

Exercise 1. One-Body Quantum Marginal Problem for N Qubits

Let $\rho = |\Psi\rangle\langle\Psi|$ be a pure quantum state of N qubits. We shall denote by $\lambda_{\max}^{(k)}$ the maximal eigenvalue of the reduced density matrix of the k -th qubit, $\rho^{(k)}$.

(a) Show that the eigenvalues satisfy the *polygonal inequalities*

$$\sum_{l \neq k} \lambda_{\max}^{(l)} \leq \lambda_{\max}^{(k)} + (N - 2). \tag{1}$$

Solution. Just as in the lecture, we use the variational principle for the largest eigenvalue. By symmetry, it suffices to consider the case where $k = N$:

$$\begin{aligned} \sum_{l < N} \lambda_{\max}^{(l)} &= \sum_{l < N} \max_{\|\phi\|=1} \langle \phi | \rho^{(l)} | \phi \rangle = \max_{\|\phi^{(l)}\|=1} \sum_{l < N} \langle \phi^{(l)} | \rho^{(l)} | \phi^{(l)} \rangle \\ &= \max_{\|\phi^{(1)}\|=1, \dots, \|\phi^{(N-1)}\|=1} \text{tr} \left(\rho^{(1 \dots N-1)} \underbrace{\sum_{l < N} \mathbf{1}^{\otimes(l-1)} \otimes |\phi^{(l)}\rangle\langle\phi^{(l)}| \otimes \mathbf{1}^{\otimes(N-l-1)}}_{\leq |\phi^{(1)} \otimes \dots \otimes \phi^{(N-1)}\rangle\langle\phi^{(1)} \otimes \dots \otimes \phi^{(N-1)}| + (N-2)\mathbf{1}^{\otimes(N-1)}} \right). \end{aligned}$$

In order to upper-bound the underbraced Hermitian operator, we have used that its eigenvalues are $N - 1, N - 2, \dots, 1, 0$ and, moreover, that the eigenvalue $N - 1$ has multiplicity one, with eigenvector given by the product of the $|\phi^{(l)}\rangle$. (More generally, it is easy to see that the multiplicity of an eigenvalue μ is equal to the binomial coefficient $\binom{N-1}{\mu}$.)

If we maximize over all pure states of $N - 1$ (instead of only over the product ones) then we find that

$$\sum_{l < N} \lambda_{\max}^{(l)} \leq \max_{\|\phi\|=1} \langle \phi | \rho^{(1 \dots N-1)} | \phi \rangle + (N - 2).$$

The left-hand side is equal to the maximal eigenvalue of $\rho^{(1 \dots N-1)}$, which in turn is the maximal eigenvalue $\lambda_{\max}^{(N)}$ of $\rho^{(N)}$, since the overall state is pure.

These inequalities are in fact the only constraints. That is, for any choice of $\lambda_{\max}^{(k)} \in [0.5, 1]$ satisfying (1) there exists a corresponding pure state.

(b) Prove this statement by explicitly constructing a global state.

Hint. Solve the problem for $N = 3$ and induct.

Solution. We follow the inductive argument of Higuchi, Sudbery and Szulec for $N \geq 3$ (arXiv:0209085); the case where $N = 2$ follows trivially from the Schmidt decomposition.

(1) $N = 3$: We consider the “ansatz”

$$|\Psi_3\rangle = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle.$$

where a, b, c, d are real and $a^2 + b^2 + c^2 + d^2 = 1$. Then,

$$\rho^{(1)} = \begin{pmatrix} a^2 + b^2 & \\ & c^2 + d^2 \end{pmatrix}, \quad \rho^{(2)} = \begin{pmatrix} a^2 + c^2 & \\ & b^2 + d^2 \end{pmatrix}, \quad \rho^{(3)} = \begin{pmatrix} a^2 + d^2 & \\ & b^2 + c^2 \end{pmatrix},$$

and we would like to solve the equations

$$\begin{aligned} \lambda_{\max}^{(1)} &= a^2 + b^2, \\ \lambda_{\max}^{(2)} &= a^2 + c^2, \\ \lambda_{\max}^{(3)} &= a^2 + d^2. \end{aligned}$$

These imply

$$\begin{aligned} b^2 &= \frac{1}{2} \left(\lambda_{\max}^{(1)} + 1 - \lambda_{\max}^{(2)} - \lambda_{\max}^{(3)} \right) \\ c^2 &= \frac{1}{2} \left(\lambda_{\max}^{(2)} + 1 - \lambda_{\max}^{(1)} - \lambda_{\max}^{(3)} \right) \\ d^2 &= \frac{1}{2} \left(\lambda_{\max}^{(3)} + 1 - \lambda_{\max}^{(1)} - \lambda_{\max}^{(2)} \right), \end{aligned}$$

which can be solved over the reals provided that the polygonal inequalities (1) are satisfied. Moreover, it follows that

$$a^2 = 1 - b^2 + c^2 + d^2 = 1 - \frac{1}{2} \left(3 - \lambda_{\max}^{(1)} - \lambda_{\max}^{(2)} - \lambda_{\max}^{(3)} \right) \geq \frac{1}{4},$$

so we can also choose a accordingly.

