Gauge/Gravity Duality
The $\text{AdS}_5/\text{CFT}_4$ Correspondence

Kevin Ferreira

ETH Zürich

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Type IIB String Theory on $AdS_5 \times S^5$ with
- radius $R$
- $N$ units of 5-form flux on $S^5$

AdS/CFT

4-dim. $N=4$ SU($N$) Gauge Field Theory (CFT)

Strong/Weak Duality

under control when

$R^2 \gg \alpha'$

small curvature

under control when

$g_{\text{SYM}}^2 N \ll 1$

small parameter

$\left( \frac{R}{\alpha'} \right)^2 = 4\pi g_{\text{SYM}}^2 N$

large curvature

large parameter
Part 0: Large N Gauge Theories as String Theories

1. The ’t Hooft limit
2. Perturbative diagram expansion
3. Link with Strings
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1. The 't Hooft limit
2. Perturbative diagram expansion
3. Link with Strings

Part I: D3-branes: Gauge Theories and Gravity Solutions
1. D3-branes and gauge theories
2. Global and Gauge Symmetries Groups
3. Black p-branes as classical SuGra solutions
Part II: AdS$_5$/CFT$_4$ Correspondence

1. Low Energy Limits
2. AdS/CFT Conjecture, Duality
3. Formulation
4. Holography
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1. Low Energy Limits
2. AdS/CFT Conjecture, Duality
3. Formulation
4. Holography

Part III: AdS$_5$/CFT$_4$ Correlation Functions

1. General method
2. 2-point functions and bulk-to-boundary propagators
3. n-point functions and boundary-to-boundary propagators
Part 0

Large N Gauge Theories as String Theories

or How special limits of special theories take you to a special place
The 't Hooft limit

Consider $U(N)$ Yang-Mills theory, coupling constant $g_{YM}$

Gauge fields in the adjoint representation of $U(N)$ $A_{\mu}^a$ with field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_{YM} [A_\mu, A_\nu]$$

Lagrangian density

$$\mathcal{L} = \frac{1}{g_{YM}^2} (Tr(F_{\mu\nu}^2) + \mathcal{L}_{matter})$$

t’Hooft parameter: $\lambda \equiv g_{YM}^2 N \implies \mathcal{L} \sim N/\lambda$

t’Hooft limit: $N \to \infty$ and $\lambda$ fixed
Double Line Notation

Represent the adjoint gauge field $A_\mu = A_\mu^a T^a$ as a direct product of fundamental and anti-fundamental fields:

$\rightarrow A^{i}_j, N \times N$ hermitian matrices

$\rightarrow U(N)$ propagator: $< A^{j}_i A^{l}_k > \propto \delta^l_i \delta^j_k$

$\rightarrow$ Feynman diagrams: network of double lines

Vacuum diagrams: compact closed oriented surfaces.
Planar Perturbative Expansion

Example: gluon self energy

\[ \text{graph} \propto O(g_{YM}^2 N), \]
finite in t’Hooft limit

\[ \mathcal{L} = \frac{N}{\lambda} \left( Tr(F_{\mu\nu}^2) + \mathcal{L}_{\text{matter}} \right) \]

Feynman rules:
- \( N/\lambda \) for each vertex (V)
- \( \lambda/N \) for each propagator (E, edge)
- \( N \) for each loop (F, face)

\[ N^{V+F-E} \lambda^{E-V} = N \times \lambda^{E-V} \quad \text{for each vacuum bubble graph} \]
Perturbative Expansion

For closed oriented surfaces, $\chi = 2 - 2g$

→ perturbative expansion

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) = \sum_{g=0}^{\infty} \left( \frac{1}{N} \right)^{2g} N^2 f_g(\lambda)$$

→ large N limit: dominated by maximal $\chi$ / minimal $g$, sphere topology

→ correspond to the perturbative theory of closed oriented strings

In general, $S \rightarrow S + N \sum_j J_j G_j$ and

$$\langle \Pi_{j=1}^n G_j \rangle = (iN)^{-n} \left[ \frac{\delta^n W}{\Pi_{j=1}^n \delta J_j} \right]_{J_j=0} \propto N^{2-n}$$

→ $1/N$ as a coupling constant $g_s$
Part I

D3-branes
Gauge Theories and Gravity Solutions

or How to describe something in two amazing different ways
Gauge Theory on D3-branes

