i.e. planar graphs. This corresponds to the usual closed interacting strings g_s expansion in terms of worldsheet topologies.

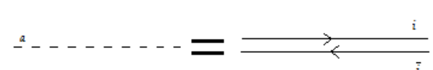
2.1 Double Line Notation and 't Hooft Limit

Consider a U(N) Yang-Mills gauge theory with gauge fields A_μ^a in the adjoint representation, coupling constant g_{YM} and some general $\mathcal{L}_{matter}$ part of the Lagrangian composed of matter fields. Write the Lagrangian density in the usual form as

$$\mathcal{L} = \frac{1}{g_{YM}^2} \left(Tr(F_{\mu\nu}F^{\mu\nu}) + \mathcal{L}_{matter} \right), \tag{1}$$

where the field strength is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_{YM} [A_\mu, A_\nu]$.

Define the 't Hooft parameter to be $\lambda \equiv g_{YM}^2 N$. The 't Hooft limit is defined as letting $N \to \infty$ while keeping λ fixed. It is possible to see that the Lagrangian density goes as $\mathcal{L} \sim N/\lambda$, and this will be used later.



As known, the $U(N)$ group adjoint representation can be written as a direct product between the fundamental and the anti-fundamental representations. Use this fact by labelling gauge fields with two indices, A^i_j, each corresponding to each representation, $i, j = 1, ..., N$. In diagram notation, a gauge field is then written as two lines with opposite orientation, corresponding to the fundamental and anti-fundamental representations indices[2], as in the figure above.

In this way, the Feynman graphs used when performing perturbative expansions on this theory will be networks of double lines, if only the gauge fields are considered, by the form of the gauge fields propagator:

$$\langle A^i_j A^k_l \rangle \propto \delta^i_l \delta^k_j .$$

The figure below is the diagram corresponding to the gauge field self-energy, in the usual and new notations.

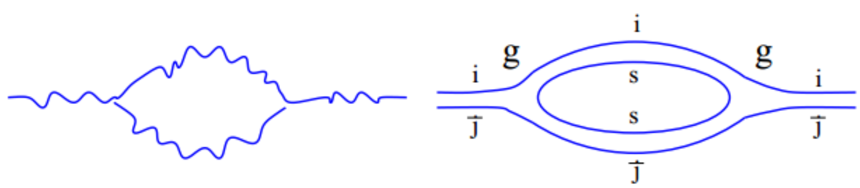


Figure 2: Gluon field self-energy, as in [9].

This diagram is of order $O(g^2_{YM}N)$, as each propagator contributes with a factor g^2_{YM}, each 3-gluon vertex with g^{-1}_{YM} and the loop index runs between N possible values. In the 't Hooft limit, this diagram receives therefore no divergent contributions.

2.2 Perturbative Expansion and Link with Strings

Keeping in mind the form of the Lagrangian as

$$\mathcal{L} = \frac{N}{\lambda} \left(Tr(F_{\mu\nu}F^{\mu\nu}) + \mathcal{L}_{matter} \right),$$

it is easy to derive a set of Feynman rules that allows to write the contributions given by connected vacuum diagrams of gauge fields.

By the considerations given before, these graphs can be seen as corresponding to compact, closed, oriented surfaces. One useful way of understanding this fact is by considering the double line diagrams as topological simplicial triangulations. In the figure below this is explained for two different vacuum bubble graphs.

[2] Each line is made to correspond to an index, and not to a field as usual in quantum field theories

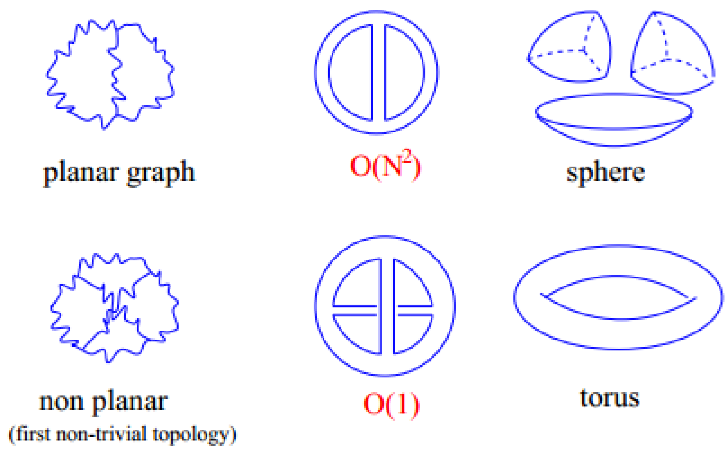


Figure 3: Some graphs in the double line notation, as in [9].

The Feynman rules read:

- N/λ for each vertex (V)

- λ/N for each propagator (E, edge)

- N for each loop (F, face)

where *edge* is used to denote propagators as it corresponds to the object connecting two vertices, and *face* is used to denote loops as it corresponds to a succession of propagators/edges that close on themselves with no propagators/edges inside. Therefore each graph contributes with

$$N^{V+F-E}\lambda^{E-V} = N^{\chi}\lambda^{E-V}, \tag{2}$$

where χ is Euler's characteristic of the 2-dimensional surface represented by the graph. For closed oriented surfaces we can write $\chi = 2 - 2g$, with g the surface genus. The sum over all Feynman graphs therefore takes the form

$$\sum_{g\ fixed} \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) = \sum_{g\ fixed} \sum_{g=0}^{\infty} \left(\frac{1}{N}\right)^{2g} N^2 f_g(\lambda),$$

and the 't Hooft limit corresponds to a perturbative expansion in even negative powers of N. This means that higher genus surfaces are suppressed by factors of

N^{-2g} and therefore the leading contributions come from planar graphs that have the topology of a sphere $g = 0$. Remark that no assumption is made about how to sum over all graphs of a given genus, but the conclusion remains.

This corresponds remarkably to the closed string perturbative g_s expansion in worldsheet topologies, with the identification $g_s \sim 1/N$. This expansion arises in string theory when considering closed oriented strings interactions by summing not only over possible worldsheet metrics but also possible topologies. We have then

$$S_{string} = S_{Polyakov} + \phi\chi, \qquad \chi = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}R,$$

and by Gauss-Bonnet theorem χ is just Euler's characteristic, a topologically invariant number for 2-dimensional surfaces. When summing over possible worldsheet topologies, $g_s \equiv e^{\phi}$ plays the role of a coupling constant and the sum may be seen as a perturbative expansion.

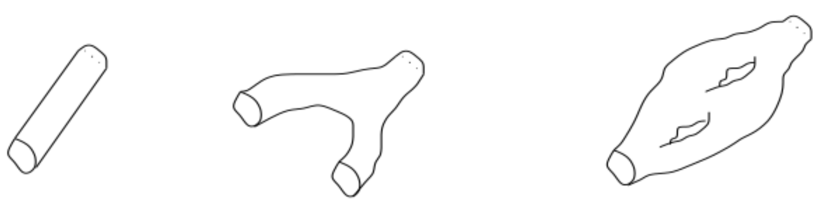


Figure 4: String interactions and topological expansion.

To compute connected n-point correlation functions for some operators O_j consider the transformed action[3] $S \to S + N\sum_j J_j O_j$, for any arbitrary sources J_j, and write in the 't Hooft limit

$$\langle\prod_{j=1}^{n} O_j\rangle = (iN)^{-n}\left[\frac{\delta^n W}{\prod_{j=1}^{n}\delta J_j}\right]_{J_j=0} \propto N^{2-n}, \tag{3}$$

where W is the generating functional of connected graphs. It is remarkable to see that 2-point functions come out canonically normalised and 3-point functions are proportional to $1/N$, so that indeed in this limit $1/N$ is the coupling constant.
One can easily be convinced that the addition of matter fields in the fundamental representation of $U(N)$ corresponds to the effect of open strings. Indeed, as

[3]For consistency the additional term is made to scale in the same way as the first original term in the 't Hooft limit.

fundamental fields carry only one index, in our notation they would correspond to the presence of boundaries in the surfaces representing the Feynman diagram expansion. Then again the expansion would be made in terms of the Euler's characteristic $\chi = 2 - 2g - b$, where b is the number of independent boundaries, and it would correspond to a g_s string theory expansion, for both open and closed strings. Remark that now, contrary to the previous case, the expansion contains terms in odd powers of $1/N$.

It is finallly important to take away from this section that it is possible to obtain a similar treatment as the one obtained in standard string theory by analysing a special limit in a gauge theory. It has not been treated, and this method does not provide it, any way of explicitly deriving such possible correspondence between the two theories, this is rather the purpose of the AdS/CFT.

3 D3-Branes: Gauge Theories and Classical Super-Gravity Solutions

This section introduces a paradigm that is central to the AdS/CFT correspondence. This paradigm is a dual description of a stack of N D3-Branes, which will then lead to the gauge/gravity duality of the correspondence.