(2) $N > 3$: We first note that the corresponding ansatz for $|\Psi_N\rangle$ as a linear combination of $|0\dots 0\rangle$ and those basis vectors which contain at least two “1”s does not work as smoothly, since the single-body reduced density matrices will no longer be diagonal in the computational basis. Hence we proceed differently:

Without loss of generality, we may assume that the given eigenvalues are ordered according to $\lambda_{\max}^{(1)} \leq \dots \leq \lambda_{\max}^{(N)}$, so that the inequality

$$\lambda_{\max}^{(2)} + \dots + \lambda_{\max}^{(N)} \leq \lambda_{\max}^{(1)} + (N - 2) \tag{S.1}$$

is the strongest among all polygonal inequalities (i.e., it implies the other ones).

Set $\Delta := 1 - (\lambda_{\max}^{(N)} - \lambda_{\max}^{(1)}) \in [0.5, 1]$. We claim that there exists a quantum state $|\Psi_{N-1}\rangle$ of $N - 1$ qubits with local eigenvalues $\Delta, \lambda_{\max}^{(2)}, \dots, \lambda_{\max}^{(N-1)}$. To see this, we consider two cases: In the case where $\Delta \leq \lambda_{\max}^{(2)}$, the strongest of the polynomial inequalities is

$$\lambda_{\max}^{(2)} + \dots + \lambda_{\max}^{(N-1)} \leq \Delta + (N - 3),$$

which follows from (S.1). If $\Delta > \lambda_{\max}^{(2)}$ then the strongest of the polynomial inequalities is

$$\Delta + \lambda_{\max}^{(3)} + \dots + \lambda_{\max}^{(N-1)} = \lambda_{\max}^{(1)} + (1 + \lambda_{\max}^{(3)} - \lambda_{\max}^{(2)}) + \dots + \lambda_{\max}^{(N-1)} \leq \lambda_{\max}^{(1)} + (N - 3) \leq \lambda_{\max}^{(2)} + (N - 3).$$

Consider now the Schmidt decomposition of $|\Psi_{N-1}\rangle$, which has the form

$$|\Psi_{N-1}\rangle = |0\rangle \otimes |\phi_{N-2}\rangle + |1\rangle \otimes |\psi_{N-2}\rangle$$

with $\langle \phi_{N-2} | \phi_{N-2} \rangle = \Delta = 1 - \langle \psi_{N-2} | \psi_{N-2} \rangle$. We make the following ansatz for the N -particle pure state:

$$|\Psi_N\rangle = \cos(\chi) |0\rangle \otimes \phi_{N-2} \otimes |0\rangle + \sin(\chi) |1\rangle \otimes \phi_{N-2} \otimes |1\rangle + |1\rangle \otimes \psi_{N-2} \otimes |0\rangle.$$

Clearly, the maximal eigenvalues of the particles 2, \dots , $N - 1$ are correct for any choice of the phase χ . On the other hand,

$$\rho^{(1)} = \begin{pmatrix} \cos^2(\chi)\Delta & \\ & \sin^2(\chi)\Delta + (1 - \Delta) \end{pmatrix}, \quad \rho^{(N)} = \begin{pmatrix} \cos^2(\chi)\Delta + (1 - \Delta) & \\ & \sin^2(\chi)\Delta \end{pmatrix}.$$

Since $\Delta \geq \lambda_{\max}^{(1)}$, we can find a phase χ such that $\cos^2(\chi)\Delta = \lambda_{\max}^{(1)}$. But then,

$$\cos^2(\chi)\Delta + (1 - \Delta) = \lambda_{\max}^{(1)} + (\lambda_{\max}^{(N)} - \lambda_{\max}^{(1)}) = \lambda_{\max}^{(N)},$$

hence the maximal eigenvalues of both the first and last particle are correct for this choice of χ .

(c) Prove this statement by using convexity of the solution.

