Exercise 1. *Entropic relations*

(a) Prove

$$H(Z^A|B)_\rho + H(X^A|B)_\rho \geq \log_2 d + H(A|B)_\rho \tag{1}$$

where ρ is over systems AB and d is the dimension of system A.

Hint. First show that for any system over $A'B'AB$ we have

$$H(A'|AB) + H(B'|AB) \geq H(A'B'|AB)\ . \tag{2}$$

Then define

$$\Omega^{A'B'AB} = \frac{1}{d^2}\sum_{jk} P_j^{A'} \otimes P_k^{B'} \otimes \rho_{jk}^{AB}$$

where $\rho_{jk}^{AB} = X_j^A Z_k^A \rho Z_k^{\dagger A} X_j^{\dagger A}$, $P_i = |i\rangle\langle i|$ and $dim(A') = dim(B') = d$ and show that

$$H(A'|AB)_\Omega + H(B'|AB)_\Omega \geq H(A'B'|AB)_\Omega$$

implies

$$H(Z^A|B)_\rho + H(X^A|B)_\rho \geq log_2 d + H(A|B)_\rho\ .$$

Solution. It is always true that $I(A' : B'|AB) \geq 0$. Using $I(A' : B'|AB) = H(A'|AB) + H(B'|AB) - H(A'B'|AB)$ we get Equation (2). From the way we defined $\Omega^{A'B'AB}$ we have

$$\begin{aligned} H(A'|AB)_\Omega &= H(A'AB)_\Omega - H(AB)_\Omega = H(Z^AB)_\rho - H(B)_\rho = H(Z^A|B)_\rho \\ H(B'|AB)_\Omega &= H(B'AB)_\Omega - H(AB)_\Omega = H(X^AB)_\rho - H(B)_\rho = H(X^A|B)_\rho \\ H(A'B'|AB)_\Omega &= \log_2 d + H(A|B)_\rho \end{aligned}$$

which concludes the proof.

(b) Prove that if $H(X^A|B)_\rho = 0$ or $H(Z^A|E)_\rho = 0$ then

$$H(Z^A|B)_\rho + H(X^A|E)_\rho = \log_2 d\ .$$

Solution. We will need the following two equations:

$$H(Z^A|B) - H(Z^A|E) = H(A|B) \tag{S.1}$$

$$H(X^A|B) - H(X^A|E) = H(A|B) \tag{S.2}$$

We give here the proof for the first equality, the proof for the second one is analogous.

$$\begin{aligned} H(Z^A|B) &= H(Z^AB) - H(B) \\ &= H(B|Z^A) + H(Z^A) - H(B) \\ &= H(E|Z^A) + H(Z^A) - H(B) \end{aligned}$$

where the last equality follows from the fact that the conditional state $\rho_{BE|Z^A}$ is pure. Similarly we have

$$\begin{aligned} H(Z^A|E) &= H(Z^AE) - H(E) \\ &= H(E|Z^A) + H(Z^A) - H(E) \end{aligned}$$

Subtracting the two equations above we get

$$\begin{aligned} H(Z^A|B) - H(Z^A|E) &= -H(B) + H(E) \\ &= -H(B) + H(AB) \\ &= H(A|B) \end{aligned}$$

as required.

We now prove that $H(Z^A|B) + H(X^A|E) = \log_2 d$ for the case $H(Z^A|E) = 0$. The proof for the case $H(X^A|B) = 0$ is similar.

First, since $H(Z^A|E) = 0$ we get from Equation (S.1) that

$$H(Z^A|B) = H(A|B)\ . \tag{S.3}$$

We then get from Equation (1) that $H(X^A|B) \geq \log_2 d$, but since $H(X^A|B) \leq \log_2 d$ we must have equality, i.e.

$$H(X^A|B) = \log_2 d\ . \tag{S.4}$$

Combining Equation (S.2) and (S.3) together we get $H(X^A|B) = H(Z^A|B) + H(X^A|E)$ and therefore by using Equation (S.4) we get $H(Z^A|B) + H(X^A|E) = \log_2 d$.

As in the lecture, consider the following states:

$$\begin{aligned} |\psi\rangle^{ABR} &= \sum_z \sqrt{p_z}|z\rangle^A \otimes |\varphi_z\rangle^{BR} \\ |\psi'\rangle^{ABCR} &= \sum_z \sqrt{p_z}|z\rangle^A|z\rangle^C \otimes |\varphi_z\rangle^{BR} \end{aligned}$$

(c) Derive the conditions

$$\begin{aligned} H(Z^A|B)_\psi &\leq \varepsilon_1^2 \\ H(Z^A|R)_\psi &\geq \log_2 d - \varepsilon_2^2 \end{aligned}$$

from

$$\begin{aligned} H(X^A|RC)_{\psi'} &\geq \log_2 d - \varepsilon_1^2 \\ H(Z^A|R)_\psi &\geq \log_2 d - \varepsilon_2^2 \end{aligned}$$

using the previous item.

Solution. In the entropic relations above we take E to be RC. Since the system C just holds z, we know that $H(Z^A|RC)_{\psi'} = 0$. Therefore we can use the previous item and get

$$H(Z^A|B)_{\psi'} + H(X^A|RC)_{\psi'} = \log_2 d\ .$$

$H(X^A|RC)_{\psi'} \geq \log d - \varepsilon_1^2$ and therefore we have

$$H(Z^A|B)_{\psi'} \leq \varepsilon_1^2\ .$$

Since the marginal system $\rho'_{AB} = \mathrm{Tr}_{RC}(\psi')$ and $\rho_{AB} = \mathrm{Tr}_R(\psi)$ are equal this also implies

$$H(Z^A|B)_\psi \leq \varepsilon_1^2\ .$$

and we are done.