System: \( N \) parallel D3-branes in type IIB string theory in 10d

\[ \rightarrow \text{low energy effective theory: } U(N) \text{ gauge theory in } (3+1) \text{-dimensions with 16 supercharges.} \]
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This gauge theory is unique: \( \mathcal{N} = 4 \) Super Yang-Mills, and contains:

\[ \frac{\lambda_a}{a} \alpha, \quad a = 1,...,4 \]

left Weyl fermionic fields

\[ \frac{X_i}{i}, \quad i = 1,...,6 \]
real scalar fields

\[ \text{SO}(6) \sim \text{SU}(4) \]

\[ F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + i[A_{\mu},A_{\nu}] \]
gauge fields with field strength

\[ \mathcal{D}_{\lambda} = \partial_{\lambda} + i[A_{\mu},\lambda] \]
covariant derivative.
Gauge Theory on D3-branes

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This gauge theory is unique: $\mathcal{N} = 4$ Super Yang-Mills, and contains:

- $\lambda^a_\alpha$, $\alpha = 1, 2$, $a = 1, \ldots, 4$ left Weyl fermionic fields
- $X^i$, $i = 1, \ldots, 6$ real scalar fields - $SO(6) \sim SU(4)$
- $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu]$, $A_\mu$ gauge fields with field strength $F_{\mu \nu}$
- $D_\mu \lambda = \partial \lambda + i [A_\mu, \lambda]$, a covariant derivative.
Formulation of $\mathcal{N} = 4$ SYM

The Lagrangian is completely settled by supersymmetry:

$$
\mathcal{L} = Tr \left( -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta I}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a \sigma^\mu D_\mu \lambda_a 
- \sum_i D_\mu X^i D^\mu X_i + \sum_{a,b,i} g C_{i}^{ab} \lambda_a [X^i, \lambda_b] 
+ \sum_{a,b,i} g \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}_b] + \frac{g^2}{2} \sum_{i,j} [X^i, X^j]^2 \right)
$$

$$
[\lambda^a] = \frac{3}{2} \quad [X^i] = 1 \quad [A_\mu] = 1, \quad [g_{YM}] = 0
$$

$\rightarrow$ renormalisable theory

Quantum level: $\mathcal{N} = 4$ SYM exhibits no UV divergences! Also, it preserves all its symmetries!
Symmetries of $\mathcal{N} = 4$ SYM

- R-symmetry $SO(6) \sim SU(4)$, with generators $T^A$, $A = 1, \ldots, 15$
- scale invariance + Poincaré invariance $\rightarrow$ conformal symmetry in 4d $SO(2, 4)$ with generators $P_\mu, L_{\mu\nu}, D, K_\mu$
- even more, $\mathcal{N} = 4$ Poincaré Supersymmetry + conformal invariance $\rightarrow$ superconformal symmetry $SU(2, 2|4)$, with superalgebra

\[
\begin{pmatrix}
P_\mu, K_\mu, L_{\mu\nu}, D & Q_\alpha^a, \bar{S}_{\dot{\alpha}}^a \\
\bar{Q}_{\dot{\alpha}a}, S_{\alpha a} & T^A
\end{pmatrix}
\]
Now for something completely different.

D3-branes are massive charged objects in a gravity theory: string theory!

What happens when taking the same low energy limit, and describe it with string theory?
D3-branes: Source of Gravity

Now for something completely different.

D3-branes are massive charged objects in a gravity theory: string theory!

What happens when taking the same low energy limit, and describe it with string theory?

Replace D-branes by a non-trivial geometry, as in everyday life physics!
Black p-Branes

Again, type IIB string theory in 10-d.

Low energy effective action

\[
S = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10} x \sqrt{-g} \left( e^{-2\phi} (R + 4(\nabla \phi)^2) - \frac{2}{(5)!} F_5^2 \right)
\]
Black p-Branes

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→ Look for solution as 3-d electric source of charge \( N \) for \( A_4 \).

\[ N = \int_{S^5} *F_5 \]

Require desirable symmetries: \( ISO(1,3) \) in 3-d along D3-brane, and spherical symmetry \( SO(6) \) in 6 transversal directions.
Classical SuGra Solution

In 4-d Gravity with point-like object: Reissner-Nördstrom black hole solution

In 10-d SuperGravity with 3-d object: not so easy…
Classical SuGra Solution

In 4-d Gravity with point-like object: Reissner-Nördstrom black hole solution

In 10-d SuperGravity with 3-d object: not so easy...

