

**Exercise 1. Entropic relations**

(a) Prove

$$H(Z^A|B)_\rho + H(X^A|B)_\rho \geq \log_2 d + H(A|B)_\rho \tag{1}$$

where  $\rho$  is over systems  $AB$  and  $d$  is the dimension of system  $A$ .

*Hint.* First show that for any system over  $A'B'AB$  we have

$$H(A'|AB) + H(B'|AB) \geq H(A'B'|AB) . \tag{2}$$

Then define

$$\Omega^{A'B'AB} = \frac{1}{d^2} \sum_{jk} P_j^{A'} \otimes P_k^{B'} \otimes \rho_{jk}^{AB}$$

where  $\rho_{jk}^{AB} = X_j^A Z_k^A \rho Z_k^{\dagger A} X_j^{\dagger A}$ ,  $P_i = |i\rangle\langle i|$  and  $\dim(A') = \dim(B') = d$  and show that

$$H(A'|AB)_\Omega + H(B'|AB)_\Omega \geq H(A'B'|AB)_\Omega$$

implies

$$H(Z^A|B)_\rho + H(X^A|B)_\rho \geq \log_2 d + H(A|B)_\rho .$$

(b) Prove that if  $H(X^A|B)_\rho = 0$  or  $H(Z^A|E)_\rho = 0$  then

$$H(Z^A|B)_\rho + H(X^A|E)_\rho = \log_2 d .$$

As in the lecture, consider the following states:

$$\begin{aligned} |\psi\rangle^{ABR} &= \sum_z \sqrt{p_z} |z\rangle^A \otimes |\varphi_z\rangle^{BR} \\ |\psi'\rangle^{ABCR} &= \sum_z \sqrt{p_z} |z\rangle^A |z\rangle^C \otimes |\varphi_z\rangle^{BR} \end{aligned}$$

(c) Derive the conditions

$$\begin{aligned} H(Z^A|B)_\psi &\leq \varepsilon_1^2 \\ H(Z^A|R)_\psi &\geq \log_2 d - \varepsilon_2^2 \end{aligned}$$

from

$$\begin{aligned} H(X^A|RC)_{\psi'} &\geq \log_2 d - \varepsilon_1^2 \\ H(Z^A|R)_\psi &\geq \log_2 d - \varepsilon_2^2 \end{aligned}$$

using the previous item.