In a first part a description of the gauge theory living in the world-volume is made, with an emphasis on its symmetries. Indeed, in a low energy limit when N parallel D-branes are made to coincide, the usual $U(1)$ gauge theory living on each brane enhances the system's gauge symmetry from $U(1)^N$ to $U(N)$. In this process enter the Chan-Paton factors, which label the brane on which each end of the strings live, and that in the end become the indices of the final gauge group. The gauge theory surviving in the D3-branes world-volume inherites the space-time supersymmetry of type IIB string theory, but not fully as D-branes break half the supersymmetries. Therefore, this gauge theory contains 16 conserved supercharges; being a 4-dimensional theory, it is constrained to have supersymmetry rank $\mathcal{N} = 4$. The Lagrangian of this theory is completely settled by these considerations and the final unique theory is called $\mathcal{N} = 4$ Super-Yang Mills theory.

In a second part of the section, same system of N D3-branes is again described but using a different analysis. Given the natural emergence of gravity from string theory, and D-branes being charged massive objects in this theory, the question naturally arises of knowing what is the geometric meaning of a stack of D-branes. In particular, using the low energy limit of superstring theory, namely supergravity, it is possible to replace the D-branes influence on the theory by a non-trivial geometry[4] and let strings react to this non-trivial background. This is much in the same way as a very familiar dicotomy in physics, which can be grasped by considering the example of an electron-proton scattering process. Indeed, there are two different ways of describing this process, at least in a low energy limit. The first one is by considering and summing over all possible Feynman diagrams that contribute to the process, find an expression for the S-matrix using the usual QED methods and obtaining a scattering amplitude. The second is to consider that the proton creates a static electric field in its surroundings and that the electron reacts to this non-trivial space by scattering through it[5], i.e. by being free in a deformed background and interacting with it. This is indeed the perfect analogy of what will be done when describing the N D3-branes by its geometry backreaction.

[4]In opposition to the usual flat 10-dimensional background considered in superstring theory

[5]Taking the simplistic and helpful consideration that the proton does not move.

3.1 $\mathcal{N} = 4$ Super-Yang Mills U(N) Theory

As discussed, the theory living on the world-volume of a stack of D3-branes in a low energy limit is a supersymmetric non-abelian gauge theory $\mathcal{N} = 4$ with gauge group U(N).
This theory contains the following fields:

- λ_α^a, $\alpha = 1, 2$, $a = 1, ..., 4$ left Weyl fermionic fields

- X^i, $i = 1, ..., 6$ real scalar fields - $SO(6) \sim SU(4)$

- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right]$, A_μ gauge fields with field strength $F_{\mu\nu}$

- $D_\mu \lambda = \partial\lambda + i\left[A_\mu, \lambda\right]$, a covariant derivative.

The 6 real scalar fields X^i transform between themselves by a SO(6) rotation, which appears from the breaking of Lorentz symmetry by the D3-branes, namely $SO(1, 9) \longrightarrow SO(1, 3) \times SO(6)$, the first part being related to the fields tangent to the D3-branes that live in a theory with Lorentz symmetry in the world-volume, and the second part representing the transversal dynamics, the fields X^i labelling these type of excitations of the branes.

The Lagrangian is completely settled by supersymmetry:

$$\mathcal{L} = Tr\left(-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a \right.$$
$$- \sum_i D_\mu X^i D^\mu X_i + \sum_{a,b,i} gC_i^{ab}\lambda_a\left[X^i, \lambda_b\right]$$
$$\left. + \sum_{a,b,i} g\bar{C}_{iab}\bar{\lambda}^a\left[X^i, \bar{\lambda}_b\right] + \frac{g^2}{2}\sum_{i,j}\left[X^i, X^j\right]^2\right),$$

where $\tilde{F}^{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is the dual field strength and θ_I is the instanton angle. The constants C_i^{ab} are Clebsch-Gordon coefficients relating fields of different spins and the Yang Mills coupling g is related to the string coupling by $g^2 \sim g_s$.

The details of this theory, including the consequences of such Lagrangian density, are not of fundamental importance in the context of this report, except for the few following points. Firstly, it can be seen by dimensional analysis that

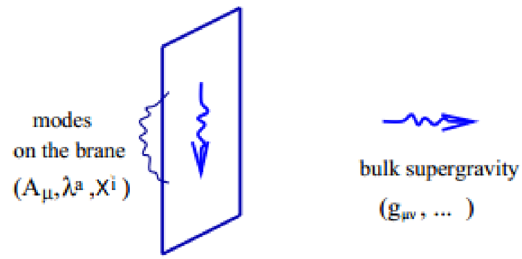


Figure 5: The brane modes composing $\mathcal{N} = 4$ SYM and low energy limit describing the bulk modes, 10d supergravity.

this theory is renormalisable. But this isn't all, it can be shown that the theory has a vanishing β-function. This means that this theory is actually a Conformal Field Theory, or CFT. Also, it preserves all its symmetries at the quantum level, namely:

- R-symmetry $SO(6) \sim SU(4)$, with 15 traceless hermitian generators $T^a_{\ b},\ a, b = 1, ..., 4$

- conformal symmetry in 4d $SO(2, 4)$ with generators $P_\mu, L_{\mu\nu}, D, K_\mu$

- $\mathcal{N} = 4$ Poincaré Supersymmetry

These symmetries compose the superconformal group $SU(2, 2|4)$ of this gauge theory, and the superalgebra may be represented by a diagonal bosonic part and an off-diagonal fermionic part:

$$\left(\begin{array}{c|c} P_\mu, K_\mu, L_{\mu\nu}, D & Q^a_\alpha, \bar{S}^a_{\dot{\alpha}} \\ \hline \bar{Q}_{\dot{\alpha}a}, S_{\alpha a} & T^a_{\ b} \end{array}\right)$$

where Q^a_α, $\bar{Q}_{\dot{\alpha}a}$ are the usual supersymmetry generators and $S_{\alpha a}$, $\bar{S}^a_{\dot{\alpha}}$ are the generators given by the commutation relations of Q^a_α, $\bar{Q}_{\dot{\alpha}a}$ with the special conformal transformations K_μ, and that therefore close the algebra.

3.2 Classical Super-Gravity Solutions

Historically this subject didn't develop as presented in this report. In fact, the conjecture of the D-branes existence was proposed in parallel of the discovery of the black p-brane solution of supergravity that is exposed here. It was only in 1995 that J. Polchinski conjectured the duality of the description, i.e. that in fact this solution corresponds to D-branes[6]. The exposition made in this report is based in a full aknowledgement of this conjecture.

3.2.1 Black p-Branes

As known, Dp-branes are massive charged objects. Consider the classical supergravity action

$$S = \frac{1}{(2\pi)^7\alpha'^4}\int d^{10}x\sqrt{-g}\left(e^{-2\phi}(R+4(\nabla\phi)^2) - \frac{2}{(8-p)!}F_{p+2}^2\right), \qquad (4)$$

where $F_{p+2} = dA_{p+1}$ is the field strength associated to a A_{p+1} potential obtained in superstring theory. In the case of interest here, type IIB string theory, there are a 2-rank tensor and a 4-rank tensor resulting from the theory, i.e. $p = 1, 3$. The case which corresponds to D3-branes, $p = 3$, which will be detailed in this report, corresponds to the 4-rank tensor in the Ramond-Ramond sector of superstring theory, with self-dual field strength $F_5 = *F_5$[7].

Look for a p-dimensional solution carrying N charges[8] with respect to this potential[9]

$$N = \int_{S^{8-p}} *F_{p+2},$$

and require some desirable symmetries: euclidean symmetry ISO(p) in p dimensions along the Dp-branes, and spherical symmetry SO(9-p) in the (9-p) transversal directions.

This reduces the problem to the finding of a spherically symmetric charged black hole static solution in (10-p) dimensions. In general relativity this corresponds to the Reissner-Nördstrom solution of a static, charged black-hole. In this case, the situation is more complex but the solution has some similarities:

$$ds^2 = -\frac{f_+(\rho)}{\sqrt{f_-(\rho)}}dt^2 + \sqrt{f_-(\rho)}\sum_{i=1}^{p} dx^i dx^i + \frac{f_-(\rho)^{-\frac{1}{2}-\frac{5-p}{7-p}}}{f_+(\rho)}d\rho^2 + \rho^2 f_-(\rho)^{\frac{1}{2}-\frac{5-p}{7-p}}d\Omega_{8-p}^2, \qquad (5)$$

[6] In *Dirichlet-Branes and Ramond-Ramond Charges*, November 1995, arXiv-hep-th/9510017v3

[7] This condition has actually to be added to the action (4), as it is not implied by it.

[8] Each Dp-brane carries one unit of charge.

[9] The dimension of the world-volume of the object and the rank of the potential coincide, as required by a minimal coupling.

where

$$f_\pm = 1 - \left(\frac{r_\pm}{\rho}\right)^{7-p},$$

and the parameters $r_\pm$ are related to the mass and charge of the N Dp-branes by

$$M = \frac{1}{(7-p)(2\pi)^7 d_p l_P^8}\left((8-p)r_+^{7-p}) - r_-^{7-p}\right), \tag{6}$$

$$N = \frac{1}{d_p g_s l_s^{7-p}}(r_+ r_-)^{\frac{7-p}{2}}. \tag{7}$$

Here ρ is the radial direction in the (9-p) spherically symmetric submanifold, and the x^i parametrise the Dp-branes. The symbol l_P is Planck's constant in 10 dimensions and d_p is the area of the unit (p-1)-sphere.

