Exercise 1. *Non-quantum correlations*

In this exercise we will use the semi-definite technique used in the lecture to show that the perfect PR-box cannot be realized quantum mechanically. For this we relabel the outcome 0 with +1 and the outcome 1 with -1, as in the following table:

	X	0		1	
Y	B \ A	+1	-1	+1	-1
0	+1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
1	+1	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	-1	$\frac{1}{2}$	0	0	$\frac{1}{2}$

Denote by P^x and Q^y the operators corresponding to observables on the systems (P^0 is the observable corresponding to the choice $X = 0$, etc.) and consider the set $\{A_i\} := \{P^0 \otimes \mathbb{1}, P^1 \otimes \mathbb{1}, \mathbb{1} \otimes Q^0, \mathbb{1} \otimes Q^1\}$ as in the lecture.

(a) Fill in as much of the 4×4 matrix M as you can. There should be 4 undefined elements.

(b) By computing the eigenvalues of M, or otherwise, show that the PR-box correlations cannot be realized quantum mechanically.

Hint. Start by showing that it is enough to consider the case where the missing elements are real by looking on the matrix $\frac{1}{2}(M + M^*)$

Exercise 2. *Tsirelson's bound*

In this exercise we will prove Tsirelson's bound using the dual semi-definite program (SDP).

In general, for any primal SDP we can define the dual SDP. One way of writing the primal (left) and the dual (right) programs is as follows:

$$\begin{aligned} \max \quad & \mathrm{Tr}(C^T X) \\ \text{s.t.} \quad & \mathrm{Tr}({A^{(i)}}^T X) = b_i \ \forall i \\ & X \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & b \cdot \lambda \\ \text{s.t.} \quad & \sum_i \lambda_i A^{(i)} \geq C \end{aligned}$$

where b and λ are real vectors and C, X and $A^{(i)}$ are real matrices.

(a) Show that any value of the dual program sets an upper bound on the value of the primal program.

In the lecture we saw that we can compute the highest violation of the CHSH inequality which can be achieved within quantum theory using the following semi-definite program:

$$
\begin{aligned}
\max \quad & \frac{1}{2}\left(M_{13}+M_{31}+M_{14}+M_{41}+M_{23}+M_{32}-M_{24}-M_{42}\right) \\
\text{s.t.} \quad & M_{ii}=1\ \forall i \\
& M \geq 0
\end{aligned}
$$

Remark. *We use here a different notation of Bell inequalities, in which the bound is given on the correlations of the system and not on the success probability of some game as the CHSH game. Using this notation, the CHSH inequality reads*

$$
|\langle X_0 \otimes Y_0\rangle + \langle X_1 \otimes Y_0\rangle + \langle X_0 \otimes Y_1\rangle - \langle X_1 \otimes Y_1\rangle| \leq 2\ .
$$

(b) Write the dual SDP for this problem and use it to show that the value $2\sqrt{2}$ is an upper bound on the violation that can be achieved within quantum theory. Since this violation can also be achieved within quantum theory this implies that Tsirelson's bound is optimal.