Exercise 1. ***Free Electron Gas.***

The Hamiltonian of a gas of N free electrons is written in the second quantization formalism as

$$H = \sum_{\boldsymbol{k},s} \xi_k \, c^\dagger_{\boldsymbol{k},s} c_{\boldsymbol{k},s} \ , \tag{1}$$

where $c_{\boldsymbol{k},s}$ (resp. $c^\dagger_{\boldsymbol{k},s}$) is the annihilation (resp. creation) operator of the electron mode $\boldsymbol{k}, s$ of energy $\xi_k = \epsilon_k - \mu$. (Here $\varepsilon_k = \hbar^2 k^2/(2m)$ and μ is the chemical potential, $\mu = E_F$ at $T = 0$.) The index s distinguishes the two spin components.

Let's look at excitations that are holes under the Fermi level and electrons above the Fermi level. We would like to rewrite the Hamiltonian in a form which involves explicitly only these excitations. We define the creation and annihilation operators of an excitation $\alpha^\dagger_{\boldsymbol{k},s}$, $\alpha_{\boldsymbol{k},s}$ by

$$\alpha_{\boldsymbol{k},\uparrow} = \begin{cases} c_{\boldsymbol{k},\uparrow} & \text{for } k > k_F \\ c^\dagger_{-\boldsymbol{k},\downarrow} & \text{for } k < k_F \end{cases} \quad ; \quad \alpha_{\boldsymbol{k},\downarrow} = \begin{cases} c_{\boldsymbol{k},\downarrow} & \text{for } k > k_F \\ c^\dagger_{-\boldsymbol{k},\uparrow} & \text{for } k < k_F \end{cases} \ . \tag{2}$$

(a) Show that the α, $\alpha^\dagger$'s obey fermionic commutation relations.

(b) Argue that eq. (2) is a unitary transformation of the creation and annihilation operators. Such a transformation is also called a *Bogoliubov transformation*. What happens if you act with the annihilators $c_{\boldsymbol{k},s}$ and $\alpha_{\boldsymbol{k},s}$ on the ground state of the gas?

(c) Rewrite the Hamiltonian (1) in the form

$$H = \sum_{\boldsymbol{k}} |\xi_k| \left(\alpha^\dagger_{\boldsymbol{k}\uparrow} \alpha_{\boldsymbol{k}\uparrow} + \alpha^\dagger_{\boldsymbol{k}\downarrow} \alpha_{\boldsymbol{k}\downarrow} \right) + E_G \quad ; \quad E_G = 2 \sum_{k<k_F} \xi_k \ . \tag{3}$$

Exercise 2. ***Correlation Functions in a Fermi Sea.***

Consider a gas of N identical fermions with spin $1/2$. The fermions are free and non-interacting. The ground state is then given by

$$|\Phi_0\rangle = \prod_{|\boldsymbol{k}| \leqslant k_F,\, s} a^\dagger_{\boldsymbol{k}\, s} \ |0\rangle \ . \tag{4}$$

One defines the *one-particle correlation function* $G_s(\boldsymbol{x} - \boldsymbol{y})$ as

$$G_s(\boldsymbol{x} - \boldsymbol{y}) = \frac{n}{2} \, g_s(\boldsymbol{x} - \boldsymbol{y}) = \langle \Phi_0 | \Psi^\dagger_s(\boldsymbol{x}) \Psi_s(\boldsymbol{y}) | \Phi_0 \rangle \ . \tag{5}$$

This is the amplitude of recreating a fermion of spin s at position $\boldsymbol{x}$ when one was annihilated at position $\boldsymbol{y}$ with same spin.

(a) Using explicit expressions for the field operators $\Psi_s(\boldsymbol{x})$, calculate $G_s(\boldsymbol{x} - \boldsymbol{y})$ and sketch its graph as function of $|\boldsymbol{x} - \boldsymbol{y}|$. Show that $\lim_{\boldsymbol{r}\to 0} G_s(\boldsymbol{r}) = \frac{n}{2}$ and $\lim_{r\to\infty} G_s(\boldsymbol{r}) = 0$.

Likewise, one can define the *pair correlation function* $g_{ss'}(\boldsymbol{x}-\boldsymbol{y})$ by

$$\left(\frac{n}{2}\right)^2 g_{ss'}(\boldsymbol{x}-\boldsymbol{y}) = \langle \Phi_0 | \Psi_s^\dagger(\boldsymbol{x}) \Psi_{s'}^\dagger(\boldsymbol{y}) \Psi_{s'}(\boldsymbol{y}) \Psi_s(\boldsymbol{x}) | \Phi_0 \rangle \ . \tag{6}$$

(b) Rewrite Eq. (6) in the form

$$\left(\frac{n}{2}\right)^2 g_{ss'}(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{V^2} \sum_{\boldsymbol{k}_1\, \boldsymbol{k}_2\, \boldsymbol{q}_1\, \boldsymbol{q}_2} \mathrm{e}^{-i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}}\, \mathrm{e}^{-i(\boldsymbol{q}_1-\boldsymbol{q}_2)\cdot\boldsymbol{y}} \langle \Phi_0 | a_{\boldsymbol{k}_1,s}^\dagger a_{\boldsymbol{q}_1,s'}^\dagger a_{\boldsymbol{q}_2,s'} a_{\boldsymbol{k}_2,s} | \Phi_0 \rangle \ . \tag{7}$$

(c) Assume first that $s \neq s'$. Calculate $g_{ss'}(\boldsymbol{x}-\boldsymbol{y})$.

(d) Now consider the case where $s = s'$ and calculate $g_{ss}(\boldsymbol{x}-\boldsymbol{y})$. Plot the quantity $g_{ss}(\boldsymbol{x}-\boldsymbol{y})$ as a function of $|\boldsymbol{x}-\boldsymbol{y}|$.