Eidgenössische Technische Hochschule Zürich
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Advanced Topics in Quantum Information Theory
Solutions 3

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Exercise 1. *Pusey–Barrett–Rudolph argument for non-orthogonal states*

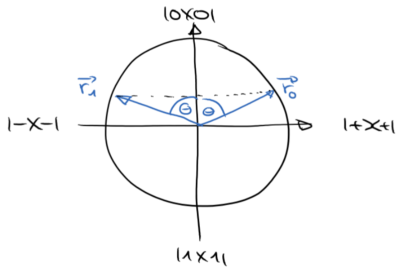
In this exercise you will generalize the argument given in the lecture to arbitrary non-orthogonal states by following the appendix of the paper by Pusey, Barrett and Rudolph (arXiv:1111.3328). For this, suppose that you are given n copies of a device that prepares a quantum system in either the state $|\psi_0\rangle$ or the state $|\psi_1\rangle$. Thus there are 2^n possible preparations, $|\psi_{\vec{x}}\rangle = |\psi_{x_1}\rangle \otimes \ldots \otimes |\psi_{x_n}\rangle$, where $\vec{x}$ is a binary string with $x_i = 0/1$ if the i-th system is prepared in state $|\psi_{0/1}\rangle$. The challenge is to find a joint measurement of the n systems such that each measurement outcome has probability zero for (at least) one of the preparations.

(a) Show that (up to global phases) there always exist orthonormal vectors $|0\rangle$, $|1\rangle$ such that

$$|\psi_0\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\,|1\rangle, \qquad |\psi_1\rangle = \cos\frac{\theta}{2}\,|0\rangle - \sin\frac{\theta}{2}\,|1\rangle,$$

with $0 < \theta \leq \frac{\pi}{2}$.

Solution. We can always rotate the Bloch sphere (i.e., the basis vectors $|0\rangle$ and $|1\rangle$) such that



Then it is clear that the states $|\psi_i\rangle\langle\psi_i|$ (whose Bloch vectors we have denoted by r_i) have the desired form (with θ as in the figure).

We shall think of $|0\rangle$ and $|1\rangle$ defining the computational basis of a qubit, and write $|\vec{x}\rangle = |x_1\rangle \otimes \ldots \otimes |x_n\rangle$ for the corresponding product basis.

(b) Consider the following measurement procedure: First, apply the unitary $Z_\beta = \begin{pmatrix} 1 & \\ & e^{i\beta} \end{pmatrix}$ to each qubit. Second, apply the unitary R_α which maps $|0\ldots 0\rangle \mapsto e^{i\alpha}|0\ldots 0\rangle$ and acts as the identity on the orthogonal complement. Third, apply a Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to each qubit. Finally, measure each qubit in the computational basis. Show that the probability of outcome $|\vec{x}\rangle$ given the initial state $|\psi_{\vec{x}}\rangle$ as predicted by quantum mechanics is equal to

$$\frac{1}{2^n}\left(\cos\frac{\theta}{2}\right)^{2n}\left|e^{i\alpha} - 1 + \left(1 + e^{i\beta}\tan\frac{\theta}{2}\right)^n\right|^2.$$

Solution. The desired probability is the *modulus squared* of

$$\begin{aligned}
&\langle\vec{x}|\,H^{\otimes n} R_\alpha Z_\beta^{\otimes n}\,|\psi_{\vec{x}}\rangle \\
=&\frac{1}{\sqrt{2^n}}\left(\sum_{\vec{y}}(-1)^{\vec{x}\cdot\vec{y}}\langle\vec{y}|\right)\; R_\alpha \bigotimes_i \left(\cos\frac{\theta}{2}\,|0\rangle + (-1)^{x_i} e^{i\beta}\sin\frac{\theta}{2}\,|1\rangle\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2^n}} \left(e^{i\alpha} \langle 0 \dots 0| + \sum_{\vec{y} \neq 0} (-1)^{\vec{x}\cdot\vec{y}} \langle \vec{y}| \right) \bigotimes_i \left(\cos\frac{\theta}{2} |0\rangle + (-1)^{x_i} e^{i\beta} \sin\frac{\theta}{2} |1\rangle \right) \\
&= \frac{1}{\sqrt{2^n}} \left(e^{i\alpha} \left(\cos\frac{\theta}{2}\right)^n + \sum_{\vec{y} \neq 0} (-1)^{\vec{x}\cdot\vec{y}} (-1)^{\vec{x}\cdot\vec{y}} \left(\cos\frac{\theta}{2}\right)^{n-\|y\|_1} \left(e^{i\beta} \sin\frac{\theta}{2}\right)^{\|y\|_1} \right) \\
&= \frac{1}{\sqrt{2^n}} \left(e^{i\alpha} \left(\cos\frac{\theta}{2}\right)^n + \sum_{y=1,\dots,n} \binom{n}{y} \left(\cos\frac{\theta}{2}\right)^{n-y} \left(e^{i\beta} \sin\frac{\theta}{2}\right)^{y} \right) \\
&= \frac{1}{\sqrt{2^n}} \left(\left(e^{i\alpha} - 1\right) \left(\cos\frac{\theta}{2}\right)^n + \sum_{y=0,\dots,n} \binom{n}{y} \left(\cos\frac{\theta}{2}\right)^{n-y} \left(e^{i\beta} \sin\frac{\theta}{2}\right)^{y} \right) \\
&= \frac{1}{\sqrt{2^n}} \left(\left(e^{i\alpha} - 1\right) \left(\cos\frac{\theta}{2}\right)^n + \left(\cos\frac{\theta}{2} + e^{i\beta} \sin\frac{\theta}{2}\right)^n \right) \\
&= \frac{1}{\sqrt{2^n}} \left(\cos\frac{\theta}{2}\right)^n \left(e^{i\alpha} - 1 + \left(1 + e^{i\beta} \tan\frac{\theta}{2}\right)^n \right)
\end{aligned}
$$

(hence the overall phases which we had ignored in (a) indeed do not matter).