It is possible to see that there is a horizon at $\rho = r_+$. Indeed, use the metric to compute

$$g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\frac{f_+(\rho)}{\sqrt{f_-(\rho)}} = 0 \text{ at } \rho = r_+.$$

Also, for $p \leq 6$, there is a curvature singularity at $\rho = r_-$. It is possible to see it by computing some frame-invariant quantity, as the Kretschmann scalar $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and see its divergence when $\rho = r_-$. This is done in the same way, and yielding a similar result, as in simpler Schwarzschild solution.

By the cosmic censorship hypothesis, which states that no naked singularities should exist[10], it is imperative to have $r_+ \geq r_-$. This implies a lower bound to the mass, using (7) and (6),

$$M \geq \frac{N}{(2\pi)^7 g_s l_s^{p+1}}.$$

where it has been used $l_P^4 = g_s l_s^4$. For supersymmetry reasons[11] we retain the solution with $r_+ = r_- \equiv R$, called the extremal solution, where the horizon and the curvature singularity are made to coincide. All care is necessary to treat this case as the supergravity description becomes inadequate at a certain distance of the singularity, as the curvature becomes too high. For the case p=3, which will be the case in interest, the situation is nevertheless more gentle as the $\rho = r_+$ surface is regular and it is possible to find a smooth analytic extension of the solution. Writing everything in this extremal case, the solution (5) becomes

[10]This is usually linked to the requirement of having a well-posed Cauchy problem.

[11]The found inequality is related to the BPS bound with respect to the 10 dimensional space-time supersymmetry.

$$ds^2 = \sqrt{f_+(\rho)}\left(-dt^2 + \sum_{i=1}^{p} dx^i dx^i\right) + f_+(\rho)^{\frac{3}{2}-\frac{5-p}{7-p}} d\rho^2 + \rho^2 f_+(\rho)^{\frac{1}{2}-\frac{5-p}{7-p}} d\Omega_{8-p}^2, \quad (8)$$

where now

$$f_+(\rho) = 1 - \left(\frac{R}{\rho}\right)^{7-p}.$$

The symmetry ISO(p) is enhanced to the Poincaré symmetry ISO(1,p), which is rather pertinent as theories in Dp-branes should carry this kind of spacetime symmetry in the limits which are of interest to us. Using equation (7) in the extremal case, it is possible to get an expression for R:

$$R^{7-p} = d_p g_s l_s^{7-p}. \quad (9)$$

Define a new coordinate $r^{7-p} = \rho^{7-p} - R^{7-p}$; notice that the horizon is now at $r = 0$. Also, define $H(r) = [f_+(\rho(r))]^{-1}$:

$$H(r) = \frac{1}{1 - \left(\frac{R}{\rho}\right)^{7-p}} = \frac{1}{1 - \frac{R^{7-p}}{r^{7-p}+R^{7-p}}} = \frac{r^{7-p} + R^{7-p}}{r^{7-p}} = 1 + \left(\frac{R}{r}\right)^{7-p}.$$

Focus on the p=3 case, corresponding to D3-branes. Then using

$$d(\rho^4) = d(r^4) \Longrightarrow d\rho^2 = \left(\frac{r^3}{\rho^3}\right)^2 dr^2 = \frac{r^6 dr^2}{(r^4 + R^4)^{3/2}}$$

$$\Longrightarrow d\rho^2 = \frac{1}{(1 + \frac{R^4}{r^4})^{3/2}} dr^2 = H(r)^{-3/2} dr^2,$$

and also $\rho^2 = (r^4 + R^4)^{1/2} = r^2 H(r)^{1/2}$, the solution (8) reads

$$ds^2 = \frac{1}{\sqrt{H(r)}} \eta_{\mu\nu} dx^\mu dx^\nu + H(r)^2 H(r)^{-3/2} dr^2 + r^2 H(r)^{1/2} d\Omega_5^2$$

$$\Longrightarrow ds^2 = \frac{1}{\sqrt{H(r)}} \eta_{\mu\nu} dx^\mu dx^\nu + \sqrt{H(r)}(dr^2 + r^2 d\Omega_5^2). \quad (10)$$

Before continuing, it is useful to remark that this supergravity description is adequate in a low curvature limit, i.e. when $R \gg l_s$. Using the relation

$$R^4 = 4\pi N g_s l_s^4, \quad (11)$$

obtained from (9) by using $d_3 = 4\pi$, it is easy to conclude that this description is valid when

$$1 \ll g_s N \ll N, \tag{12}$$

where the last relation arises from the fact that we are always implicitly considering g_s to be small and a perturbative expansion on g_s to be meaningful. Therefore by this relation this is also a large N limit of the system[12].

It is interesting, as well as fundamental, to remark that this decription is dual to the one obtained in the previous section via a gauge theory in the world-volume of the D3-branes. Indeed, a perturbative treatment of this gauge theory is comprehensible and meaningful when the effective parameter is small enough, i.e. when $g_s N \ll 1$. This is perfectly incompatible, and thus *dual*, with the regime found in (12). This is a central statement to the gauge/gravity duality.

3.2.2 Near Horizon Limit and Redshift

In the next chapter the near horizon limit of this geometry will be needed. As already mentioned, the horizon is at $r = 0$, so that a meaningful limit to take near the horizon is $r/R \ll 1$. In this limit,

$$\sqrt{H(r)} = \sqrt{1 + \frac{R^4}{r^4}} = \frac{R^2}{r^2}\sqrt{\frac{r^4}{R^4} + 1} \simeq \frac{R^2}{r^2},$$

and therefore the solution represents a geometry which in this limit is

$$ds^2 = \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2. \tag{13}$$

This is the metric of the product space of 5-dimensional Anti-de Sitter space with a 5-dimensional sphere, $\mathrm{AdS}_5 \times \mathrm{S}^5$, both with radius R. The 5-sphere is trivial to see in this solution, as it constitutes the third term. For those unfamiliar with Anti-de Sitter space, the next subsection should be useful to clarify this aspect.

Another important aspect of this geometry is the redshift it produces. Consider two observers (r, t) and (r', t'), describing some line element such that

$$ds^2 = ds'^2 \Longrightarrow \frac{-dt^2}{\sqrt{H(r)}} = \frac{-dt'^2}{\sqrt{H(r')}}$$

[12] It is evident that it is easier to treat the system when the number N of D-branes, and thus the total charge and mass, are large, since in this way it is easier to clear-cut it from quantum influences.

$$\Longrightarrow \frac{dt'}{(H(r'))^{1/4}} = \frac{dt}{(H(r))^{1/4}}.$$

Send one of the radial coordinates to infinity, so that $H(r) \simeq 1$, and remark that there will be an associated redshift $[H(r)]^{-1/4}$. Since the redshift goes to zero when $r \longrightarrow 0$, i.e. near the horizon, an observer at infinity see states in this region as having a very low energy no matter the energy the state has in the string frame. This will be important to keep in mind in the next section.

3.3 5-dimensional Anti-de Sitter Space

Anti-de Sitter space has a variety of descriptions, one of the most interesting ones being that it is a maximally symmetric[13] solution to Einstein field equations with a negative cosmological constant. Nevertheless, for the sake of brevity and objectivity, the description used here is the one of a 5-dimensional hypersurface in a 2+4-dimensional Minkowski space.

Consider for that matter the metric in $\mathbb{M}^{2,4}$ in cartesian coordinates $(x^i)_{i=0}^5$

$$ds^2 = -dx_0^2 - dx_5^2 + \sum_{i=1}^{4} dx_i^2,$$

and the hypersurface given by

$$x_0^2 + x_5^2 - \sum_{i=1}^{4} x_i^2 = R^2,$$

for some real positive R^2.

3.3.1 Global Coordinates and General Features

Introduce a new set of coordinates (τ, ρ, Ω_i), $\rho \geq 0$, $\tau \in [0, 2\pi)$, defined by

$$x_0 = R \cosh\rho \cos\tau, \qquad x_5 = R\cosh\rho \sin\tau$$

$$x_i = R \sinh\rho \; \Omega_i, \quad i = 1, ..., 4 : \quad \sum_i \Omega_i^2 = 1,$$

that solve the quadric equation defining AdS_5. Then

$$dx_0^2 = R^2 \left(\sinh\rho \cos\tau \; d\rho - \cosh\rho \sin\tau \; d\tau\right)^2$$

[13]Carrying the same number of isometry generators as Euclidean space.

$$dx_5^2 = R^2 \left(\sinh\rho \sin\tau \; d\rho - p + \cosh\rho \cos\tau \; d\tau\right)^2$$
$$dx_i^2 = R^2 \left(\cosh\rho \; \Omega_i \; d\rho + \sinh\rho \; d\Omega_i\right)^2 .$$

Substituting this results in the Minkowski metric, and carrying the computation using trigonometric identities and

$$\Omega_i \Omega^i = 1 \Longrightarrow \Omega_i d\Omega^i = 0,$$

it is possible to find the intrinsic metric of the AdS$_5$ hypersurface:

$$ds^2_{AdS} = R^2 \left(d\rho^2 + \sinh^2\rho \; d\Omega_i^2 - \cosh^2\rho \; d\tau^2\right).$$

These are called the *global coordinates*. Two things should be remarked by the use of these coordinates: first, the time coordinate is 2π-periodic. This unpleasant feature can be discarded by unwarping the τ coordinate to $-\infty < \tau < \infty$ without any identifications. By this means the result is the universal cover of AdS$_5$, and this is what will be used throughout this report. Secondly, at $\rho \sim 0$ the metric reads

$$ds^2_{AdS} \simeq R^2(d\rho^2 + \rho^2 \; d\Omega_i^2 - d\tau^2)$$

which indicates that AdS$_5$ is isomorphic to a cylinder $S^1 \times \mathbb{R}^4$, as τ is periodic.