Extremal Solution:

\[ ds^2 = \frac{1}{\sqrt{H(r)}} \eta_{\mu\nu} dx^\mu dx^\nu + \sqrt{H(r)} (dr^2 + r^2 d\Omega_5^2) \]

with \( H(r) = 1 + \frac{R^4}{r^4} \) \quad \quad R^4 = 4\pi g_s \alpha'^2 N
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\end{align*}
\]

with \( H(r) = 1 + \frac{R^4}{r^4} \)

\( R^4 = 4\pi g_s \alpha'^2 N \)

This description is appropriate at **low curvature** of the 3-brane geometry:

\[
R \gg l_s \quad \implies \quad 1 \ll g_s N \ll N
\]
Near Horizon Limit

Duality coming from this **double-description**:

- $g_s N \gg 1$: use the SuGra solution
- $g_s N \ll 1$: use perturbative string theory
Near Horizon Limit

Duality coming from this double-description:
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Take the near horizon limit: $r \ll R$

$$ds^2 = \frac{r^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + R^2 \frac{dr^2}{r^2} + R^2 d\Omega_5^2$$

$$R^4 = 4\pi g_s \alpha'^2 N$$

Claim: this is the metric of the product space $AdS_5 \times S^5$!
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Take a deep breath:

N D3-branes can consistently be described in a low energy limit by
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- a $U(N)$ gauge theory on the world-volume
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- \( g_s N \ll 1 \): use perturbative string theory

Take the near horizon limit: \( r \ll R \)

\[
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\]

\[
R^4 = 4\pi g_s \alpha'^2 N
\]

Claim: this is the metric of the product space \( AdS_5 \times S^5 \)!

Take a deep breath:

N D3-branes can consistently be described in a low energy limit by
- a U(N) gauge theory on the world-volume
- a string theory in a background which near the horizon is \( AdS_5 \times S^5 \) with radius \( R^4 = 4\pi g_s \alpha'^2 N \)
Part II

\( \text{AdS}_5/\text{CFT}_4 \) Correspondence

or How to get to the point
Low Energy Limit Action

N parallel D3-branes
N parallel D3-branes

String theory:
→ closed strings, excitations of empty space
→ open strings ending on D-branes, excitations of the D-branes
Low Energy Limit Action

$N$ parallel D3-branes

String theory:
→ closed strings, excitations of empty space
→ open strings ending on D-branes, excitations of the D-branes

Take low energy limit, so that only massless states survive:
→ closed strings states: give type IIB SuGra
→ open strings states: give $\mathcal{N} = 4 \ U(N) \ SYM$ up to higher derivative terms

$$\Rightarrow \quad S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}$$
N parallel D3-branes

String theory:
→ closed strings, excitations of empty space
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$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}$$

Take $\alpha' \to 0$, $g_s$, $N$ fixed:
→ $S_{\text{int}}$ vanishes
→ $S_{\text{bulk}}$ becomes free gravity
→ $S_{\text{brane}}$ becomes pure $\mathcal{N} = 4$ $U(N)$ $SYM$
Two decoupled systems:
- 10d free gravity in the bulk
- 4d gauge theory in the branes

Energies are redshifted by a factor $\frac{1}{1 + \frac{r_4}{r_4} R_4}$ → states of any mass close to the horizon $r = 0$ will survive → massless states away from the horizon also survive → these two sets of states are decoupled from each other.
Two decoupled systems:
- 10d free gravity in the bulk
- 4d gauge theory in the branes

Now replace the N D3-branes by the geometry we found, and take the low energy limit:

$$ds^2 = \frac{1}{\sqrt{H(r)}} \eta_{\mu\nu} dx^\mu dx^\nu + \sqrt{H(r)} (dr^2 + r^2 d\Omega_5^2)$$

$$H(r) = 1 + \frac{R^4}{r^4} \quad R^4 = 4\pi g_s \alpha'^2 N$$

Energies are redshifted by a factor $1/ \left(1 + \frac{R^4}{r^4}\right)^{\frac{1}{4}}$
Two decoupled systems:

- 10d free gravity in the bulk
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Second Low Energy Description

⇒ Two decoupled systems:

- 10d free gravity in the bulk
- near horizon full theory
Second Low Energy Description

⇒ Two decoupled systems:
- 10d free gravity in the bulk
- near horizon full theory

Remember: near horizon region is $AdS_5 \times S^5$

Conclusion:
In both approaches, we get two decoupled theories:
- 10d free gravity in the bulk
- 4d gauge theory in the branes
- 10d free gravity in the bulk
- near horizon full theory
AdS/CFT Conjecture

