Exercise 1. ***Density Operator Overdose.***

Write out the density matrices of the following systems in the standard $|\uparrow\rangle$, $|\downarrow\rangle$ basis (or the relevant basis for the system).

(i) A spin $1/2$ particle in its "down" state $|\downarrow\rangle$;

(ii) A spin $1/2$ particle in the superposition state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - i\,|\downarrow\rangle\right)$;

(iii) A spin $1/2$ particle, randomly produced with probability $1/2$ in either state $|\uparrow\rangle$ or $|\downarrow\rangle$.

Suppose you are given either many copies of system (ii) or many copies of system (iii). Devise a procedure to distinguish both cases.

(iv) A spin $1/2$ particle in the superposition state $|\phi\rangle = \frac{\sqrt{3}}{2}|\uparrow\rangle + \frac{1}{2}|\downarrow\rangle$;

(v) The particle described in (iv), after having measured it and observed it in the state $|\uparrow\rangle$;

(vi) The particle described in (iv), after someone else measured it in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ but didn't tell you the measurement result ;

(vii) The particle described in (iv), after someone else measured it in the basis $\{|+\rangle, |-\rangle\}$ (where $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \pm |\downarrow\rangle\right)$) but didn't tell you the measurement result ;

(viii) The spin state of an electron coming out of a source that produces particles with uniformly random spin. Assume that the spins are created in a random direction given by the spherical angles θ, ϕ, evenly distributed on the surface of the sphere. The state with "spin in direction θ, ϕ" is given by

$$|\theta,\phi\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + \mathrm{e}^{i\phi}\sin\frac{\theta}{2}\,|\downarrow\rangle\ . \tag{1}$$

(This representation is known as the Bloch sphere, or Poincaré sphere for photon polarizations.)

(ix) Two (distinguishable) spin $1/2$ particles in the entangled state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right)\ ; \tag{2}$$

(x) The two particles of system (ix), after particle #1 was measured in the basis $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \pm |\downarrow\rangle\right)$, and was found in the $|+\rangle$ state. In particular, what is the state of particle #2?

(xi) Two (distinguishable) spin $1/2$ particles with their state chosen at random between $|\uparrow\rangle_1|\downarrow\rangle_2$ and $|\downarrow\rangle_1|\uparrow\rangle_2$ with probability $1/2$;

(xii) The two particles of system (xi), after particle #1 was measured in the basis $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \pm |\downarrow\rangle\right)$, and was found in the $|+\rangle$ state. In particular, what is the state of particle #2?

(xiii) A harmonic oscillator in thermodynamic equilibrium at temperature T.

Hint. At thermodynamic equilibrium, each Hamiltonian eigenstate $|n\rangle$ of energy ϵ_n is populated with probability proportional to $\mathrm{e}^{-\epsilon_n/kT}$.