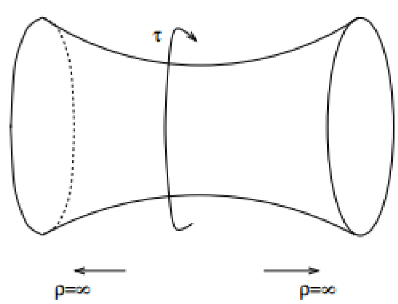


Figure 6: Anti-de Sitter as a hypersurface in Minkowski space, in global coordinates.

3.3.2 Poincaré Coordinates

To get the form of the metric found in the near horizon region in the last subsection, a new set of coordinates $(u, \vec{x}, t)$, called *Poincaré coordinates*, must be used:

$$x_0 = \frac{1}{2u}(1 + u^2(R^2 + \vec{x}^2 - t^2)), \quad x_5 = R \; u \; t$$

$$x_4 = \frac{1}{2u}(1 - u^2(R^2 - \vec{x}^2 + t^2)),\ x_i = R\ u\ x_i.$$

In this new patch the differentials read

$$dx_0 = \left(\left[-\frac{x_0}{u} + R^2 + \vec{x}^2 - t^2\right] du - ut\ dt + ux^i\ dx_i\right)$$

$$dx_5 = Rt\ du + Ru\ dt$$

$$dx_i = Rx_i\ du + Ru\ dx_i$$

$$dx_4 = \left(\left[-\frac{x_4}{u} - R^2 + \vec{x}^2 - t^2\right] du - ut\ dt + ux^i\ dx_i\right).$$

Inserting these results on the metric and using the easily confirmed and helpful results

$$-x_0^2 + x_4^2 = -\frac{R^2}{u^2}\left(-t^2 + \vec{x}^2 + u^2\right),$$

and

$$x_0(\vec{x}^2 - t^2) + x_4(t^2 - \vec{x}^2) = u^2 R^2(\vec{x}^2 - t^2),$$

one easily finds in a first computation

$$-dx_0^2 + dx_4^2 = -\frac{R^2}{u^4}(-t^2 + \vec{x}^2 - u^2)du^2 - 2R^2(ux^i\ dx_i du - ut\ dtdu),$$

and

$$-dx_5^2 + dx_1^2 + dx_2^2 + dx_3^2 = -R^2t^2\ du^2 - R^2u^2\ dt^2 - 2R^2tu\ dudt + R^2x_i^2\ du^2$$
$$+R^2u^2\ dx_i^2 + 2R^2x^iu\ dudx_i.$$

Therefore it is straightforward to see that

$$ds^2_{AdS} = \frac{R^2}{u^2}du^2 + u^2R^2\eta_{\mu\nu}dx^\mu dx^\nu,$$

where the Lorentz notation is here used to specify the four-vector $(t, \vec{x})$.
This is the metric of AdS_5 expressed in the Poincaré coordinates, but it is not yet the result found in (13). To find it, define

$$r = R^2 u, \qquad dr = R^2 du,$$

so that

$$ds^2_{AdS} = \frac{R^2}{r^2}R^4\frac{dr^2}{R^4} + \frac{r^2}{R^4}R^2\eta_{\mu\nu}dx^\mu dx^\nu$$
$$\Longrightarrow ds^2_{AdS} = \frac{R^2}{r^2}dr^2 + \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu,$$

which is exactly the first two terms in (13). Remark that u has dimension $(\text{lenght})^{-1}$ so that r is a length coordinate, as R has dimension $(\text{length})^1$

3.3.3 Boundary and Conformal Boundary

Introduce a new change in coordinates by $z = u^{-1}$. Then $dz^2 = u^{-4}du^2$ and the AdS metric reads

$$ds^2_{AdS} = R^2 z^2 z^{-4} dz^2 + \frac{R^2}{z^2}\eta_{\mu\nu}dx^\mu dx^\nu = R^2\,\frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2}. \tag{14}$$

The boundary of AdS is placed on the limit $r \longrightarrow \infty$, i.e. $u \longrightarrow \infty$ or $z \longrightarrow 0$. The horizon, on the other hand, as seen is at $r \longrightarrow 0$, i.e. $u \longrightarrow 0$ or $z \longrightarrow \infty$.

Using this coordinate patch, it is possible to compute the time needed for a light-ray to travel radially to the boundary. First, see that

$$0 = ds^2 = \frac{R^2}{u^2}du^2 - u^2R^2dt^2 \Longrightarrow \frac{du}{u} = udt.$$

Integrate from some region in spacetime $u = a$, $a \neq 0$ so that it is not at the horizon (from where it cannot escape), to the boundary $u = \infty$ to get

$$\Delta t = \int_a^\infty u^{-2}du < \infty.$$

The light ray can reach the boundary at a finite time, and therefore each point in AdS is in causal contact with the boundary of the space. This is a very special feature, clearly not present in Minkowski space; it becomes important to know what the boundary of AdS_5 looks like. To do that, there are at least two basic constructions, which are presented in the following.

Consider the AdS_5 metric above; perform a Wick rotation $t \longrightarrow it_E$ so that the Minkowski part of the metric becomes euclidean in the coordinates $(t_E, \vec{x})$. Now change the names of the coordinates to something more familiar, $x_0 = z$, $x_4 = t_E$. Then the metric is

$$ds^2_{AdS_E} = \frac{1}{x_0^2}\sum_{i=0}^{4} dx_i^2. \tag{15}$$

These coordinates are called *half-plane coordinates* as in two dimensions it describes the half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ with Poincaré metric as given above. The half plane can be mapped to a ball in euclidean space, as in the figure below in 2-dimensions, where the hole $|z| = \infty$ is mapped to a point $u = 0$ in the boundary of the ball.

The boundary $u = \infty$ is a 4-sphere with one point removed; this is precisely the result of the compactification of 4-dimensional Euclidean space $\mathbb{R}^4$. Perform

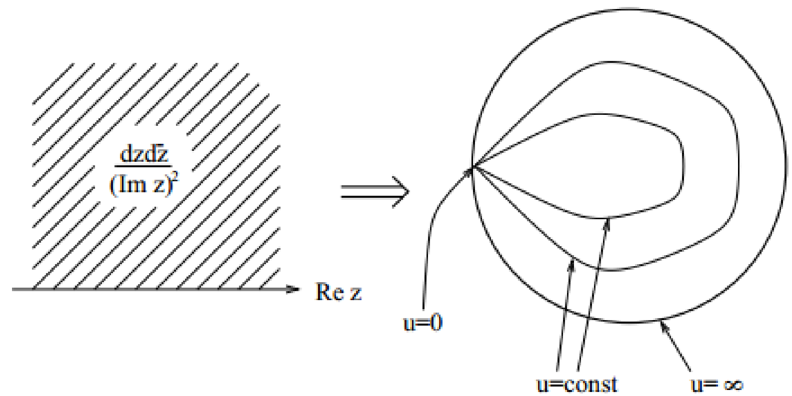


Figure 7: Mapping AdS_2 to a disc, as in [1].

the Wick rotation again to get the initial AdS space and finally the conclusion is that the conformal boundary of AdS_5 is compactified Minkowski space $\mathbb{M}^4$. Escher's painting on the title page of this report is a visualisation of this feature of hyperbolic spaces.

The second, more precise, way of seeing this is by considering the problem in terms of quadrics. Start with a set of coordinates (u, v, x^i) in $\mathbb{M}^{2,4}$ and consider the quadric equation

$$uv - \eta_{ij}x^i x^j = 0. \tag{16}$$

This quadric is invariant under an overall rescaling $u \longrightarrow su$, $v \longrightarrow sv$, $x^i \longrightarrow sx^i$. Use this scaling to fix $v = 1$ and solve for u

$$u = \eta_{ij}x^i x^j.$$

This parametrises the portion of the quadric with $v \neq 0$, not taken into account in the previous considerations. So the quadric describes a parametrisation of (3+1) dimensional Minkowski space plus some points "at infinity" $v = 0$. This is precisely what is obtained from a compactification of Minkowski space.
Indeed, it is analogous to the euclidean case; take for example the compactification of the real line into a circle. We may use the stereographic projection as a map from every point in the real line to every point in the circle, except the pole P. This point P would correspond to a "point at infinity" in the real line. The full circle can be obtained by this procedure simply by adding this point P and demanding that every sequence $(x_n) \subset \mathbb{R}$ such that $x_n \longrightarrow \infty$ when $n \longrightarrow \infty$, now in the circle satisfies $x_n \longrightarrow P$ when $n \longrightarrow \infty$, thus closing the compactification

completely.