Identify the two approaches and be forced to make the conjecture:

\[ \mathcal{N} = 4 \ U(N) \ SYM \] in flat 3+1 dimensions

is "equivalent" to

Type IIB superstring theory on \( AdS_5 \times S^5 \)
String theory

- $AdS_5$ has isometry group $SO(2, 4)$
- $S^5$ has rotational symmetry $SO(6)$

$N = 4$ SYM

- 4d conformal symmetry $\simeq SO(2, 4)$
- it also has global symmetry $SO(6)$

In fact, the whole supersymmetric group match on both sides

$\rightarrow$ both sides of the conjecture have the same spacetime symmetries!
Regimes

Remember:

\[ R^4 = 4\pi g_s \alpha'^2 N \]
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$$R^4 = 4\pi g_s \alpha'^2 N$$

A perturbative analysis in the YM part can be trusted when

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4} \ll 1 \quad \text{Perturbative FT Regime}$$
Regimes

Remember:

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A perturbative analysis in the YM part can be trusted when

\[ g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4} \ll 1 \quad \text{Perturbative FT Regime} \]

The classical gravity description can be trusted when

\[ \frac{R^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1 \quad \text{Gravity Regime} \]

\[ \Rightarrow \quad \text{perfectly incompatible: duality} \]
Problem: how to link $AdS_5$ fields to CFT$_4$ operators?
Boundary Considerations

**Problem**: how to link $AdS_5$ fields to CFT$_4$ operators?

**Fact**: $AdS$ space has a **boundary**; any Field Theory on $AdS$ needs boundary conditions on its fields to be solvable. The conformal boundary of $AdS_5$ is compactified 4d Minkowski space $\mathbb{M}^4$.

The CFT$_4$ **lives precisely** on $\mathbb{M}^4$. 
Boundary Considerations

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The CFT$_4$ lives precisely on $\mathbb{M}^4$.

Solution: for an operator $O$ in CFT$_4$, add to the Lagrangian

$$\int d^4x \; \phi_0(\vec{x}) O(\vec{x})$$

where a field $\phi(\vec{x}, z)$ in $AdS_5$ has boundary condition

$$\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})$$
value of $\phi(\vec{x}, z)$ on the boundary act as source to operator $O$. 
Formulation

\[ \Rightarrow \text{ value of } \phi(\vec{x}, z) \text{ on the boundary act as source to operator } O. \]

So, naturally propose:

\[ \langle e^{\int d^4x \phi_0(\vec{x})O(\vec{x})} \rangle_{\text{CFT}}^{\text{AdS/CFT}} = Z_{\text{AdS}}^{\text{string}} [\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})] \]

which matches

the generating functional of \( O \)-correlation functions

with

the full partition function of string theory (in AdS) with boundary condition.

This is the way correlation functions may be computed.

**Note:** The coupling requires field \( \phi \) and operator \( O \) to have the same quantum numbers of the theories’ symmetry group!
Take a Breath: what have we done?

N D3-branes in string theory
\[ \rightarrow \] decoupled system with states in 4d
\[ N = 4 \]
\[ SU(N) \text{SYM} \otimes \text{free 10d SuGra} \]

N D3-branes as geometry
\[ \rightarrow \] decoupled system with states in Full type IIB string theory in AdS
\[ 5 \times S^5 \otimes \text{free 10d SuGra} \]

Identify the two views, conjecture equivalence

Remark the equivalence as a duality: appreciate it as powerful; question how to test it

Let the CFT live on the boundary of the string theory; propose link by letting boundary values of fields on the bulk couple as sources to operators on the boundary CFT

Despair
Take a Breath: what have we done?

1. \( N \) D3-branes in string theory + low energy limit
   \( \rightarrow \) decoupled system with states in
   \[ 4d \mathcal{N} = 4 \, SU(N) \, SYM \otimes \text{free 10d SuGra} \]
Take a Breath: what have we done?

1. N D3-branes in string theory + low energy limit
   \[ \rightarrow \text{decoupled system with states in} \]
   \[ 4d \mathcal{N} = 4 \ SU(N) \ SYM \otimes \text{free 10d SuGra} \]

2. N D3-branes as geometry + low energy limit
   \[ \rightarrow \text{decoupled system with states in} \]
   \[ \text{Full type IIB string theory in} \ AdS_5 \times S^5 \otimes \text{free 10d SuGra} \]
Take a Breath: what have we done?

