Advanced Topics in Quantum Information Theory — Spring 2013
Series 1

Prof. M. Christandl
Prof. R. Renner

Exercise 1. *Convex Combinations*

(a) Prove that any convex combination of non-signaling systems is a valid non-signaling system. That is, for any two non-signaling systems $\mathrm{P}^{(1)}_{AB|XY}$ and $\mathrm{P}^{(2)}_{AB|XY}$ and for all $p \in [0,1]$, consider the system $\mathrm{Q}_{AB|XY}$ defined by

$$\mathrm{Q}_{AB|XY}(ab|xy) = p \cdot \mathrm{P}^{(1)}_{AB|XY}(ab|xy) + (1-p) \cdot \mathrm{P}^{(2)}_{AB|XY}(ab|xy) \quad \forall x, y, a, b.$$

Show that it is a valid conditional probability distribution and that the non-signaling conditions hold:

$$\sum_a \mathrm{Q}_{AB|XY}(ab|0y) = \sum_a \mathrm{Q}_{AB|XY}(ab|1y) \quad \forall b, y$$
$$\sum_b \mathrm{Q}_{AB|XY}(ab|x0) = \sum_b \mathrm{Q}_{AB|XY}(ab|x1) \quad \forall a, x$$

(b) If for the CHSH game the winning probability using system $\mathrm{P}^{(1)}_{AB|XY}$ is w_1 and the winning probability using system $\mathrm{P}^{(2)}_{AB|XY}$ is w_2, what is the winning probability using the system $\mathrm{Q}_{AB|XY}$ as defined above?

In the exercise class, we have considered the PR-box (left) and the quantum system $\mathrm{Q}_{AB|XY}$ (right) which are given by the measurement statistics displayed in the following tables:

Y \ B (rows), X \ A (columns)		X = 0, A = 0	X = 0, A = 1	X = 1, A = 0	X = 1, A = 1
Y = 0	B = 0	0	$\frac{1}{2}$	0	$\frac{1}{2}$
	B = 1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
Y = 1	B = 0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	B = 1	$\frac{1}{2}$	0	0	$\frac{1}{2}$

Y \ B (rows), X \ A (columns)		X = 0, A = 0	X = 0, A = 1	X = 1, A = 0	X = 1, A = 1
Y = 0	B = 0	$\frac{\sin^2(\frac{\pi}{8})}{2}$	$\frac{\cos^2(\frac{\pi}{8})}{2}$	$\frac{\sin^2(\frac{\pi}{8})}{2}$	$\frac{\cos^2(\frac{\pi}{8})}{2}$
	B = 1	$\frac{\cos^2(\frac{\pi}{8})}{2}$	$\frac{\sin^2(\frac{\pi}{8})}{2}$	$\frac{\cos^2(\frac{\pi}{8})}{2}$	$\frac{\sin^2(\frac{\pi}{8})}{2}$
Y = 1	B = 0	$\frac{\sin^2(\frac{\pi}{8})}{2}$	$\frac{\cos^2(\frac{\pi}{8})}{2}$	$\frac{\cos^2(\frac{\pi}{8})}{2}$	$\frac{\sin^2(\frac{\pi}{8})}{2}$
	B = 1	$\frac{\cos^2(\frac{\pi}{8})}{2}$	$\frac{\sin^2(\frac{\pi}{8})}{2}$	$\frac{\sin^2(\frac{\pi}{8})}{2}$	$\frac{\cos^2(\frac{\pi}{8})}{2}$

(c) Denote by $\mathrm{D}^{(i,j)}_{AB|XY}$ the deterministic strategy which outputs (i,j) for every input. For example $\mathrm{D}^{(0,0)}_{AB|XY}(00|xy) = 1$ for every x, y. Find $p \in [0,1]$ such that the quantum system above is given by

$$\mathrm{Q}_{AB|XY}(ab|xy) = (1-p) \cdot \mathrm{PR}_{AB|XY}(ab|xy) + \sum_{(i,j)} \frac{p}{4} \cdot \mathrm{D}^{(i,j)}_{AB|XY}(ab|xy) \quad \forall x, y, a, b$$

where $\mathrm{PR}_{AB|XY}$ is the perfect PR-box.

Exercise 2. ***IP Game***

Consider the following game. Alice gets a bit string $x \in \{0,1\}^n$ of length n and Bob gets a bit string $y \in \{0,1\}^n$ of the same length. Alice and Bob can share as many PR-boxes as they wish and can communicate classically. The goal of the game is to calculate the following function

$$\mathrm{IP}^\star(x,y) = (\overline{x_1 \cdot y_1}) \oplus (\overline{x_2 \cdot y_2}) \oplus \ldots \oplus (\overline{x_n \cdot y_n})$$

where $\overline{x \cdot y}$ is the negation of $x \cdot y$, with as little communication as possible (measured in classical bits). Only one of the parties needs to know the result of the calculation. Give a strategy for this game which allows Alice and Bob to win the game with just one bit of communication.

Remark: The amount of communication needed for such a distributed calculation of a function is called the communication complexity of the function. There is a classical result which shows that the distributed calculation of any binary function can be reduced to a calculation of some inner product function. Together with what you prove here, this implies that the communication complexity of any binary function is at most one bit if Alice and Bob are allowed to share PR-boxes.

Exercise 3. ***Mermin–GHZ Game***

In this game, Alice, Bob and Charlie receive input bits x, y and z, with the promise that $x \oplus y \oplus z = 0$. Their goal is to output bits a, b and c, respectively, such that

$$a \oplus b \oplus c = x \vee y \vee z.$$

(a) Show that there is no classical winning strategy (i.e., no classical strategy that wins with probability one). What is the maximal probability of winning using a classical strategy assuming that all valid inputs are equally likely?

(b) Show that there exists a quantum winning strategy in which Alice, Bob and Charlie share a GHZ state, $|\Psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + |111\rangle_{ABC})$.

(c) Find a non-signaling winning strategy in which Alice and Bob share a PR-box.

(d) Is there a quantum winning strategy in which only Alice and Bob share a quantum state?

Exercise 4. ***Mermin–Peres Magic Square Game***

A *magic square* is a three-by-three grid with entries in ± 1, such that the product of each row is equal to $+1$ while the product of each column is equal to -1. The *magic square game* now is the following game of two players, Alice and Bob. Alice receives the index x of a row and has to output three numbers in ± 1 which look like the row of a magic square (i.e., their product is equal to $+1$). Bob receives the index y of a column and has to output three numbers in ± 1 which look like the column of a magic square (i.e., their product is equal to -1). Crucially, their output has to agree on the intersection, as in the following example:

$x \backslash y$	1	2	3	
1			1	
2	-1	-1	1	$\prod = 1$
3			-1	

$\prod = -1$

(a) Show that there is no classical strategy that wins with probability one. *Hint: Do magic squares exist?*

(b) Find a quantum winning strategy. *Hint: Let Alice and Bob share two entangled pairs of qubits and consider products of Pauli operators.*