## Exercise 1. *One-Body Quantum Marginal Problem for $N$ Qubits*

Let $\rho = |\Psi\rangle\langle\Psi|$ be a pure quantum state of $N$ qubits. We shall denote by $\lambda_{\max}^{(k)}$ the maximal eigenvalue of the reduced density matrix of the $k$-th qubit, $\rho^{(k)}$.

(a) Show that the eigenvalues satisfy the *polygonal inequalities*

$$\sum_{l \neq k} \lambda_{\max}^{(l)} \leq \lambda_{\max}^{(k)} + (N-2). \tag{1}$$

**Solution.** Just as in the lecture, we use the variational principle for the largest eigenvalue. By symmetry, it suffices to consider the case where $k = N$:

$$\begin{aligned}\sum_{l<N} \lambda_{\max}^{(l)} &= \sum_{l<N} \max_{\|\phi\|=1} \langle\phi|\rho^{(l)}|\phi\rangle = \max_{\|\phi^{(l)}\|=1} \sum_{l<N} \langle\phi^{(l)}|\rho^{(l)}|\phi^{(l)}\rangle \\ &= \max_{\|\phi^{(1)}\|=\ldots\|\phi^{(N-1)}\|=1} \operatorname{tr}\Big(\rho^{(1\ldots N-1)} \underbrace{\sum_{l<N} \mathbf{1}^{\otimes(l-1)} \otimes |\phi^{(l)}\rangle\langle\phi^{(l)}| \otimes \mathbf{1}^{\otimes(N-l-1)}}_{\leq |\phi^{(1)}\otimes\ldots\otimes\phi^{(N-1)}\rangle\langle\phi^{(1)}\otimes\ldots\otimes\phi^{(N-1)}| + (N-2)\mathbf{1}^{\otimes(N-1)}}\Big).\end{aligned}$$

In order to upper-bound the underbraced Hermitian operator, we have used that its eigenvalues are $N-1, N-2, \ldots, 1, 0$ and, moreover, that the eigenvalue $N-1$ has multiplicity one, with eigenvector given by the product of the $|\phi\rangle^{(l)}$. (More generally, it is easy to see that the multiplicity of an eigenvalue $\mu$ is equal to the binomial coefficient $\binom{N-1}{\mu}$.)

If we maximize over all pure states of $N-1$ (instead of only over the product ones) then we find that

$$\sum_{l<N} \lambda_{\max}^{(l)} \leq \max_{\|\phi\|=1} \langle\phi|\rho^{(1\ldots N-1)}|\phi\rangle + (N-2).$$

The left-hand side is equal to the maximal eigenvalue of $\rho^{(1\ldots N-1)}$, which in turn is the maximal eigenvalue $\lambda_{\max}^{(N)}$ of $\rho^{(N)}$, since the overall state is pure.

These inequalities are in fact the only constraints. That is, for any choice of $\lambda_{\max}^{(k)} \in [0.5, 1]$ satisfying (1) there exists a corresponding pure state.

(b) Prove this statement by explicitely constructing a global state.
*Hint. Solve the problem for $N = 3$ and induct.*

**Solution.** We follow the inductive argument of Higuchi, Sudbery and Szulc for $N \geq 3$ (arXiv:0209085); the case where $N = 2$ follows trivially from the Schmidt decomposition.

(1) $N = 3$: We consider the "ansatz"

$$|\Psi_3\rangle = a\,|000\rangle + b\,|011\rangle + c\,|101\rangle + d\,|110\rangle\,.$$

where $a, b, c, d$ are real and $a^2 + b^2 + c^2 + d^2 = 1$. Then,

$$\rho^{(1)} = \begin{pmatrix} a^2+b^2 & \\ & c^2+d^2 \end{pmatrix}, \quad \rho^{(2)} = \begin{pmatrix} a^2+c^2 & \\ & b^2+d^2 \end{pmatrix}, \quad \rho^{(3)} = \begin{pmatrix} a^2+d^2 & \\ & b^2+c^2 \end{pmatrix},$$

and we would like to solve the equations

$$\begin{aligned}\lambda_{\max}^{(1)} &= a^2 + b^2, \\ \lambda_{\max}^{(2)} &= a^2 + c^2, \\ \lambda_{\max}^{(3)} &= a^2 + d^2.\end{aligned}$$

These imply

$$
\begin{aligned}
b^2 &= \frac{1}{2}\left(\lambda^{(1)}_{\max} + 1 - \lambda^{(2)}_{\max} - \lambda^{(3)}_{\max}\right)\\
c^2 &= \frac{1}{2}\left(\lambda^{(2)}_{\max} + 1 - \lambda^{(1)}_{\max} - \lambda^{(3)}_{\max}\right)\\
d^2 &= \frac{1}{2}\left(\lambda^{(3)}_{\max} + 1 - \lambda^{(1)}_{\max} - \lambda^{(2)}_{\max}\right),
\end{aligned}
$$

which can be solved over the reals provided that the polygonal inequalities (1) are satisfied. Moreover, it follows that

$$a^2 = 1 - b^2 + c^2 + d^2 = 1 - \frac{1}{2}\left(3 - \lambda^{(1)}_{\max} - \lambda^{(2)}_{\max} - \lambda^{(3)}_{\max}\right) \geq \frac{1}{4},$$

so we can also choose $a$ accordingly.

