Exercise 1. One-Body Quantum Marginal Problem for N Qubits

Let $\rho = |\Psi\rangle\langle\Psi|$ be a pure quantum state of N qubits. We shall denote by $\lambda_{\max}^{(k)}$ the maximal eigenvalue of the reduced density matrix of the k-th qubit, $\rho^{(k)}$.

(a) Show that the eigenvalues satisfy the *polygonal inequalities*

$$\sum_{l \neq k} \lambda_{\max}^{(l)} \le \lambda_{\max}^{(k)} + (N-2).$$
(1)

These inequalities are in fact the only constraints. That is, for any choice of $\lambda_{\max}^{(k)} \in [0.5, 1]$ satisfying (1) there exists a corresponding pure state.

- (b) Prove this statement by explicitly constructing a global state. Hint. Solve the problem for N = 3 and induct.
- (c) Prove this statement by using convexity of the solution.

Exercise 2. Isotypical Projectors

Recall from the lecture that any finite-dimensional unitary representation \mathcal{H} of SU(2) can be decomposed into a direct sum of irreducible representations which are all of the same spin, i.e.

$$\mathcal{H} = \bigoplus_{j=0,\frac{1}{2},1,\dots} \mathcal{H}_j, \quad \mathcal{H}_j \cong \underbrace{V_j \oplus \ldots \oplus V_j}_{m_j \text{ times}}$$

Here, V_j denotes the irreducible representation of SU(2) with spin $j \in \{0, \frac{1}{2}, 1, \ldots\}$. The subspace \mathcal{H}_j is called an *isotypical component* of \mathcal{H} ; it is canonically defined (i.e., does not depend on any choices). The corresponding *isotypical projector* is the orthogonal projection onto \mathcal{H}_j , and we denote it by P_j . Similarly, the irreducible components of the product group $K = \mathrm{SU}(2)^N$ are just the tensor products $V_{j_1} \otimes \ldots \otimes V_{j_N}$, and hence the isotypical projectors are given by $P_{j_1} \otimes \ldots \otimes P_{j_N}$.

As in the lecture, let $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ be the Hilbert space of N qubits and denote by $\mathbb{C}[\mathcal{H}]_{(k)}$ the space of polynomial functions on \mathcal{H} of degree k. Show that the following two statements are equivalent:

- 1. There exists a pure state $|\psi\rangle \in \mathcal{H}$ such that $(P_{j_1} \otimes \ldots \otimes P_{j_N}) |\psi\rangle^{\otimes k} \neq 0$.
- 2. $V_{j_1}^* \otimes \ldots \otimes V_{j_N}^* \subseteq \mathbb{C}[\mathcal{H}]_{(k)}.$

Discuss how this connects the spectrum estimation theorem from the last lecture with the representation-theoretic description of the one-body quantum marginal problem presented in the lecture before.