1. N D3-branes in string theory + low energy limit
   $\rightarrow$ decoupled system with states in
   $\mathcal{N} = 4 \ SU(N)\ SYM \otimes\ free\ 10d\ SuGra$

2. N D3-branes as geometry + low energy limit
   $\rightarrow$ decoupled system with states in
   Full type IIB string theory in $AdS_5 \times S^5 \otimes\ free\ 10d\ SuGra$

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   \[ \rightarrow \text{decoupled system with states in} \]
   \[ \text{Full type IIB string theory in } AdS_5 \times S^5 \otimes \text{free 10d SuGra} \]

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→ decoupled system with states in

   Full type IIB string theory in \( AdS_5 \times S^5 \otimes \text{free 10d SuGra} \)

3. Identify the two views, conjecture **equivalence**

4. Remark the equivalence as a **duality**: appreciate it as powerful; question how to test it

5. Let the CFT live on the boundary of the string theory; propose link by letting boundary values of fields on the bulk **couple as sources** to operators on the boundary CFT
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6. Despair
Small detour on Holography.

The AdS/CFT correspondence is an explicit realisation of holography.

By matching every observable on AdS$_5$ to every observable on CFT$_4$, we apparently reduce the dimension of our theory by 1 without any problem.

Where do the mismatching degrees of freedom go?
Small detour on Holography.

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Where do the mismatching degrees of freedom go?

Solution:

Holographic principle: in a full quantum gravity description this is not a problem, it is even required.
Black-hole entropy: \( S_{BH} = \frac{A}{4} \)

\( \rightarrow \) generalised 2nd law of thermodynamics: \( dS_{BH+\text{matter}} \geq 0 \)

This law requires an entropy bound for matter.
Black-hole entropy: $S_{BH} = \frac{A}{4}$

→ generalised 2nd law of thermodynamics: $dS_{BH+matter} \geq 0$

This law requires an entropy bound for matter.

Spherical entropy bound:

$$S_{matter} \leq \frac{A}{4G_N}$$
In a quantum system, $e^S$ is the **number of independent quantum states** compatible with certain macroscopic parameters, e.g. $E, T, V, ...$

This is also the number of degrees of freedom of the quantum theory where the system is described.
Entropy and degrees of freedom

In a quantum system, $e^S$ is the **number of independent quantum states** compatible with certain macroscopic parameters, e.g. $E, T, V, ...$

This is also the number of degrees of freedom of the quantum theory where the system is described.

More precisely,

\[ N \equiv \#dof = \log \dim(\mathcal{H}) \]

\[ \dim(\mathcal{H}) = e^S \]

For a QFT: $N = \infty$

For a regularised QFT (e.g. quantum gravity): $N \sim V$

Contradiction: $S \leq A/4 \implies N \sim A$
We acknowledge the previous result as a law of nature and generalise the concept.

Holographic Principle:

"In a quantum theory of gravity all physics within some volume can be described in terms of some theory on the boundary which has less than one d.o.f. per Planck area"

→ so that it satisfies the required bound.

In a nutshell, our theories have a redundancy in the form of useless d.o.f. ; this redundancy can be solved by considering some dual theories on the boundary of the space our theories live in.
The AdS/CFT correspondence explicitly realises this principle.

How: count the degrees of freedom and compare with the area.

Problem: CFT has infinitely many d.o.f.; boundary of AdS has infinite area...

→ regularise infinities and compare them.
The AdS/CFT correspondence explicitly realises this principle.

How: count the degrees of freedom and compare with the area.

Problem: CFT has infinitely many d.o.f.; boundary of AdS has infinite area...

→ regularise infinities and compare them.

- Regularise the boundary: discrete cells of size $\delta$, 1 d.o.f. per cell
- $\# \text{ cells } \sim \delta^{-3}$
- $\# \text{ d.o.f. in U(N) with this UV cut-off: } S \sim N^2 \delta^{-3}$
- $\implies$ we have a IR cut-off at $z \sim \delta$ on the bulk
Area in $AdS_5$ of the surface $z \sim \delta$:

$$A \sim \frac{R^3}{\delta^3}$$

Given that $G_5 \sim N^{-2} R^3$,

$$\implies S \sim \frac{A}{G_5}$$

The holographic bound is thus saturated by the AdS/CFT correspondence.