Solution. The polygonal inequalities (1) together with the “trivial” inequalities $\lambda_{\max}^{(k)} \geq 0.5$ cut out a convex polytope, whose vertices are all points of the form

$$\left(\underbrace{0.5, \dots, 0.5}_{\text{none or at least two}}, 1, \dots, 1 \right)$$

together with their permutations. Since the solution of the one-body quantum marginal problem (i.e., the set of achievable maximal eigenvalues) is convex, it suffices to show that each vertex is achievable:

$$\begin{aligned} |0\dots 0\rangle &\mapsto (1, \dots, 1), \\ \frac{1}{\sqrt{2}} (|0\dots 0\rangle + |1\dots 1\rangle) \otimes |0\dots 0\rangle &\mapsto (0.5, \dots, 0.5, 1, \dots, 1). \end{aligned}$$

Exercise 2. Isotypical Projectors

Recall from the lecture that any finite-dimensional unitary representation \mathcal{H} of $SU(2)$ can be decomposed into a direct sum of irreducible representations which are all of the same spin, i.e.

$$\mathcal{H} = \bigoplus_{j=0, \frac{1}{2}, 1, \dots} \mathcal{H}_j, \quad \mathcal{H}_j \cong \underbrace{V_j \oplus \dots \oplus V_j}_{m_j \text{ times}}$$

Here, V_j denotes the irreducible representation of $SU(2)$ with spin $j \in \{0, \frac{1}{2}, 1, \dots\}$. The subspace \mathcal{H}_j is called an *isotypical component* of \mathcal{H} ; it is canonically defined (i.e., does not depend on any choices). The corresponding *isotypical projector* is the orthogonal projection onto \mathcal{H}_j , and we denote it by P_j . Similarly, the irreducible components of the product group $K = SU(2)^N$ are just the tensor products $V_{j_1} \otimes \dots \otimes V_{j_N}$, and hence the isotypical projectors are given by $P_{j_1} \otimes \dots \otimes P_{j_N}$.

As in the lecture, let $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ be the Hilbert space of N qubits and denote by $\mathbb{C}[\mathcal{H}]_{(k)}$ the space of polynomial functions on \mathcal{H} of degree k . Show that the following two statements are equivalent:

1. There exists a pure state $|\psi\rangle \in \mathcal{H}$ such that $(P_{j_1} \otimes \dots \otimes P_{j_N}) |\psi\rangle^{\otimes k} \neq 0$.
2. $V_{j_1}^* \otimes \dots \otimes V_{j_N}^* \subseteq \mathbb{C}[\mathcal{H}]_{(k)}$.

Discuss how this connects the spectrum estimation theorem from the last lecture with the representation-theoretic description of the one-body quantum marginal problem presented in the lecture before.

Solution. The vector $|\psi\rangle^{\otimes k}$ is an element of the symmetric subspace $\text{Sym}^k(\mathcal{H})$, which is not only a subspace of $\mathcal{H}^{\otimes k}$, but also a representation of $U(\mathcal{H})$ —indeed, if $|\phi\rangle$ is a fully symmetric tensor then so is $U^{\otimes k} |\phi\rangle$ for any global unitary $U \in U(\mathcal{H})$. The same is of course true if we restrict to the “subgroup” of local unitaries $K = SU(2)^N$.

Thus, if $(P_{j_1} \otimes \dots \otimes P_{j_N}) |\psi\rangle^{\otimes k} \neq 0$ then the corresponding isotypical component of $\text{Sym}^k(\mathcal{H})$ is necessarily non-zero (since $|\psi\rangle^{\otimes k}$ has non-zero overlap with it!), hence $V_{j_1} \otimes \dots \otimes V_{j_N} \subseteq \text{Sym}^k(\mathcal{H})$.

Abstractly, it follows that $V_{j_1}^* \otimes \dots \otimes V_{j_N}^* \subseteq (\text{Sym}^k(\mathcal{H}))^* \cong \mathbb{C}[\mathcal{H}]_{(k)}$ (Q: What is the last isomorphism?).

To make this concrete, choose a basis $|j_1, m_1\rangle \otimes \dots \otimes |j_N, m_N\rangle$ of $V_{j_1} \otimes \dots \otimes V_{j_N}$. Since $V_{j_1} \otimes \dots \otimes V_{j_N} \subseteq \text{Sym}^k(\mathcal{H})$, we can find a corresponding basis in $\text{Sym}^k(\mathcal{H})$, which we shall denote by the same symbol. But then,

$$|\psi'\rangle \mapsto (\langle j_1, m_1 | \otimes \dots \otimes \langle j_N, m_N |) |\psi'\rangle^{\otimes k}$$

defines a family of polynomials of degree k which transform in the same way as the *dual* basis $\langle j_1, m_1 | \otimes \dots \otimes \langle j_N, m_N |$. Note that the family is non-trivial, since we know that at least one of the functions is non-zero when we plug in $|\psi\rangle$. It follows that $V_{j_1}^* \otimes \dots \otimes V_{j_N}^* \subseteq \mathbb{C}[\mathcal{H}]_{(k)}$.

The converse statement follows by reversing the above argument.

Coincidentally, $V_j \cong V_j^*$ for representations of $SU(2)$. The irreducible representations of $SU(d)$, $d > 2$, are in general no longer self-dual.