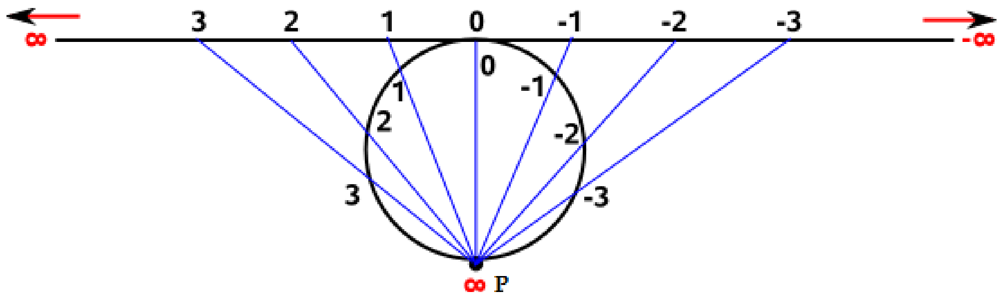


Figure 8: Compactification of the real line into a circle.

Use the same coordinates to write[14] AdS_5 as

$$uv - \eta_{ij} x^i x^j = 1.$$

Let $u, v, x^i \longrightarrow \infty$ and do a conformal transformation by dividing by a positive constant factor, so that the right-hand side can be neglected and the quadric (16) is obtained. Then this quadric, the compactification of $\mathbb{M}^4$, is the conformal boundary of AdS_5.

[14]Set $R = 1$.

4 The AdS_5/CFT_4 Correspondence

In this section the results of the previous sections are taken in order to formulate the correspondence. The strong form of the conjecture is suggested, and its formalisation is made using the previous results on the conformal boundary of AdS. Finally, the holographic principle is derived from black hole thermodynamics considerations and its realisation in the AdS/CFT correspondence is analysed.

4.1 The Dual Descriptions

In the previous section it was seen that a stack of N D3-branes can be described by the means of two very different descriptions: on the one hand we obtain $\mathcal{N} = 4$ Super-Yang Mills U(N) theory in the 4-dimensional world-volume. On the other hand, we can replace the effect of the D3-branes by a non-trivial geometry which is the manifold $AdS_5 \times S^5$ in the near horizon region. A closer analysis of the theories emerging from this dual description is the next step.

Consider the first encountered description. Taking the low energy limit, the system is completely described by two theories: a type IIB supergravity theory living in the bulk, and $\mathcal{N} = 4$ U(N) SYM up to some higher derivative correction terms. The total action has the form

$$S = S_{bulk} + S_{brane} + S_{int},$$

where the last term represents the interaction between the bulk states and the brane states. It can be shown that these interactions are proportional to positive powers of α', so that if the limit $\alpha' \to 0$ is taken while keeping g_s, N fixed, these interactions vanish. Plus, S_{bulk} becomes type IIB free gravity and S_{brane} becomes pure $\mathcal{N} = 4$ U(N) SYM theory. Therefore the states that survive to this limit can be fully described by

- 10-dimensional free gravity in the bulk

- 4-dimensional gauge theory in the branes

Consider now the second description of the N D3-branes:

$$ds^2 = \frac{1}{\sqrt{H(r)}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{H(r)}(dr^2 + r^2 d\Omega_5^2),$$

$$H(r) = 1 + \frac{R^4}{r^4}, \qquad R^4 = 4\pi g_s \alpha'^2 N.$$

As seen, energies are redshifted by a factor $\left(1+\frac{R^4}{r^4}\right)^{-\frac{1}{4}}$, so that when taking the low energy limit states of any mass close to the horizon $r=0$ will survive. Indeed, any state sufficiently close to the horizon will have sufficiently low energy to survive to the limit. Plus, there are also the massless states away from the horizon, and these two sets decouple from each other in the limit that is being taken.
Again, the states that survive can be fully described by

- 10d free gravity in the bulk

- near horizon full theory

Therefore in both approaches two decoupled theories describing all states in the prescribed limit are obtained. Remember that the near horizon region is $\mathrm{AdS}_5\times\mathrm{S}^5$. If the two approaches are seen as different descritpions of the same object, as it has been conjectured until now, then this identification forces to identify the two obtained results so that

$\mathcal{N}=4$ $U(N)$ SYM in flat 3+1 dimensions

is "equivalent" to

Type IIB superstring theory on $\mathrm{AdS}_5\times\mathrm{S}^5$.

The strong form of this conjecture applies for any value of g_s and N, and is the one analysed in this report. Remark that, by fully aknowledging this correspondence, no reference is needed from now on to how the argument that was exposed and leads to this conjecture actually works. In this way, for example when reading *type IIB superstring theory on* $AdS_5\times S^5$, it should be read *type IIB superstring theory living in a background which asymptotically is as* $AdS_5\times S^5$.

4.1.1 Matching of Symmetries

The conjecture just made seems in a first regard to be more enigmatic than what it turns out to be. These two theories are completely different: on one hand there is 10-dimensional string theory, a gravity theory, in a very special spacetime. On the other hand, there is a U(N) gauge theory, which is, as seen, also a conformal theory, living in a flat 4-dimensional spacetime!
A first question that should be asked is whether the global symmetries of these two theories match, and the answer is yes. Indeed, on the string theory side, the product space $\mathrm{AdS}_5\times\mathrm{S}^5$ has isometry group $SO(2,4)\times SO(6)$, the first being the

manifest isometry group of AdS$_5$, the second rotating the 5-sphere.
On the CFT side, there is a 4-dimensional conformal symmetry $SO(2,4)$ and a $SO(6)$ symmetry as seen in section 3.1. So indeed, the bosonic part is seen to coincide perfectly. In fact, the whole supersymmetric group $SU(2,2|4)$ matches in both theories, the strange geometry on the string theory side being responsible for the breaking of half of the supersymmetries. Gauge symmetries, being redundancies of the description, do not need to match.

4.1.2 Tractable Limits, Validity Regimes and Duality

The strong equivalence relation conjectured previously should be substantiated with strong evidence for it[15]. This is the purpose of the remaining of this report. For that, it is useful to start by considering the consequences of such correspondence between the two theories. The most fundamental one is that there should exist a map between states and fields on the AdS string side and local gauge invariant operators on the Yang-Mills CFT side, as well as a correspondence between correlators. Indeed, the matching of observables between theories is the ultimate request and a necessity for the theories to be *different descriptions* of the same theory, where these observables would take their full meaning.
Even if the conjecture concerns the two complete theories, there are special limits in which it is possible to see this correspondence somehow closer. One of these limits is the 't Hooft limit described in the first part of this report. As it was seen, a large N limit of a U(N) gauge theory with $\lambda = g^2 N$ fixed leads to a topological expansion as the one found in classical string theory. In the context of the correspondence, the large N limit of $\mathcal{N} = 4$ U(N) SYM corresponds to a genus expansion in powers of g_s in type IIB string theory.
Another regime expected to have a meaningful result is an α' expansion in type IIB supergravity. What is the corresponding behaviour in the CFT part? Use the AdS metric in the form

$$ds^2_{AdS} = R^2 \; \frac{dz^2 + dx_\mu dx^\mu}{z^2},$$

and include it in the string theory non-linear sigma model

$$S_G = \frac{1}{4\pi\alpha'} \int \sqrt{\gamma}\gamma^{mn} G_{MN}(x;R)\partial_m X^M \partial_n X^N$$
$$\Longrightarrow S_G = \frac{R^2}{4\pi\alpha'} \int \sqrt{\gamma}\gamma^{mn} \overline{G}_{MN}(x;R)\partial_m X^M \partial_n X^N$$

where

$$R^2 \; \frac{dz^2 + dx_\mu dx^\mu}{z^2} = ds^2_{AdS} = G_{MN} dx^M dx^N = R^2 \; \overline{G}_{MN} dx^M dx^N.$$

[15] In the perspective of today's lack of a rigorous proof.

Keeping in mind equation (11), write $R^2/4\pi\alpha' = \sqrt{4\pi\lambda}$ so that the supergravity α' expansion corresponds now to a $\lambda^{-1/2}$ expansion, meaningful in a large λ limit.

Issuing from the dual description decribed in the last section, the AdS/CFT correspondence inherits a similar dual behaviour. A perturbative analysis in the Yang-Mills CFT part of the relation is meaningful when the coupling effective parameter g^2N is small enough. Using relation (11), this is

$$g^2N \sim g_sN \sim \frac{R^4}{l_s^4} \ll 1.$$

On the other hand, the classical gravity description is valid when the typical radius of curvature R is large when compared to the string length. Again by equation (11) this means

$$\frac{R^4}{l_s^4} \sim g_sN \sim g_{YM}^2N \gg 1.$$

These validity regimes are perfectly incompatible, again we find this correspondence as a duality relation. Two important consequences arise from this fact: first, the duality gives the correspondence a much larger importance from a theoretical point of view. Clearly, this duality is stating that it is possible to solve a string theory in a strongly curved background[16] by easily solving a gauge theory in a perturbative expansion. Similarly, it should be possible to solve strongly coupled gauge theories[17] by solving a low curvature string theory, which includes gravity. It is hard to overestimate the potential importance of such statement.