AdS/CFT is a realisation of the Holographic Principle.
Part III:
AdS$_5$/CFT$_4$ Correlation Functions

or How to find old CFT friends
Correlation Functions

So,

\[ Z_{CFT} [\phi_0] \equiv \langle e^{\int d^4x \, \phi_0(\vec{x}) O(\vec{x})} \rangle_{CFT} = Z_{AdS}^{string} [\phi(\vec{x}, z)|_z=0 = \phi_0(\vec{x})] \]

we meet again.
Correlation Functions

So,

$$Z_{\text{CFT}}[\phi_0] \equiv \langle e^{\int d^4x \; \phi_0(\vec{x}) O(\vec{x})} \rangle_{\text{CFT}} = Z^{\text{string}}_{\text{AdS}}[\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})]$$

we meet again.

Computing $O$-correlation functions:

$$\langle O...O \rangle = \frac{\delta^n Z_{\text{CFT}}[\phi_0]}{\delta \phi^n_0} \bigg|_{\phi_0=0}$$

$\rightarrow \phi(\vec{x}, z)$ solves e.o.m. derived from $S_{\text{AdS}_5}$ (on-shell), with given boundary conditions

$\rightarrow$ the extension from $\phi_0$ to $\phi$ is unique
Consider

\[ S_{\text{AdS}}[\phi] = \frac{1}{2} \int d^{d+1}x \sqrt{g} \ (\partial \phi)^2 \]

**Idea**

- solve \( \phi \) in terms of \( \phi_0 \) with regularised boundary condition
- evaluate \( S[\phi] \) on \( \phi \to S_{\text{AdS}}[\phi_0] \)
- take functional derivatives w.r.t. \( \phi_0 \)

\( \to \) \( \phi \) is on-shell: integrate \( S_{\text{AdS}} \) by parts. The term giving the e.o.m. is zero, the remaining regularised boundary term is

\[ S_{\text{AdS}}[\phi] = \lim_{\epsilon \to 0} \frac{1}{2} \int_{T_\epsilon} d^d x \sqrt{\mathcal{h}} \ \phi \ \partial_n \phi \]
Scalar Field 2-pt. Function

\[ S_{AdS}[\phi_0] = \frac{cd}{2} \int d\vec{x} \, d\vec{x}' \frac{\phi_0(\vec{x})\phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2d}} \]

2-pt function in a CFT of operator \( O \) with conformal dimension

\[ \Delta = d \]

Also:

\[ \langle O \rangle = \left. \frac{\delta S_{AdS}}{\delta \phi_0} \right|_{\phi_0=0} = 0 \]
Scalar Field 2-pt. Function

\[ S_{AdS} [\phi_0] = \frac{cd}{2} \int d\vec{x} \; d\vec{x}' \; \phi_0(\vec{x})\phi_0(\vec{x}') \frac{1}{|\vec{x} - \vec{x}'|^{d+2}} \]

2-pt function in a CFT of operator \( O \) with conformal dimension

\[ \Delta = d \]

Also:

\[ \langle O \rangle = \left. \frac{\delta S_{AdS}}{\delta \phi_0} \right|_{\phi_0=0} = 0 \]

Massive case: more complicated. Result:

\[ \Delta = \frac{1}{2} \left( d + \sqrt{d^2 + 4m^2 R^2} \right) \]
Massive interacting scalar fields:

\[ S_{\text{AdS}} = \int d^5x \sqrt{g} \left( \frac{1}{2} (\partial \phi_i)^2 + \frac{1}{2} m_i^2 \phi_i^2 + \sum_{k=3}^{m} \lambda_{i_1...i_k} \phi_{i_1}...\phi_{i_k} \right) \]

Typical e.o.m.: \((\Box + m^2)\phi = \lambda \phi^n\)
Massive interacting scalar fields:

\[ S_{\text{AdS}} = \int d^5x \sqrt{g} \left( \frac{1}{2} (\partial \phi_i)^2 + \frac{1}{2} m_i^2 \phi_i^2 + \sum_{k=3}^m \lambda_{i_1...i_k} \phi_{i_1}...\phi_{i_k} \right) \]

Typical e.o.m.: \((\Box + m^2)\phi = \lambda \phi^n\)

Perturbative expansion on \(\lambda\):

at \(\lambda^0\) : \(\phi^{\text{zero}}(x, z) = \int K(x - x', z) \phi_0(x')\)

at \(\lambda^1\) : \(\phi^{\text{one}}(x, z) = \int G(x - x', z - z') (\phi^{\text{zero}}(x', z'))^n\)

where \(G(x - x', z - z')\) is a bulk-to-bulk Green’s function.
THANK YOU FOR YOUR ATTENTION!