11.1 The group of SLOCC operations

Last time we talked about SLOCC (stochastic LOCC), where we can post-select on particular outcomes. Given a class of states that can be interconverted by SLOCC,

$$C_\psi = \{|\phi\rangle : |\phi\rangle \overset{\text{SLOCC}}{\longleftrightarrow} |\psi\rangle\},$$

a result by Dür, Vidal and Cirac says that

$$C_\psi := \{|\phi\rangle : |\phi\rangle \propto (A \otimes B \otimes C)|\psi\rangle : A, B, C \in \mathrm{SL}(d)\}$$

For three qubits there is a simple classification of all such classes of entanglement. Apart from product states and states with only bipartite entanglement, there are two classes, with the following representative states:

$$|GHZ\rangle = \frac{1}{2}(|000\rangle + |111\rangle)$$
$$|W\rangle = \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle)$$

Note that the class of SLOCC operations forms a group, which we denote by

$$G = \{A \otimes B \otimes C : A, B, C \in \mathrm{SL}(d)\}.$$

In this language, we can rephrase the result of Dür, Vidal and Cirac as follows: Any SLOCC entanglement class is simply the orbit $G \cdot |\psi\rangle_{ABC}$ of a representative quantum state $|\psi\rangle$ under the group of SLOCC operations G (up to normalization).

An easy-to-check fact is that $\mathrm{SL}(d) = \{e^X : \mathrm{tr}(X) = 0\}$. Therefore,

$$G = \{e^A \otimes e^B \otimes e^C = e^{A \otimes I \otimes I + I \otimes B \otimes I + I \otimes I \otimes C} : \mathrm{tr}\, A = \mathrm{tr}\, B = \mathrm{tr}\, C = 0\},$$

and we find that the Lie algebra of G is spanned by the traceless local Hamiltonians.

11.2 Quantum marginal problem for an entanglement class

What are the possible ρ_A, ρ_B, ρ_C that are compatible with a pure state *in a given entanglement class*? Note that this only depends on the spectra λ_A, λ_B and λ_C of the reduced density matrices, as one can always apply local unitaries and change the basis without leaving the SLOCC class.

Are there any new constraints? Yes! For example, the reduced density matrices of the class of product states are always pure, hence its local eigenvalues satisfy $\lambda^A_{\max} = \lambda^B_{\max} = \lambda^C_{\max} = 1$.

A more interesting example is the W-class. Here, the set of compatible spectra is given by the equation $\lambda^A_{\max} + \lambda^B_{\max} + \lambda^C_{\max} \geq 2$, as we will discuss in the problem session (see Figure 1).

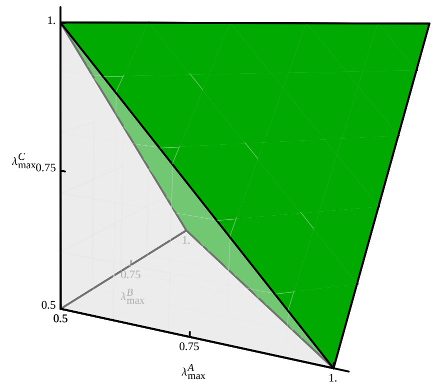

Figure 1: The entanglement polytope of the W class (blue) is the region of all local eigenvalues that are compatible with a state from the W class or its closure.

11.3 The case of the origin

Let us start with the following special case of the problem: Given an entanglement class $G \cdot |\phi\rangle_{ABC}$, does it contain a state $|\psi\rangle_{ABC}$ with $\rho_A = \rho_B = \rho_C = I/d$? (Such a state is also called *locally maximally mixed*; it corresponds to the "origin" in the coordinate system of Figure 1.) This is equivalent to

$$\operatorname{tr}(\rho_A A) = \operatorname{tr}(\rho_B B) = \operatorname{tr}(\rho_C C) = 0$$

for all traceless Hermitian matrices A, B, C, which in turn is equivalent to

$$\operatorname{tr}(\rho_A A) + \operatorname{tr}(\rho_B B) + \operatorname{tr}(\rho_C C) = 0$$

for all traceless Hermitian matrices A, B, C. We can write it as

$$\langle\psi_{ABC}| A \otimes I \otimes I + I \otimes B \otimes I + I \otimes I \otimes C |\psi_{ABC}\rangle = 0.$$

Thus the norm of the state $|\psi_{ABC}\rangle$ should not change (to 1st order) when we apply an arbitrary infinitesimal SLOCC operation. Indeed:

$$\begin{aligned}&\frac{\partial}{\partial t}\Big|_{=0} \|e^{At} \otimes e^{Bt} \otimes e^{Ct} |\psi_{ABC}\rangle\|^2 = \frac{\partial}{\partial t}\Big|_{=0} \langle\psi_{ABC}| e^{2At} \otimes e^{2Bt} \otimes e^{2Ct} |\psi_{ABC}\rangle \\ &= 2\,\langle\psi_{ABC}| A \otimes I \otimes I + I \otimes B \otimes I + I \otimes I \otimes C |\psi_{ABC}\rangle.\end{aligned}$$

In particular, if $|\psi_{ABC}\rangle$ is a closest point to the origin (i.e., a vector of minimal norm) in the orbit $G \cdot |\phi_{ABC}\rangle$ then $\rho_A = \rho_B = \rho_C = I/d$.