On the other hand, no rigorous proof or real test was given here, so given this complete opposition of comprehensible regimes of validity a legitimate question would be how could this correspondence ever be tested if each side is understandable in non-overlapping regimes. One important result would be to compare quantities from both theories which do not depend on the coupling/curvature radius. This is possible and has been done. Another more significant result obtained from the research made on this topic in the past 10 years has to do with the subject of integrability. Briefly stated, the result obtained allows to interpolate between the two regimes described above to a region where the overlapping happens and the two theories can be directly compared. This has been done and yields very promising confirmations of the AdS/CFT correspondence.

[16]Something which today is not at all obvious in a full quantum level, at least in the author's understanding of it.

[17]Low energy QCD would be rather useful.

4.2 Formulation of the Correspondence

The precise formulation of the AdS/CFT correspondence may be made using the results obtained until now. The main problem concerning this formulation is that it should provide a link between the fields living in AdS_5 and the operators in the CFT_4.

As seen, because of the particular form of Anti-de Sitter space, any field theory in this spacetime needs boundary conditions on its fields for the theory to be completely determined. This is by no means an easy problem, as boundary and points in the bulk are causally connected, as seen. It is possible nevertheless to use this, together with the fact that the conformal boundary of AdS_5 is compactified 4-dimensional Minkowski space $\mathbb{M}^4$, and introduce a coupling

$$\int d^4x \ \phi_0(\vec{x})O(\vec{x}) \tag{17}$$

in the CFT_4 side, where O is a CFT operator and $\phi_0(\vec{x})$ is an arbitrary source for this operator such that there exists a field $\phi(\vec{x}, z)$[18] in AdS_5 with boundary condition

$$\phi(\vec{x}, z)\big|_{z=0} = \phi_0(\vec{x}).$$

In this way, a link between AdS-fields and CFT-operators is realised as a boundary-bulk relation, which is by all means possible with what was studied. Of course, mass dimensions should agree in the appropriate way so that expression (17) is dimensionless; also the quantum numbers with respect to the full symmetry group of both the operator O and of the field ϕ should agree so that the coupling as a whole is meaningful in the theories.

Since this bulk-boundary relation is also a source-operator correspondence, it is natural to propose the following statement:

$$\langle e^{\int d^4x \ \phi_0(\vec{x})O(\vec{x})} \rangle_{CFT} \ \overset{\text{AdS/CFT}}{=} \ Z^{string}_{AdS}\left[\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})\right], \tag{18}$$

that is, the generating functional of O-correlation functions in the CFT side matches the full partition function on the AdS side where the field corresponding to the operator O is solved having a specific boundary condition. In the end, when $\phi(\vec{x})$ is solved in terms of $\phi_0(\vec{x})$, i.e. it is on-shell with respect to the action in AdS_5, both sides are functionals of $\phi_0(\vec{x})$. O-correlation functions on the CFT side may be computed by taking functional derivatives on the left hand side, and then letting the sources vanish, as usual by the QFT path integral methods. This relation then states that by computing the partition function in the AdS side and taking functional derivatives with respect to the boundary field $\phi_0(\vec{x})$ the result should be the same. This will be done in the next section for a massless scalar

[18] The used coordinate system is the one introduced in (14) after a Wick rotation.

field, but this is a general proposition applicable in principle to fields of any spin and mass.

4.3 Holography

Before proceeding to take on hands the problem of computing correlation functions in the context of the correspondence, this section ends with one of the most promising and fascinating preliminary results of the AdS/CFT correspondence. This correspondence in its full form is an explicit statement about the equivalence of two theories, one living in a 5-dimensional spacetime and the other in the usual 4-dimensional Minkowski spacetime, on the boundary of the first as seen. This equivalence apparently reduces the dimension of the spacetime where the descriptions take its forms by 1 without any further problems. But this mismatching of dimensions in a boundary-bulk relation seems in a first view very problematic, as there are degrees of freedom in the 5-dimensional theory which seem to be utterly redundant.

Throughout this section it will be seen that this is not a problem, it is even a necessity if a full quantum gravity description is ever to be obtained, by means of the *holographic principle*. By the end of the section, the explicit realisation of this principle in the context of the AdS/CFT correspondence is fully analysed.

4.3.1 Matter Entropy Bound

Black hole thermodynamics, a subject firstly introduced by Hawking and his semi-classical analysis resulting on what is now called Hawking radiation, yields the important result concerning black hole entropy:

$$S_{BH} = \frac{A}{4G_N},$$

where A is the black hole's event horizon area and G_N is Newton's gravitational constant. Another feature which is important for the following is the fact that the black hole's event horizon area is proportional to the second power of the total mass[19] of the black hole, $A \sim M^2$. Finally, it is important to state the *generalised 2^{nd} law of thermodynamics*, which aknowledges black hole entropy as a part of the total entropy of the universe and the unitarity of the processes involving black holes, e.g. black hole scattering. In this way, it states

$$dS_{BH+matter} \geq 0,$$

[19]The subject presented in this fashion is only concerned with static black-holes.

where here and hereon *matter* refers to *non-collapsed* matter systems.

Following these assumptions, consider a system composed of matter of mass m and entropy S_{matter} and a black hole of mass M and entropy $S_{BH} \sim A \sim M^2$. Let the total system evolve in such a way that the matter enters the event horizon created by the black hole. Then, the total entropy before and after the collapse is given by

$$S_{before} = S_{matter} + \frac{A}{4},$$
$$S_{after} = S'_{BH} \sim A' \sim (M+m)^2.$$

The generalised 2$^{\text{nd}}$ law requires $S_{before} \leq S_{after}$. However, as the black hole after the collapsing of the in-falling matter keeps no information about its entropy S_{matter}, but only takes its mass m to enlarge its area and thus its entropy, it is straightforward to see that there must exist an upper limit to S_{matter} if the generalised 2$^{\text{nd}}$ law is to be true. In other words, a large enough matter entropy for a small enough mass would violate the 2$^{\text{nd}}$ law as presented.
Using these considerations, Bekenstein and Susskind derived generalised matter entropy bounds by considering specific processes. In particular, by considering the collapse of a mass shell around a matter system in order to create a black hole, Susskind derived what is called the spherical entropy bound

$$S \leq \frac{A}{4G_N},$$

where A is the area of the smallest volume inclosing the system. Black holes saturate this inequality, and therefore have the highest possible entropy for the given occupied volume.

4.3.2 The Holographic Principle

Using the thermodynamical relation $S = k_B \ln W$, it is then easy to view e^S as the number of independent quantum states compatible with certain macroscopic parameters. In a full quantum theory of nature, this corresponds to the number of degrees of freedom of the theory. More precisely,

$$N \equiv \#dof = log\ dim(\mathcal{H}),$$

$$dim(\mathcal{H}) = e^S,$$

where $\mathcal{H}$ is the Hilbert space where the theory is formulated, so that in the end the entropy S corresponds to the number of degrees of freedom.

In a standard QFT, the number of degrees of freedom is infinite. Indeed, it is costumary to view the framework of a QFT as consisting of an harmonic oscillator in each point in space, which yields an infinite number of degrees of freedom as a system. When regularised, e.g. to describe some quantum gravity theory, as to have 1 d.o.f. per Planck volume, the number of degrees of freedom grows as the volume considered. This is however in contradiction to what has been exposed, as the correct relation given by the matter entropy bound is that the entropy, i.e. the number of d.o.f., should grow as the area of the considered volume, $S \sim A$.

The holographic principle is obtained by fully aknowledging the previous results as a principle of nature and by generalising it to all frameworks susceptible of describing nature. It states

> *In a quantum theory of gravity all physics within some volume can be described in terms of some theory on the boundary which has less than one d.o.f. per Planck area.*

This principle is formulated so that every system satisfies the required entropy bound. For conclusion, our usual theories have a redundacy in the form of useless degrees of freedom. This redundancy can be solved by considering some *corresponding theory on the boundary* of the space our theories live in. Even if stated in a purposely intended fashion so to be used in the AdS/CFT correspondence, this principle in its general form, and by what has been presented as argument to it, can be seen to have a faithful realisation in the bulk-boundary relation exposed to formulate the AdS/CFT correspondence.

