Exercise 1. *One-Body Quantum Marginal Problem for N Qubits*

Let $\rho = |\Psi\rangle\langle\Psi|$ be a pure quantum state of N qubits. We shall denote by $\lambda_{\max}^{(k)}$ the maximal eigenvalue of the reduced density matrix of the k-th qubit, $\rho^{(k)}$.

(a) Show that the eigenvalues satisfy the *polygonal inequalities*

$$\sum_{l\neq k} \lambda_{\max}^{(l)} \leq \lambda_{\max}^{(k)} + (N-2). \tag{1}$$

These inequalities are in fact the only constraints. That is, for any choice of $\lambda_{\max}^{(k)} \in [0.5, 1]$ satisfying (1) there exists a corresponding pure state.

(b) Prove this statement by explicitely constructing a global state.

Hint. Solve the problem for $N = 3$ and induct.

(c) Prove this statement by using convexity of the solution.

Exercise 2. *Isotypical Projectors*

Recall from the lecture that any finite-dimensional unitary representation $\mathcal{H}$ of SU(2) can be decomposed into a direct sum of irreducible representations which are all of the same spin, i.e.

$$\mathcal{H} = \bigoplus_{j=0,\frac{1}{2},1,\ldots} \mathcal{H}_j, \quad \mathcal{H}_j \cong \underbrace{V_j \oplus \ldots \oplus V_j}_{m_j \text{ times}}$$

Here, V_j denotes the irreducible representation of SU(2) with spin $j \in \{0, \frac{1}{2}, 1, \ldots\}$. The subspace $\mathcal{H}_j$ is called an *isotypical component* of $\mathcal{H}$; it is canonically defined (i.e., does not depend on any choices). The corresponding *isotypical projector* is the orthogonal projection onto $\mathcal{H}_j$, and we denote it by P_j. Similarly, the irreducible components of the product group $K = \mathrm{SU}(2)^N$ are just the tensor products $V_{j_1} \otimes \ldots \otimes V_{j_N}$, and hence the isotypical projectors are given by $P_{j_1} \otimes \ldots \otimes P_{j_N}$.

As in the lecture, let $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ be the Hilbert space of N qubits and denote by $\mathbb{C}[\mathcal{H}]_{(k)}$ the space of polynomial functions on $\mathcal{H}$ of degree k. Show that the following two statements are equivalent:

1. There exists a pure state $|\psi\rangle \in \mathcal{H}$ such that $(P_{j_1} \otimes \ldots \otimes P_{j_N}) |\psi\rangle^{\otimes k} \neq 0$.
2. $V_{j_1}^* \otimes \ldots \otimes V_{j_N}^* \subseteq \mathbb{C}[\mathcal{H}]_{(k)}$.

Discuss how this connects the spectrum estimation theorem from the last lecture with the representation-theoretic description of the one-body quantum marginal problem presented in the lecture before.