What happens when there is no point in the class with $\rho_A = \rho_B = \rho_C = I/d$? That might seem strange, as it implies by the above that there is no closest point to the origin. But since the group G is not compact, such situations can indeed occur. For example,

$$\begin{pmatrix} \varepsilon & \\ & \frac{1}{\varepsilon} \end{pmatrix} \otimes \begin{pmatrix} \varepsilon & \\ & \frac{1}{\varepsilon} \end{pmatrix} \otimes \begin{pmatrix} \varepsilon & \\ & \frac{1}{\varepsilon} \end{pmatrix} |W\rangle = \epsilon |W\rangle \tag{11.1}$$

and when ϵ goes to zero, we approaches the origin 0 of the Hilbert space. However, 0 is *not* an element of the W-class $G \cdot |W\rangle$ (in fact, $\{0\}$ is a G-orbit on its own).

Although so far we have only proved the converse, this observation is in fact enough to conclude that there exists no W-class state which is locally maximally mixed. We have the following general result, which can be proved using the Hilbert–Mumford criterion:

Theorem 11.1 (Kempf–Ness)**.** *The following are equivalent:*

- *There exists a vector of minimal norm in* $G \cdot |\phi\rangle_{ABC}$*.*
- *There exists a quantum state in the class* $G \cdot |\phi\rangle_{ABC}$ *with* $\rho_A = \rho_B = \rho_C = I/d$*.*
- $G \cdot |\phi\rangle_{ABC}$ *is closed.*

Thus there exists no quantum state in the W-class that is maximally mixed.

How about if we look at the closure of the W-class? States in the closure of class are those which can be approximated arbitrarily well by states from the class. Thus they can in principle be used for the same tasks as any state in the class itself (as long as the task is "continuous").

It is a fact that the closure of any orbit $G \cdot |\phi\rangle_{ABC}$ is a disjoint union of orbits, among which there is a *unique* closed orbit. There are two options: Either this orbit $\{0\}$, or it is the orbit through some proper (unnormalized) quantum state. Therefore:

Corollary 11.2. *There exists a quantum state in the closure of* $G \cdot |\phi\rangle$ *that is locally maximally mixed if, and only if,* $0 \notin \overline{G \cdot |\phi_{ABC}\rangle}$*.*

We saw before that 0 is in the closure of the W-class. Therefore, the corollary shows that we cannot even approximate a locally maximally mixed state by states from the W-class. This agrees with Figure 1, which shows that the set of eigenvalues that are compatible with the closure of the W class does not contain the locally maximally mixed point.

11.4 Invariant polynomials

If we have two closed sets—such as $\{0\}$ and an orbit closure $\overline{G \cdot |\phi\rangle}$ which not contain the origin—then we can always find a continuous function which separates these sets. Since we are working in the realm of algebraic geometry, we can always choose this function to be a G-invariant homogeneous polynomial P, such that $P(0) = 0$ and $P(|\psi\rangle) \neq 0$. The converse is obviously also true, and so we find that:

Theorem 11.3. *There exists a quantum state in the closure of* $G \cdot |\phi\rangle$ *that is locally maximally mixed if, and only if, there exists a non-constant* G*-invariant homogeneous polynomial such that* $P(|\phi_{ABC}\rangle) \neq 0$*.*

At first sight, this new characterization does not look particularly useful, since we have to check all G-invariant homogeneous polynomials. However, since the algebra of G-invariant polynomials is finitely generated, we only have to check a finite number of polynomials. For three qubits, e.g., there is only a single generator: Every G-invariant polynomial is a linear combination of powers of

Cayley's hyperdeterminant

$$\begin{aligned} P(|\psi\rangle) = &\psi_{000}^2\psi_{111}^2 + \psi_{100}^2\psi_{011}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{001}^2\psi_{110}^2 \\ &- 2\psi_{000}\psi_{111}\psi_{100}\psi_{011} - 2\psi_{000}\psi_{111}\psi_{010}\psi_{101} - 2\psi_{000}\psi_{111}\psi_{001}\psi_{110} \\ &- 2\psi_{100}\psi_{011}\psi_{010}\psi_{101} - 2\psi_{100}\psi_{011}\psi_{001}\psi_{110} - 2\psi_{010}\psi_{101}\psi_{001}\psi_{110} \\ &+ 4\psi_{000}\psi_{110}\psi_{101}\psi_{011} + 4\psi_{111}\psi_{001}\psi_{010}\psi_{100}. \end{aligned}$$

It is non-zero precisely on the quantum states of GHZ class.

We conclude this lecture with some remarks. The characterization in terms of invariant polynomials brings us into the realm of *representation theory*. Indeed, the space of polynomials on the Hilbert space is a G-representation, and the invariant polynomials are precisely the trivial subrepresentations of this representation. Now, it is natural to ask about the meaning of the other irreducible representations. It turns out that, in the same way as trivial representations correspond to the local eigenvalues $1/d, \dots, 1/d$, these other representations correspond to the other spectra that are compatible with the class (i.e., the points besides the "origin" in Figure 1). Although we do not have the time to discuss this, this can also be proved using the techniques we have discussed in this lecture (there is a "shifting trick" based on the Borel–Weil theorem that can be used to replace the origin by any other triple of local spectra).

As a direct corollary, one can show that the solution to the quantum marginal problem for the closure of an entanglement class is always convex. It is in fact a convex polytope, which we might call the *entanglement polytope* of the class. Thus, the object in Figure 1 is the entanglement polytope of the W class.

Of course, one can always *ignore* the entanglement class in the above discussion. In this way, one gets a representation-theoretic characterization of the solution of the ordinary one-body quantum marginal problem. In the next lecture, we will discuss an alternative way of arriving at this characterization that starts with representation theory rather than geometry.