4.3.3 AdS/CFT and Holography

The explicit analysis of the holographic character of the AdS/CFT correspondence is of no trivial matter, contrary to what the explicit boundary-bulk relation may lead to think. Indeed, the CFT living on the boundary, being a field theory, has an infinity of degrees of freedom. In the same way, the area of the boundary of AdS space is infinite, so that in these terms a comparison between the two is impossible.
In order to be able to make such comparison, a regularisation of the 4-dimensional boundary space has to be achieved. For that purpose, a discretisation of its spatial part in terms of small cells of size $\delta \ll 1$ is made, and 1 d.o.f. is considered per cell. As the total number of cells is proportional to δ^{-3}, and the theory living on the boundary is a U(N) gauge theory, the total number of degrees of freedom goes as

$$S_{CFT}^{reg} \sim \delta^{-3} N^2. \tag{19}$$

This UV cut-off introduced in the boundary relates to the bulk AdS theory by imposing a IR cut-off. This is part of a broader subject usually refered to as *IR-UV connection*. It can be seen by considering a string stretching in the bulk to a point in the boundary. In the boundary theory, it will be seen as a point particle, and being charged it yields a divergent self-energy. This can be solved in the CFT context by imposing a UV cut-off to regularise these divergences. In the AdS string theory part, the string has infinite length, yielding therefore a divergent energy content. In order to regularise this divergence it is possible to cut the length of the string so that it does not touch the boundary anymore, resulting by these means in a finite length string. Therefore, the IR-UV connection is a relation that allows the relating of UV-effects on the boundary with IR-effects in the bulk. This is a striking feature of such boundary-bulk correspondences.

Then, using the coordinate frame described in (14), this amounts to introduce a cut-off at $z \sim \delta$. The metric as given by (15) allows to compute the area of the hypersurface $z \sim \delta$:

$$A = \int_{z\sim\delta} ds \sim \int d^3x \, \frac{R}{\delta} \sim \frac{R^3}{\delta^3}.$$

the 3 coordinates x^i being periodic and describing the tangent coordinates to the ball at a given radius, i.e. for a fixed z. Then, by substituting for δ in (19),

$$S \sim N^2 A R^{-3}.$$

It should be enough to see that $N^2 R^{-3} \sim G_N^{-1}$ in order to obtain the result required by the holographic principle and conclude that the AdS/CFT correspondence as formulated realises this principle.

Start with the 10-dimensional bulk gravitational constant $G_N^{(10)}$. In d dimensions, Newton's constant has dimensions[20] $(\text{length})^{d-2}$ and therefore it may be written as $G_N^{(d)} \sim \alpha'^{(d-2)/2}$. In $d = 10$ this yields $G_N^{(10)} \sim \alpha'^4$, and using the identity (11) to find a relatiom between α' and R, N, it is easy to find

$$G_N^{(10)} \sim R^8 N^{-2}.$$

Nevertheless the 10-dimensional bulk space is effectively 5-dimensional, as 5 dimensions are compactified into the 5-sphere. Remark the result

$$\frac{G_N^{(d)}}{G_N^{(d')}} \sim l_c^{d-d'},$$

[20]It couples gravity to the system via a term $\frac{1}{G_N}\int d^dx\sqrt{g}R$ where R is the Ricci scalar which has dimensions $(\text{length})^{-2}$.

where l_c is the length of the compactified dimensions[21]. In this case l_c is proportional to R as it is the radius of the 5-sphere. Then it yields

$$G_N^{(5)} \sim R^{-5} G_N^{(10)} \sim R^3 N^{-2},$$

which is wanted result. The AdS/CFT correspondence is therefore a holographic relation.

[21] See [13] for a derivation of this result.

5 AdS/CFT Correlation Functions

Consider the formulation of the correspondence given in equation (18), namely

$$Z_{CFT}\left[\phi_0\right] \equiv \langle e^{\int d^4x \ \phi_0(\vec{x})O(\vec{x})}\rangle_{CFT} \ = \ Z_{AdS}^{string}\left[\phi(\vec{x},z)|_{z=0} = \phi_0(\vec{x})\right].$$

The computation of O-correlatiom functions is achieved by

$$\langle O...O\rangle = \left.\frac{\delta^n Z_{CFT}\left[\phi_0\right]}{\delta\phi_0^n}\right|_{\phi_0=0}.$$

By the above statement Z_{CFT} can be made to correspond to Z_{AdS}^{string}. In particular, as $\phi(\vec{x},z)$ solves the equations of motion derived from S_{AdS_5} with the given boundary condition, Z_{AdS}^{string} is a functional of $\phi_0(\vec{x})$ and the functional derivative as stated has a meaning.
The uniqueness of the extension of ϕ_0 to ϕ is guaranteed if suitable boundary conditions are chosen. As the equations of motion in AdS are usually second-order differential equations, two boundary conditions are needed. On the other hand, it is not a valid request to ask for $\phi(\vec{x},z=0)=\phi_0(\vec{x})$ since the AdS metric blows up at the boundary $z=0$. Instead, the appropriate is to ask

$$\phi(\vec{x},z)\sim f(z)\phi_0(\vec{x})$$

for a well suited function $f(z)$. This is usually done by requiring some normalisability condition. The second boundary condition is usually imposed on the interior of AdS, namely some requirement at the horizon $z\longrightarrow\infty$.

5.1 Massless Scalar Field 2-point Function

Consider the AdS action in $d+1$ dimensions of a massless scalar field:

$$S_{AdS}\left[\phi\right] = \frac{1}{2}\int d^{d+1}x\sqrt{g}\ (\partial\phi)^2.$$

Following the presented method, the idea is to solve ϕ in terms of ϕ_0 with a regularised boundary condition, then evaluate $S\left[\phi\right]$ on ϕ such that the action is a functional of ϕ_0 and finally take functional derivatives with respect to ϕ_0, hoping to obtain CFT correlation functions by the AdS/CFT correspondence formulation.

A first simplification can be done by integrating S_{AdS} by parts. The term giving the equations of motion is zero by the vanishing of the action variation and the remaining regularised-boundary term is

$$S_{AdS}\left[\phi\right] = \lim_{\epsilon\to 0}\frac{1}{2}\int_{T_\epsilon} d^d x\sqrt{h}\ \phi\ \partial_n\phi, \tag{20}$$

where the boundary has been regularised so that the integral is made on the surface $T_\epsilon = \{(\vec{x}, z) :\ \ z \sim \epsilon\}$. Furthermore, h is the determinant of the induced metric on T_ϵ and ∂_n denotes the derivative normal to this surface.

Start by recalling the Euclidean form of the AdS_{d+1} metric:

$$ds^2 = \frac{1}{z^2}\left(\sum_{i=1}^{d} dx_i^2 + dz^2\right),$$

where it is set $R = 1$ for simplicity. Solve the equations of motion $\Delta\phi(\vec{x}, z) = 0$ by the use of a Green's function $K(\vec{x} - \vec{x}', z)$ such that

$$\phi(\vec{x}, z) = \int d\vec{x}'\ K(\vec{x} - \vec{x}', z)\phi_0(\vec{x}'),$$

where the Green's function must satisfy

$$\Delta_{\vec{x},z}K(\vec{x} - \vec{x}', z) = 0,$$

by the equations of motion and

$$K(\vec{x} - \vec{x}', z) \longrightarrow f(z)\delta(\vec{x} - \vec{x}'),$$

by the boundary condition.
The symmetry of the metric rotating the x^i, together with the fact that the boundary condition will be taken at $\vec{x}'$ being the point at infinity, so that K is invariant under translations of $|\vec{x}|$, implies that $K(\vec{x}, z) = K(z)$. Therefore the first condition for $K(z)$ reads

$$0 = \Delta K = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu K = \frac{1}{\sqrt{g}}\partial_z\sqrt{g}g^{zz}\partial_z K,$$

where the last relation is obtained by the form of the metric and because of the symmetry consideration just referred.
In view of the metric it is easy to infer the results

$$\sqrt{g} = z^{-d-1},$$

$$g^{zz} = z^2,$$

as the computation is made in $d{+}1$ dimensions and the last identity being achieved not for the metric but its inverse. Then the Green's function must solve

$$0 = z^{d+1}\partial_z\left(z^{-d-1}z^2\partial_z K(z)\right)$$

$$\begin{aligned}
&\Longrightarrow\ z^{-d+1}\partial_z K(z) = c\\
&\Longrightarrow\ \partial_z K(z) = cz^{d-1}\\
&\Longrightarrow\ K(z) = \frac{cz^d}{d} + b
\end{aligned}$$

where b, c are integration constants. Fix the constants by demanding the solution to vanish at the boundary[22] $z = 0$, so that $b = 0$, and by setting the singularity at $z = \infty$ to be a δ-function at some $\vec{x}$. To clearly see this, perform a SO(1,d) transformation[23], which is the isometry group of the space the field lives in, defined by

$$x_\mu \longrightarrow \frac{x_\mu}{z^2 + \sum_{i=1}^{d} x_i^2}, \qquad \mu = 0, ..., d \text{ and } z = x_0,$$

so that the $z = \infty$ point is mapped to $z = 0$, and therefore

$$K(\vec{x}, z) = \frac{c}{d}\frac{z^d}{\left(z^2 + \sum_{i=1}^{d} x_i^2\right)^d}.$$

When $z \longrightarrow 0$, this Green's function vanishes everywhere except at $x_1 = ... = x_d = 0$. It can be shown by integration that K is actually a δ-function supported at this point, with a suitable choice of the constant c, with $f(z) = 1$[24].