(2) $N > 3$: We first note that the corresponding ansatz for $|\Psi_N\rangle$ as a linear combination of $|0\ldots0\rangle$ and those basis vectors which contain at least two "1"s does not work as smoothly, since the single-body reduced density matrices will no longer be diagonal in the computational basis. Hence we proceed differently:

Without loss of generality, we may assume that the given eigenvalues are ordered according to $\lambda^{(1)}_{\max} \leq \ldots \leq \lambda^{(N)}_{\max}$, so that the inequality

$$\lambda^{(2)}_{\max} + \ldots + \lambda^{(N)}_{\max} \leq \lambda^{(1)}_{\max} + (N-2) \tag{S.1}$$

is the strongest among all polygonal inequalities (i.e., it implies the other ones).

Set $\Delta := 1 - (\lambda^{(N)}_{\max} - \lambda^{(1)}_{\max}) \in [0.5, 1]$. We claim that there exists a quantum state $|\Psi_{N-1}\rangle$ of $N-1$ qubits with local eigenvalues $\Delta, \lambda^{(2)}_{\max}, \ldots, \lambda^{(N-1)}_{\max}$. To see this, we consider two cases: In the case where $\Delta \leq \lambda^{(2)}_{\max}$, the strongest of the polynomial inequalities is

$$\lambda^{(2)}_{\max} + \ldots + \lambda^{(N-1)}_{\max} \leq \Delta + (N-3),$$

which follows from (S.1). If $\Delta > \lambda^{(2)}_{\max}$ then the strongest of the polynomial inequalities is

$$\Delta + \lambda^{(3)}_{\max} + \ldots + \lambda^{(N-1)}_{\max} = \lambda^{(1)}_{\max} + (1 + \lambda^{(3)}_{\max} - \lambda^{(N)}) + \ldots + \lambda^{(N-1)}_{\max} \leq \lambda^{(1)}_{\max} + (N-3) \leq \lambda^{(2)}_{\max} + (N-3).$$

Consider now the Schmidt decomposition of $|\Psi_{N-1}\rangle$, which has the form

$$|\Psi_{N-1}\rangle = |0\rangle \otimes |\phi_{N-2}\rangle + |1\rangle \otimes |\psi_{N-2}\rangle$$

with $\langle\phi_{N-2}|\phi_{N-2}\rangle = \Delta = 1 - \langle\psi_{N-2}|\psi_{N-2}\rangle$. We make the following ansatz for the $N$-particle pure state:

$$|\Psi_N\rangle = \cos(\chi)\,|0\rangle \otimes \phi_{N-2} \otimes |0\rangle + \sin(\chi)\,|1\rangle \otimes \phi_{N-2} \otimes |1\rangle + |1\rangle \otimes \psi_{N-2} \otimes |0\rangle\,.$$

Clearly, the maximal eigenvalues of the particles 2, ..., $N-1$ are correct for any choice of the phase $\chi$. On the other hand,

$$\rho^{(1)} = \begin{pmatrix} \cos^2(\chi)\Delta & \\ & \sin^2(\chi)\Delta + (1-\Delta) \end{pmatrix}, \quad \rho^{(N)} = \begin{pmatrix} \cos^2(\chi)\Delta + (1-\Delta) & \\ & \sin^2(\chi)\Delta \end{pmatrix}.$$

Since $\Delta \geq \lambda^{(1)}_{\max}$, we can find a phase $\chi$ such that $\cos^2(\chi)\Delta = \lambda^{(1)}_{\max}$. But then,

$$\cos^2(\chi)\Delta + (1-\Delta) = \lambda^{(1)}_{\max} + (\lambda^{(N)}_{\max} - \lambda^{(1)}_{\max}) = \lambda^{(N)}_{\max},$$

hence the maximal eigenvalues of both the first and last particle are correct for this choice of $\chi$.

(c) Prove this statement by using convexity of the solution.

