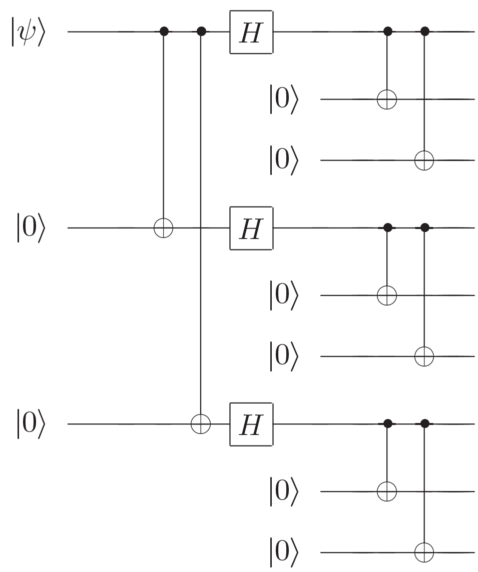
Exercise 1. ***Shor code***

In the lecture we saw how to encode a qubit using Shor's code in order to protect it from arbitrary error on one qubit.

(a) Construct the encoding circuit of the code using Hadamard and CNOT gates.

Solution.



Let $|\Psi\rangle$ be the nine qubit Shor encoding of the qubit $\alpha|0\rangle + \beta|1\rangle$. Assume that $|\Psi\rangle$ is exposed to a noise process which introduces a bit and a phase flip error on the fourth qubit yielding the faulty state $Z_4 X_x |\Psi\rangle$

(b) What measurements will help you to infer the location and the type of the error (that is, is it a bit flip, a phase flip or both)?

Solution. By measuring Z_4Z_5 and Z_5Z_6 you can see that there is a bit flip error on the fourth bit. By measuring $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$ you can see the phase flip in the second block of three qubits.

(c) Given the syndrome of the error above, how can you reconstruct the original state $|\Psi\rangle$?

Solution. In order to fix the bit flip on the fourth qubit we can apply X_4. In order to fix the phase flip in the second block we can apply $Z_4Z_5Z_6$.

Exercise 2. ***Error analysis***

Let $|\Psi\rangle$ be the nine qubit Shor encoding of the qubit $\alpha|0\rangle+\beta|1\rangle$. Assume that the depolarization channel $\mathcal{N}$, which is given by $\mathcal{N}(\rho) = (1-p)\rho+\frac{p}{3}(X\rho X+Y\rho Y+Z\rho Z)$, is acting simultaneously, but independently, on each qubit of $|\Psi\rangle$. Hence, the noise process is formally described by $\mathcal{N}^{\otimes 9}$.

(a) What is the probability that an error occurs that cannot be corrected by the Shor code? Neglect higher order terms in the calculation, i.e. do not take into account when three or more errors occur simultaneously on different qubits.

Hint. Can we fix errors in two qubits in some special cases?

Solution. We compute a lower bound on the probability that an error occurs which can be corrected. Of course, if no error occurs we are fine. This happens with probability $(1-p)^9$ as the noise is acting independently on each qubit. The probability that a single X,Y or Z flip occurs is

$$9 \cdot \frac{p}{3}(1-p)^8 + 9 \cdot \frac{p}{3}(1-p)^8 + 9 \cdot \frac{p}{3}(1-p)^8 \ .$$

Exactly two bit flips can be corrected in 27 out of the total 36 cases (the bit flips have to be in different blocks). Hence, the probability that these errors can be corrected is

$$27 \cdot \left(\frac{p}{3}\right)^2 (1-p)^7 \ .$$

Exactly two phase flips can be corrected in only 9 out of the total 36 cases (the phase flips have to be in the same block). Hence, the probability that these errors can be corrected is

$$9 \cdot \left(\frac{p}{3}\right)^2 (1-p)^7 \ .$$

2 Y flips cannot be corrected by the Shor code. Note that even if there are more than two errors it is still possible that the Shor code protects against these errors.

Combining everything together, the probability that the error can be corrected is

$$(1-p)^9 + 9 \cdot p \cdot (1-p)^8 + 4 \cdot p^2 \cdot (1-p)^7 \approx 1 - 32 \cdot p^2$$

where we neglected higher order terms. Therefore the probability that an error which cannot be corrected occurs is $\lessapprox 32 \cdot p^2$.

(b) How large can p be such that the concatenation of the Shor code still reduces the error probability?

Solution. First note that we cannot use the analysis we did in the previous item to solve this one. The reason is that although we have the error process $\mathcal{N}$ at the first concatenation level this does not imply that the same error process is acting on the second concatenation level as well (where each of the nine qubits in the Shor code is represented itself by nine qubits, i.e., total of 81 qubits at the second level). At each concatenation level, we have a different error process $\mathcal{N}^i$ acting on the (logical) qubits given by

$$\mathcal{N}^i(\rho) = (1-p_i) \cdot \rho + p_i \cdot \tilde{\mathcal{N}}^i(\rho)$$

with $p_1 = p$. Never the less, the error process is still acting independently on the (logical) qubits and therefore the overall noise process at the i'th concatenation level is described by $\left(\mathcal{N}^i\right)^{\otimes 9}$.

The goal now is to determine a lower bound on the probability that an error occurred which can be corrected at the i'th concatenation level, in which the noise process $\left(\mathcal{N}^i\right)^{\otimes 9}$ is acting on the nine logical qubits. Taking into account no error or one error, the probability the the error can be corrected is aprox. $1 - 36 \cdot p_i^2$ and hence the next concatenation is useful only if the error probability is given by

$$p_{i+1} \lessapprox 36 \cdot p_i^2 \ .$$

Solving this recursive formula yields for the error probability, given n concatenation levels, the following upper bound

$$p_n \leq \frac{1}{36} \cdot (36 \cdot p)^{2^n} \ .$$

If we set $p < \frac{1}{36}$ this expression goes to zero when increasing the number of concatenation levels.