(c) Show that for large enough n, the phases α, β can be chosen such that these probabilities are all zero.

Solution. We would like to find n, α and β such that

$$
e^{i\alpha} - 1 + \left(1 + e^{i\beta} \tan\frac{\theta}{2}\right)^n = 0.
$$

Of course, it suffices to find n and β such that

$$
|1 - \left(1 + e^{i\beta} \tan\frac{\theta}{2}\right)^n| = 1. \tag{S.1}
$$

Geometrically, $f_n(\beta) = 1 - \left(1 + e^{i\beta} \tan\frac{\theta}{2}\right)^n$ defines a curve in the complex plane for any fixed n, and the condition (S.1) amounts to showing that this curve intersects the unit circle. Note that $\tan\frac{\theta}{2} \in (0, 1]$ since $\theta \in (0, \frac{\pi}{2}]$ (see picture), and therefore

$$
f_n(\pi) = 1 - \left(1 - \tan\frac{\theta}{2}\right)^n \in (0, 1]
$$

is always contained in the unit disk. On the other hand, for large enough $n \gg 1$

$$
f_n(0) = 1 - \left(1 + \tan\frac{\theta}{2}\right)^n < -1.
$$

For such n, continuity of $f_n(\beta)$ ensures that there exists β with $|f_n(\beta)| = 1$.

(d) Conclude that, under the assumptions 1–3 stated in the lecture, there does not exist a physical state z compatible with both $|\psi_0\rangle$ and $|\psi_1\rangle$.

Solution. This works precisely as you saw in the lecture.

Exercise 2. *Reality of the wave function from different assumptions*

In this exercise, your task is to understand an alternative argument due to Colbeck and Renner (arXiv:1111.6597), which derives the reality of the wave function from a different set of assumptions. As in the lecture, we consider a system prepared in a state described by a wave function Ψ; the experimenter then chooses a measurement setting A and records the measurement outcome X. Mathematically, Ψ, A, and X are modelled by random variables on some underlying probability space. Let $\Gamma \ni \Psi$ be a collection of random variables on the same probability space, modeling all information that is in principle available before the measurement setting is chosen. Technically, we shall assume that *measurement settings can be chosen freely*; in particular, this implies that $\mathbb{P}_{A|\Gamma} = \mathbb{P}_A$. We shall also assume that *quantum theory is correct*, so that e.g. $\mathbb{P}_{X|A\Psi}$ is given by the predictions of quantum mechanics.

Under these assumptions, it has been shown that the wave function Ψ is complete for the description of this system (arXiv:1005.5173). Here and in the following, a subset of random variables $\Gamma_0 \subseteq \Gamma$ is said to be *complete* for the description of the system if $\mathbb{P}_{X|\Gamma A} = \mathbb{P}_{X|\Gamma_0 A}$, i.e. if

$$\Gamma \to (\Gamma_0, A) \to X$$

is a Markov chain.

(a) Compare this notion of completeness with the one discussed in last semester's quantum information theory lecture. *(This part of the exercise is optional.)*

Let us now consider another subset of random variables $Z \subseteq \Gamma$ (a "list of elements of reality") that is also complete for the description of the system.

(b) Show that

$$\mathbb{P}_{X|Z=z,A=a} = \mathbb{P}_{X|\Psi=\psi,A=a}$$

whenever $\mathbb{P}(Z = z, \Psi = \psi) > 0$ and $\mathbb{P}(A = a) > 0$.

Solution. It follows from the completeness of Z and Ψ that we have two Markov chains

$$\Gamma \ni \Psi \to (Z, A) \to X,$$
$$\Gamma \supseteq Z \to (\Psi, A) \to X.$$

Therefore,

$$\mathbb{P}_{X|ZA} = \mathbb{P}_{X|\Psi ZA} = \mathbb{P}_{X|\Psi A},$$

or, equivalently,

$$\mathbb{P}_{X|Z=z,A=a} = \mathbb{P}_{X|\Psi=\psi,A=a}$$

for all z, ψ and a for which

$$\mathbb{P}(Z = z, \Psi = \psi, A = a) = \mathbb{P}(Z = z, \Psi = \psi)\,\mathbb{P}(A = a) > 0.$$

(the last equality follows from the assumption of free measurement settings).

(c) Conclude that the wave function Ψ is determined uniquely by Z. In this sense, the system's wave function is in one-to-one correspondence with its elements of reality.

Solution. Suppose ψ and ψ' are both compatible with z, i.e.

$$\mathbb{P}(Z = z, \Psi = \psi) > 0, \qquad \mathbb{P}(Z = z, \Psi = \psi') > 0.$$

Then (b) implies that

$$\mathbb{P}_{X|\Psi=\psi,A=a} = \mathbb{P}_{X|\Psi=\psi',A=a}$$

whenever $\mathbb{P}(A = a) > 0$. Clearly, there exists a measurement setup for which this implies that $\psi = \psi'$ (e.g., the projective measurement $\{|\psi\rangle\langle\psi|, \mathbf{1} - |\psi\rangle\langle\psi|\}$). Thus there exists at most a single wave function $\Psi = \psi$ compatible with each $Z = z$, i.e. Ψ is uniquely determined by Z.