Exercise 1. ***Irreducible Tensor Operators: Wigner-Eckart Theorem.***

Tensor operators can be seen as a set of operators that collectively transform into each other in a specific way under rotations of space. (For example, the components of a vector observable transform into each other under rotations like those of a usual vector in $\mathbb{R}^3$.)

Formally, a collection of $2k+1$ operators $T_q^{(k)}$ (with $k \geqslant 0$ integer and $q = -k, -k+1, \ldots, k$) form an *irreducible tensor operator of rank k* if they satisfy the following commutation relations with the total angular momentum $\boldsymbol{J}$ of the physical system:

$$[J_z, T_q^{(k)}] = \hbar\, q\, T_q^{(k)} \ , \tag{1}$$

$$[J_+, T_q^{(k)}] = \hbar\, \sqrt{k(k+1) - q(q+1)}\, T_{q+1}^{(k)} \ , \tag{2}$$

$$[J_-, T_q^{(k)}] = \hbar\, \sqrt{k(k+1) - q(q-1)}\, T_{q-1}^{(k)} \ . \tag{3}$$

The *Wigner-Eckart theorem*, which will be proven in this exercise, states that the matrix elements of any such tensor operator are actually proportional to the Clebsch-Gordon coefficients. That is, if $\{|n,j,m\rangle\}$ is a standard basis of common eigenstates to $\boldsymbol{J}^2$ and J_z,

$$\langle n,j,m|T_q^{(k)}|n',j',m'\rangle = \alpha \cdot \langle j',k;m',q|j,m\rangle \ , \tag{4}$$

where the proportionality constant α only depends on n, j, k, n', j' (and not q, m, m') and is usually written in the form:

$$\alpha = \frac{1}{\sqrt{2j+1}} \left\langle n,j \right\| T^{(k)} \left\| n',j' \right\rangle \ . \tag{5}$$

(a) Show that a scalar observable is an irreducible tensor operator of rank $k=0$, and that the three standard components of a vector observable are the components of an irreducible tensor operator of rank $k=1$. Show that the angular momentum operators J_x, J_y, J_z are themselves a vector observable.

The *standard components* of a vector observable $\boldsymbol{V}$ are

$$V_1^{(1)} = -\frac{1}{\sqrt{2}}\,(V_x + iV_y) \tag{6}$$

$$V_0^{(1)} = V_z \tag{7}$$

$$V_{-1}^{(1)} = \frac{1}{\sqrt{2}}\,(V_x - iV_y) \tag{8}$$

Hint. A scalar *observable is invariant under rotations. A* vector *observable* $\boldsymbol{V}$ *transforms like a vector under rotations of space, i.e.* $\boldsymbol{V}' = R^{-1}\boldsymbol{V}$*. Consider infinitesimal rotations and remember that the rotations are generated by the* J *operators.*

(b) An alternate definition for a collection of operators $T_q^{(k)}$ to be an irreducible tensor operator is that they transform under rotations as

$$R\, T_q^{(k)}\, R^\dagger = \sum_{q'} D_{q'q}^{(k)} T_{q'}^{(k)} \ , \tag{9}$$

where $D^{(j)}_{m'm} = \langle j, m'|R|j, m\rangle$ are the matrix elements of the rotation in the standard angular momentum basis.

Show that objects $T^{(k)}_q$ that transform as (9) fulfill the commutation relations (1)–(3).

(c) Let $T^{(k)}_q$ be the components of an irreducible tensor operator of rank k. Using (1), show that $\langle n, j, m|T^{(k)}_q|n', j', m'\rangle$ is zero if m is not equal to $q + m'$.

(d) Proceeding in the same way with relations (2) and (3), show that the $(2j+1)(2k+1)(2j'+1)$ matrix elements $\langle n, j, m|T^{(k)}_q|n', j', m'\rangle$ corresponding to fixed values of n, j, k, n', j' satisfy recurrence relations identical to those satisfied by the $(2j+1)(2k+1)(2j'+1)$ Clebsch-Gordan coefficients $\langle j', k; m', q|j, m\rangle$ corresponding to fixed values of j, k, j'.

Hint. Recall the recursion relations for the Clebsch-Gordan coefficients for the addition of two angular momenta $|j_1, m_1\rangle$ *and* $|j_2, m_2\rangle$ *into a global angular momentum* $|J, M\rangle$,

$$\begin{aligned}
\sqrt{J(J+1) - M(M-1)}\,&\langle j_1, j_2; m_1, m_2|J, M-1\rangle \\
&= \sqrt{j_1(j_1+1) - m_1(m_1+1)}\,\langle j_1, j_2; m_1+1, m_2|J, M\rangle \\
&+ \sqrt{j_2(j_2+1) - m_2(m_2+1)}\,\langle j_1, j_2; m_1, m_2+1|J, M\rangle \ , \qquad (10)\\
\sqrt{J(J+1) - M(M+1)}\,&\langle j_1, j_2; m_1, m_2|J, M+1\rangle \\
&= \sqrt{j_1(j_1+1) - m_1(m_1-1)}\,\langle j_1, j_2; m_1-1, m_2|J, M\rangle \\
&+ \sqrt{j_2(j_2+1) - m_2(m_2-1)}\,\langle j_1, j_2; m_1, m_2-1|J, M\rangle \ . \qquad (11)
\end{aligned}$$

(e) Show that:

$$\langle n, j, m|T^{(k)}_q|n', j', m'\rangle = \alpha \cdot \langle j', k; m', q|j, m\rangle \ ,$$

for some α depending only on n, j, k, n', j'.

(f) Use the Wigner-Eckart theorem to show that the state $\hat{\boldsymbol{r}}|n, l, m\rangle$ (where $\hat{\boldsymbol{r}} = \sum_j \hat{\boldsymbol{r}}_j$) lives in the subspace of representations $l-1$, l, and $l+1$ of $SU(2)$. This is used in Section 2.3 of the script.

Hint. Consider the dipole moment operators

$$D^{(1)}_{+1} = -\frac{1}{\sqrt{2}}\sum_j \hat{x}_j + i\hat{y}_j \ , \qquad (12)$$

$$D^{(1)}_{0} = \sum_j \hat{z}_j \ , \qquad (13)$$

$$D^{(1)}_{-1} = \frac{1}{\sqrt{2}}\sum_j \hat{x}_j - i\hat{y}_j \ . \qquad (14)$$