The Density Operator. & a little publicity for QIT...

So far, in QM I&II: system in state $|\psi\rangle \in \mathcal{H}$
 ↳ Hilbert space, say finite

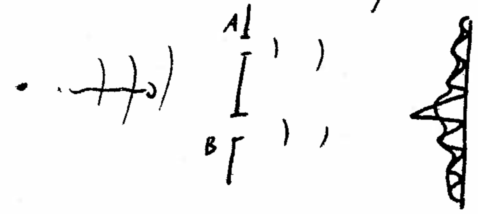
Measurement → observable A with eigenstates $|a\rangle$ (for eigenvalue a).
(say non-degenerate)

$P_a = |a\rangle\langle a|$ is the projector onto the corresponding eigenspace to eigenvalue a.

The measurement gives "a" with probability $|\langle a|\psi\rangle|^2 = \langle\psi|P_a|\psi\rangle$

Superposition ≠ Mixture. Take a particle (e^-, photon, ...) passing through Young slits. → interference pattern [after performing the experiment many times]

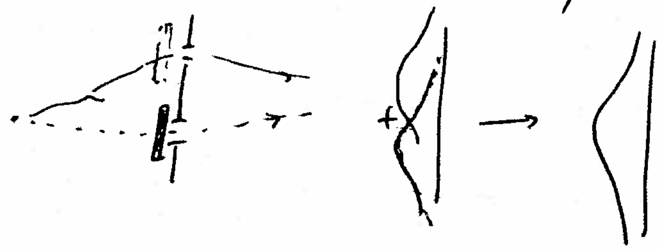
(*)



"add amplitudes"
$\psi = \psi_A + \psi_B \rightarrow$ prob $= |\psi|^2 = \ldots +$ interf.

But: if someone blocks randomly one or the other slit
→ no interference

(**)



"add probabilities"
prob. $= \frac{1}{2}|\psi_A|^2 + \frac{1}{2}|\psi_B|^2$

The relation Q.M. ⟷ probabilities

Consider a more general scenario:

$$\boxed{\text{system}} \rightarrow \begin{cases} p_1 : |\psi_1\rangle \\ p_2 : |\psi_2\rangle \\ \vdots \\ p_n : |\psi_n\rangle \end{cases}$$

[Both scenarios above are covered: (*) → $n=1$, $p_1 = 1$, $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\psi_A\rangle + |\psi_B\rangle)$
(**) → $p_1 = \frac{1}{2}$, $|\psi_1\rangle = |\psi_A\rangle$; $p_2 = \frac{1}{2}$, $|\psi_2\rangle = |\psi_B\rangle$]

Consider a measurement $A = \sum_i a\, |a\rangle\langle a|$ (eig. vect. $|a\rangle$, eig. val. a).
What is the probability of measuring "a"?

$$\text{Prob}(a) = p_1 \cdot |\langle a|\psi_1\rangle|^2 + p_2 \cdot |\langle a|\psi_2\rangle|^2 + \dots + p_n |\langle a|\psi_n\rangle|^2$$

$$= \langle a| \underbrace{\left[p_1 \cdot |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2| + \dots + p_n |\psi_n\rangle\langle\psi_n| \right]}_{\equiv \rho} |a\rangle$$

We define the density operator ρ ~~in general~~ as

$$\boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$$

Then

$$\text{Prob.}(a) = \langle a|\rho|a\rangle \quad \left(= tr[P_a \cdot \rho]\right)$$

The density operator gives you all the measurement probabilities, and nothing more!
→ ρ contains exactly all information you can obtain from a system. [⟶ link to information theory]

Illustration: no measurement will tell you the individual p_i's and ψ_i's.
example: spin-½ particle, basis $|\uparrow\rangle, |\downarrow\rangle \in \mathbb{C}^2$. 2 scenarios:

(i) randomly prepared in $|\uparrow\rangle$ or $|\downarrow\rangle$ with prob. ½.
(ii) randomly prepared in $|+\rangle$ or $|-\rangle$

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle \pm |\downarrow\rangle\right]$$

Take any measurement $A = a|a\rangle\langle a| + a'|a'\rangle\langle a'|$ with $\langle a|a'\rangle = 0$

$$\text{Prob}_{(i)}(a) = \tfrac{1}{2}|\langle a|\uparrow\rangle|^2 + \tfrac{1}{2}|\langle a|\downarrow\rangle|^2 = \tfrac{1}{2} \quad (\forall |a\rangle)$$

Bessel eq/Parseval (or Pythagoras)

$\sum_i |\langle a|e_i\rangle|^2 = \||a\rangle\|$ o.n. basis

$$\text{Prob}_{(ii)}(a) = \tfrac{1}{2}|\langle a|+\rangle|^2 + \tfrac{1}{2}|\langle a|-\rangle|^2 = \tfrac{1}{2}$$

same!
$|+\rangle, |-\rangle$ is basis!

→ Whatever measurement I choose, you get in both scenarios "a" with prob ½ and "a'" with prob. ½.

Look at their density operators:

(i) $$\rho_{(i)} = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| = \begin{pmatrix} 1/2 & \\ & 1/2 \end{pmatrix}$$ with $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

(ii) $$\rho_{(ii)} = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-|$$ $$|\pm\rangle = \begin{pmatrix} 1/\sqrt{2} \\ \pm 1/\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & +1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & \\ & 1/2 \end{pmatrix}$$

→ same density operators!

<u>Another example</u>. A spin-½ particle again.

(i) randomly prepared in $|\uparrow\rangle$ or $|\downarrow\rangle$ with prob ½.

(ii) prepared in the state $|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$

We know these two can be distinguished by experiment, e.g. Young slits interference. → they must have different density operators.

(i) $\rho_{(i)} = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| = \begin{pmatrix} 1/2 & \\ & 1/2 \end{pmatrix}$ — "fully mixed"

(ii) $\rho_{(ii)} = |+\rangle\langle+| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ — off-diagonal terms: interference between $|\uparrow\rangle$ & $|\downarrow\rangle$

but in the basis $\{|\pm\rangle\}$, $\rho_{(ii)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\text{basis } |\pm\rangle}$ ← actually a pure state!

What measurement can distinguish (i) from (ii)? → exercise.

<u>Density Operator as State</u>. Actually, one can say that ρ is (a more general) <u>state</u> of the system. → density operator formalism (cf. QIT)

* expected value of A: $\mathrm{tr}(A\rho)$
* meas. probability of outcome a: $\mathrm{tr}(P_a\rho)$
* post-meas. state: $\rho' = \frac{P_a \rho P_a}{\mathrm{tr}(P_a\rho)}$
* ...