Write now the solution for ϕ using this Green's function as

$$\phi(\vec{x}, z) = \frac{c}{d}\int d\vec{x}'\ \frac{z^d}{(z^2 + |\vec{x} - \vec{x}'|^2)^d}\ \phi_0(\vec{x}').$$

Take the derivative of this solution with respect to z (it will be needed when computing (20)):

$$\partial_z\phi(\vec{x}, z) = \frac{c}{d}\int d\vec{x}'\left(\frac{d\ z^{d-1}}{(z^2 + |\vec{x} - \vec{x}'|^2)^d} - \frac{d\ z^d}{(z^2 + |\vec{x} - \vec{x}'|^2)^{d+1}}2z\right)\ \phi_0(\vec{x}').$$

[22] As it was seen in the ball form of the metric, the boundary includes a $z = 0$ surface and a point $z = \infty$.

[23] This is simply a Euclidean version of a special conformal transformation in SO(2,d-1), as we are in Euclidean AdS space.

[24] This is a particular property of the massless scalar field, and is not true in general.

The evaluation of the action (20) only concerns the limit $z \longrightarrow 0$. In this limit this derivative becomes

$$\begin{aligned}
\partial_z \phi(\vec{x}, z) &= c \int d\vec{x}' \left(\frac{z^{d-1}(z^2 + |\vec{x} - \vec{x}'|^2) - 2z^{d+1}}{(z^2 + |\vec{x} - \vec{x}'|^2)^{d+1}} \right) \phi_0(\vec{x}') \\
&= c \int d\vec{x}' \left(\frac{z^{d+1} + z^{d-1}(|\vec{x} - \vec{x}'|^2) - 2z^{d+1}}{(z^2 + |\vec{x} - \vec{x}'|^2)^{d+1}} \right) \phi_0(\vec{x}') \\
&= cz^{d-1} \int d\vec{x}' \, \frac{\phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2d}} + O(z^{d+1}).
\end{aligned}$$

Consider now the normal derivative. The normal vector is a vector whose components are zero except in the z-direction:

$$n_\mu = \frac{1}{N}(1, 0, ..., 0),$$

where N is a normalisation factor which can be computed:

$$1 = g^{\mu\nu} n_\mu n_\nu = g^{zz} \frac{1}{N^2} = z^2 \frac{1}{N^2},$$

so that $N = z$ by chosing the right orientation. Then it is easy to conclude

$$\partial_n \phi = n_\mu g^{\mu\nu} \partial_\nu \phi = n_z g^{zz} \partial_z \phi = \frac{1}{z} z^2 \partial_z \phi,$$

so that $\partial_n = z\partial_z$.

It is now possible to evaluate the action (20) using these results. First, remark that

$$\lim_{\epsilon \to 0} \phi = \phi_0$$

so that inside (20) the ϕ term can be replaced by ϕ_0, in view of the limit being taken. The induced metric on the hypersurface T_ϵ is simply

$$\sqrt{h} = z^{-d},$$

and therefore (20) reads

$$\begin{aligned}
S_{AdS}\left[\phi_0\right] &= \frac{1}{2} \int d\vec{x} z^{-d} \ \phi_0(\vec{x}) \ zcz^{d-1} \int d\vec{x}' \ \frac{\phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2d}} \\
&= \frac{c}{2} \int d\vec{x} d\vec{x}' \ \frac{\phi_0(\vec{x})\phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2d}}.
\end{aligned}$$

By the formulation of the AdS/CFT correspondence in (18) the functional derivative of this action with respect to ϕ_0 should give O CFT correlation functions. Indeed:

$$\left(-i\frac{\delta}{\delta\phi_0(\vec{y})}\right)\ \left(-i\frac{\delta}{\delta\phi_0(\vec{z})}\right)\ e^{-iS_{AdS}[\phi_0]}\ |_{\phi_0=0} =$$

$$= \frac{c}{2}\int d\vec{x}d\vec{x}'\left[\ \frac{1}{|\vec{x}-\vec{x}'|^{2d}}\delta(\vec{x}-\vec{y})\delta(\vec{x}'-\vec{z}) + (\vec{z}\leftrightarrow\vec{y})\right]$$

$$\Longrightarrow \langle O(\vec{y})O(\vec{z})\rangle = \frac{c}{|\vec{y}-\vec{z}|^{2d}}.$$

This is precisely the 2-point correlation function of a CFT operator with conformal dimension $\Delta = d$.
Also, the 1-point function is

$$\langle O(\vec{y})\rangle = \left(-i\frac{\delta}{\delta\phi_0(\vec{y})}\right)\ e^{-iS_{AdS}[\phi_0]}\ |_{\phi_0=0} = 0,$$

as required by conformal invariance, since a non-zero vacuum expectation value of some operator would explicitly break dilation-invariance.

This same treatment, but in a more complex form where the boundary behaviour has to be carefully chosen and where the equations of motion do not admit such a simple form, can be made for a massive scalar field. The result is the same, where the conformal dimension of the associated CFT operator is

$$\Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2R^2}\right),$$

where the AdS radius R is recovered. There is therefore an explicit relation between the mass of a field propagating in AdS space and the corresponding CFT operator's conformal dimension in the boundary. For a more profound discussion of this result, see [9]. For an exposition of the correspondence between fields and operators for any quantum numbers (and thus in particular any spin and mass) see [5] and [9].

5.2 General Method for Massive Scalar Field n-point Function

Consider a general Lagrangian density for interacting scalar fields in AdS$_5$:

$$S_{AdS} = \int d^5x\sqrt{g}\left(\frac{1}{2}(\partial\phi_i)^2 + \frac{1}{2}m_i^2\phi_i^2 + \sum_{k=3}^{m}\lambda_{i_1\ldots i_k}\phi_{i_1}\ldots\phi_{i_k}\right).$$

Each scalar field has a typical equation of motion of the form

$$(\Box + m^2)\phi = \lambda\phi^n.$$

It is possible to treat this system perturbatively in λ by means of a similar method of standard QFT. Expand the field solution as

$$\phi = \phi^{zero} + \lambda\phi^{one} + \lambda^2\phi^{two} + ...$$

and insert it in the equations of motion,

$$(\Box + m^2)(\phi^{zero} + \lambda\phi^{one} + ...) = \lambda(\phi^{zero} + \lambda\phi^{one} + ...)^n.$$

Match equal powers of λ on both sides and get:

$$\text{at } \lambda^0 : \quad \phi^{zero}(x,z) = \int K(x-x',z)\ \phi_0(x')$$

$$\text{at } \lambda^1 : \quad \phi^{one}(x,z) = \int G(x-x',z-z')(\phi^{zero}(x',z'))^n$$

where $G(x-x',z-z')$ is called *bulk-to-bulk* Green's function, satisfying

$$(\Box + m^2)G(x-x',z-z') = \frac{1}{\sqrt{g}}\delta(z-z')\delta(\vec{x}-\vec{x}').$$

The expansion can be carried to any desired power of the coupling, and its meaning (or lack of it) depends of course on the details of the theory at work. The method can be generalised, with more or less effort, to any theory, containing also fields of other spin quantum numbers, where such a perturbation analysis is useful. The corresponding Feynman diagrams, called *Witten diagrams*, are shown in the figure below.

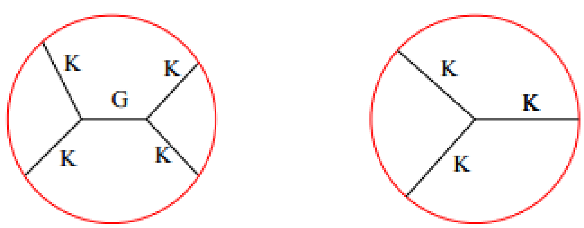


Figure 9: Witten diagrams: Feynman diagrams in the context of the AdS/CFT correspondence.

6 Conclusion

The AdS/CFT conjecture is a powerful and promising correspondence issuing from a dual description of a stack of N D3-branes. A first description uses string theory on D3-branes to see emerge a gauge theory on the world-volume of the stack. In the second description, the reaction of the geometry of spacetime to this charged and massive stack of objects is used as influence on string states. In a low energy limit, this double description gets its full meaning as presented here, and the analysis of such systems in such limit allows a detailed account for the states described by both views.
The assumption of such theories as being each on its own right a good description of the system allows to conjecture an equivalence between the emerging complete theories. These two theories are very different in format and content, one is a string theory living in a curved spacetime with half of the dimensions compactified into a sphere, the other a conformal gauge theory in a flat spacetime. The spacetime symmetries are nevertheless seen to match, and in the roll of consequences of such conjecture it is possible to match also correlation functions, and a precise link with the holographic principle is obtained. This issues from the formulation of the correspondence as a bulk-boundary relation, that allows for a statement about the coupling of observables on both sides of the equivalence.

Nevertheless no proof is given in any circumstance, and it is not obvious it should be an attainable objective. Recent works on integrability showed it is possible to interpolate the dual behaviour of the theories involved in this correspondence and compare them in a meaningful way, with successful results. The strong evidence for a full aknowledgement of the conjecture, presented in this report, as well as many others more detailed arguments not presented, give an insight on this theoretical result which could in the best of hypothesis be used to describe and solve many problems of today's physical description of nature.

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