Sheet 0

Due date: 2 March 2012

Exercise 1 [*Identities of vector analysis*]: Prove the following basic identities of vector analysis:

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) &= \mathbf{b}\cdot(\mathbf{c}\wedge\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\wedge\mathbf{b}) \\
\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} \\
(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{c}\wedge\mathbf{d}) &= (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c}) \\
\mathrm{curl}\,\mathrm{grad}\,\psi &= 0 \\
\mathrm{div}(\mathrm{curl}\,\mathbf{A}) &= 0 \\
\mathrm{curl}(\mathrm{curl}\,\mathbf{A}) &= \mathrm{grad}(\mathrm{div}\,\mathbf{A}) - \Delta\mathbf{A} \\
\mathrm{div}(\psi\mathbf{A}) &= \mathbf{A}\cdot\mathrm{grad}\,\psi + \psi\,\mathrm{div}\,\mathbf{A} \\
\mathrm{curl}(\psi\mathbf{A}) &= (\mathrm{grad}\,\psi)\wedge\mathbf{A} + \psi\,\mathrm{curl}\,\mathbf{A} \\
\mathrm{grad}(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\wedge\mathrm{curl}\,\mathbf{B} + \mathbf{B}\wedge\mathrm{curl}\,\mathbf{A} \\
\mathrm{div}(\mathbf{A}\wedge\mathbf{B}) &= \mathbf{B}\cdot\mathrm{curl}\,\mathbf{A} - \mathbf{A}\cdot\mathrm{curl}\,\mathbf{B}\,,
\end{aligned}$$

where $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\mathbb{R}^3$ are fixed vectors, ψ is a scalar field, and $\mathbf{A},\mathbf{B}$ are vector fields on $\mathbb{R}^3$.

[**Hint**: the i-th component of the wedge product $\mathbf{a}\wedge\mathbf{b}$ is given by

$$(\mathbf{a}\wedge\mathbf{b})_i = \sum_{jk}\epsilon_{ijk}\,a_j\,b_k\ ,$$

where ϵ_{ijk} is the totally antisymmetric tensor in three dimensions and $\epsilon_{123}=1$. Show that the totally antisymmetric tensor satisfies the following identities

$$\sum_i \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \qquad \text{and} \qquad \sum_{ij}\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}\ .$$

Exercise 2 [*Stokes' Theorem*]: Let $\mathbf{D}(\mathbf{x})$ be a vector field pointing in the same direction at each point $\mathbf{x}\in\mathbb{R}^3$.

(i) Under which condition does the curl of $\mathbf{D}(\mathbf{x})$ vanish?

(ii) Choose a simple example of a non-rotational vector field $\mathbf{D}(\mathbf{x})$ as above, and consider a closed curve along which the line integral of $\mathbf{D}(\mathbf{x})$ does not vanish. Show that Stokes' theorem holds true in this case by direct computation.