**Solution.** The polygonal inequalities (1) together with the "trivial" inequalities $\lambda^{(k)}_{\max} \geq 0.5$ cut out a convex polytope, whose vertices are all points of the form

$$(\underbrace{0.5, \ldots, 0.5}_{\text{none or at least two}}, 1, \ldots, 1)$$

together with their permutations. Since the solution of the one-body quantum marginal problem (i.e., the set of achievable maximal eigenvalues) is convex, it suffices to show that each vertex is achievable:

$$
\begin{aligned}
|0\ldots0\rangle &\mapsto (1, \ldots, 1),\\
\frac{1}{\sqrt{2}}\left(|0\ldots0\rangle + |1\ldots1\rangle\right) \otimes |0\ldots0\rangle &\mapsto (0.5, \ldots, 0.5, 1, \ldots, 1).
\end{aligned}
$$

**Exercise 2.** ***Isotypical Projectors***

Recall from the lecture that any finite-dimensional unitary representation $\mathcal{H}$ of SU(2) can be decomposed into a direct sum of irreducible representations which are all of the same spin, i.e.

$$\mathcal{H} = \bigoplus_{j=0,\frac{1}{2},1,\ldots} \mathcal{H}_j, \quad \mathcal{H}_j \cong \underbrace{V_j \oplus \ldots \oplus V_j}_{m_j \text{ times}}$$

Here, $V_j$ denotes the irreducible representation of SU(2) with spin $j \in \{0, \frac{1}{2}, 1, \ldots\}$. The subspace $\mathcal{H}_j$ is called an *isotypical component* of $\mathcal{H}$; it is canonically defined (i.e., does not depend on any choices). The corresponding *isotypical projector* is the orthogonal projection onto $\mathcal{H}_j$, and we denote it by $P_j$. Similarly, the irreducible components of the product group $K = \mathrm{SU}(2)^N$ are just the tensor products $V_{j_1} \otimes \ldots \otimes V_{j_N}$, and hence the isotypical projectors are given by $P_{j_1} \otimes \ldots \otimes P_{j_N}$.

As in the lecture, let $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ be the Hilbert space of $N$ qubits and denote by $\mathbb{C}[\mathcal{H}]_{(k)}$ the space of polynomial functions on $\mathcal{H}$ of degree $k$. Show that the following two statements are equivalent:

1. There exists a pure state $|\psi\rangle \in \mathcal{H}$ such that $(P_{j_1} \otimes \ldots \otimes P_{j_N}) |\psi\rangle^{\otimes k} \neq 0$.
2. $V_{j_1}^* \otimes \ldots \otimes V_{j_N}^* \subseteq \mathbb{C}[\mathcal{H}]_{(k)}$.

Discuss how this connects the spectrum estimation theorem from the last lecture with the representation-theoretic description of the one-body quantum marginal problem presented in the lecture before.

**Solution.** The vector $|\psi\rangle^{\otimes k}$ is an element of the symmetric subspace $\mathrm{Sym}^k(\mathcal{H})$, which is not only a subspace of $\mathcal{H}^{\otimes k}$, but also a representation of $\mathrm{U}(\mathcal{H})$—indeed, if $|\phi\rangle$ is a fully symmetric tensor then so is $U^{\otimes k} |\phi\rangle$ for any global unitary $U \in \mathrm{U}(\mathcal{H})$. The same is of course true if we restrict to the "subgroup" of local unitaries $K = \mathrm{SU}(2)^N$.

Thus, if $(P_{j_1} \otimes \ldots \otimes P_{j_N}) |\psi\rangle^{\otimes k} \neq 0$ then the corresponding isotypical component of $\mathrm{Sym}^k(\mathcal{H})$ is necessarily non-zero (since $|\psi\rangle^{\otimes k}$ has non-zero overlap with it!), hence $V_{j_1} \otimes \ldots \otimes V_{j_N} \subseteq \mathrm{Sym}^k(\mathcal{H})$.

Abstractly, it follows that $V_{j_1}^* \otimes \ldots \otimes V_{j_N}^* \subseteq (\mathrm{Sym}^k(\mathcal{H}))^* \cong \mathbb{C}[\mathcal{H}]_{(k)}$ (Q: What is the last isomorphism?).

To make this concrete, choose a basis $|j_1, m_1\rangle \otimes \ldots \otimes |j_N, m_N\rangle$ of $V_{j_1} \otimes \ldots \otimes V_{j_N}$. Since $V_{j_1} \otimes \ldots \otimes V_{j_N} \subseteq \mathrm{Sym}^k(\mathcal{H})$, we can find a corresponding basis in $\mathrm{Sym}^k(\mathcal{H})$, which we shall denote by the same symbol. But then,

$$|\psi'\rangle \mapsto (\langle j_1, m_1| \otimes \ldots \otimes \langle j_N, m_N|) |\psi'\rangle^{\otimes k}$$

defines a family of polynomials of degree $k$ which transform in the same way as the *dual* basis $\langle j_1, m_1| \otimes \ldots \otimes \langle j_N, m_N|$. Note that the family is non-trivial, since we know that at least one of the functions is non-zero when we plug in $|\psi\rangle$. It follows that $V_{j_1}^* \otimes \ldots \otimes V_{j_N}^* \subseteq \mathbb{C}[\mathcal{H}]_{(k)}$.

The converse statement follows by reversing the above argument.

Coincidentally, $V_j \cong V_j^*$ for representations of SU(2). The irreducible representations of SU($d$), $d > 2$, are in general no longer self-dual.