4. Second quantization

4.2 Field operators

... but before we start...

Recap

- $\mathcal{H}$ Hilbert space

  example: for spin-½ particles

$$\mathcal{H} = \underbrace{\mathbb{C}^2}_{\text{spin}} \otimes \underbrace{L^2(\mathbb{R}^3)}$$

spin $\begin{pmatrix} \alpha(\uparrow) \\ \beta(\downarrow) \end{pmatrix}$

functions over $\mathbb{R}^3$ that are "square-integrable", ie

$$f, g \in L^2(\mathbb{R}^3) \Leftrightarrow \int d^3\vec{r}\; f^*(\vec{x})\, g(\vec{x}) < \infty$$

↳ example: $\psi_k(\vec{x}) = \frac{1}{V} e^{+i\vec{k}\cdot\vec{r}}$

turns out $\mathcal{H} \cong L^2(\mathbb{R}^3 \otimes \{\uparrow, \downarrow\})$

so instead of $\mathcal{H} \ni \psi = \begin{pmatrix} \psi_\uparrow(\vec{x}) \\ \psi_\downarrow(\vec{x}) \end{pmatrix}$

we can write $\mathcal{H} \ni \psi(\vec{x}, s) =: \psi_s(\vec{x})$

- kets and bras

... are functions :

$$|\psi\rangle : \mathbb{C} \longrightarrow \mathcal{H}$$
$$\alpha \longmapsto \alpha\psi \qquad (\psi \in \mathcal{H}, \text{ eg } \psi(\vec{x}) = \frac{1}{V} e^{i\vec{k}\cdot\vec{x}})$$

$$\langle\phi| : \mathcal{H} \longrightarrow \mathbb{C}$$
$$\psi \longmapsto \langle\phi, \psi\rangle \quad , \phi \in \mathcal{H}$$

example ; if $\mathcal{H} = L^2(\mathbb{R}^3)$

$$\langle\phi| : \psi \longrightarrow \int d^3\vec{r}\ \phi^*(\vec{r})\,\psi(\vec{r})$$

So the "inner product" between bra and ket is also a function

$$\langle\phi|\psi\rangle : \mathbb{C} \longrightarrow \mathbb{C}$$
$$\alpha \longmapsto \alpha\,\langle\phi, \psi\rangle$$
$$\text{eg } \alpha \longmapsto \alpha \int d^3\vec{r}\ \phi^*(\vec{r})\,\psi(\vec{r})$$

↳ if $\{\psi_i\}_i$ form an orthonormal basis of $\mathcal{H}$,

$$\langle\psi_i|\psi_j\rangle : \mathbb{C} \longmapsto \mathbb{C}$$
$$\alpha \longmapsto \delta_{ij}\,\alpha$$
$$=: \delta_{ij} \text{ , delta function}$$

- Commutators

$$[A, B]_{\mp} := AB \mp BA \qquad \text{(we'll use } - \text{ for Bosons and } + \text{ for Fermions)}$$

- Permutations

$$\mathcal{P}_{\pm} := \sum_{P \in S_n} (\pm 1)^{|P|}\, P \qquad (+ \text{ for Bosons, } - \text{ for Fermions})$$

↳ Bosons always on top! (alphabetical order)

• a especially convenient basis for $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3)$

Function: $\delta_{\vec{y}}(\vec{x}) := \delta(\vec{y} - \vec{x})$ (delta function)

in particular:

$$\langle \delta_y | \psi \rangle = \int d^3\vec{r}\; \delta(\vec{y} - \vec{r})\, \psi(\vec{r})$$
$$= \psi(\vec{y})$$

this notation will allow us to skip many annoying calculations
(I mean simplify)

eg to prove Eq. 4.2.5 (page 58 of the script):
$\{\psi_i\}_i$ on. basis of $\mathcal{H}$

$$\sum_i \psi_i^*(\vec{x})\, \psi_i(\vec{y}) = \sum_i \langle \psi_i | \delta_x \rangle \langle \delta_y | \psi_i \rangle$$

$$= \sum_i \langle \delta_y | \psi_i \rangle \langle \psi_i | \delta_x \rangle$$

$$= \langle \delta_y | \underbrace{\Big(\sum_i | \psi_i \rangle \langle \psi_i | \Big)}_{\mathbb{1}} | \delta_x \rangle$$

$$= \langle \delta_y | \delta_x \rangle = \int d^3\vec{r}\; \delta(y - r)\, \delta(x - r)$$
$$= \delta(y - x)$$

Now we can define the <u>Field operators</u>

$\{\psi_i\}$ basis of $L^2(\mathbb{R}^3)$

$$\Psi_s(\vec{x}) = \sum_i \psi_i(\vec{x})\, a_{is}$$

Capital Psi
(I know...)

$\{a_{is}\}_{is}$ for Bosons or Fermions

$$= \sum_i \langle \delta_x | \psi_i \rangle\, a_{is}$$

$$\Psi_s^\dagger(\vec{x}) = \sum_i \psi_i^*(\vec{x})\, a_{is}^\dagger$$

$$= \sum_i \langle \psi_i | \delta_x \rangle\, a_{is}^\dagger$$

Physical interpretation: creates ($\Psi^\dagger$) or destroys (Ψ) a particle at position $\vec{x}$ (with spin s)

Commutation relations

$$[\Psi_s(x), \Psi_z(y)]_\mp = \left[\sum_i \langle \delta_x|\psi_i\rangle a_{is}, \sum_j \langle \delta_y|\psi_j\rangle a_{jz}\right]_\mp$$

$$= \sum_{i,j} \langle \delta_x|\psi_i\rangle\langle \delta_y|\psi_j\rangle [a_{is}, a_{jz}]_\mp$$

$$= 0$$

, since $[a_{is}, a_{jz}]_\mp = 0$ (again, we're doing − for Bosons, + for Fermions)

$$[\Psi_s^\dagger(x), \Psi_z^\dagger(y)]_\mp = 0 \quad (\text{same}: [a_{is}^\dagger, a_{jz}^\dagger]_\mp = 0)$$

$$[\Psi_s(x), \Psi_z^\dagger(y)]_\mp = \sum_{i,j} \langle \delta_x|\psi_i\rangle\langle \psi_j|\delta_y\rangle [a_{is}, a_{jz}^\dagger]_\mp$$

$$= \sum_{i,j} \langle \delta_x|\psi_i\rangle\langle \psi_j|\delta_y\rangle\, \delta_{sz}\,\delta_{ij}$$

$$= \delta_{sz}\, \delta(\vec{x}-\vec{y})$$

We can use field operators to create states of ~~a~~ many particles:

$$|x_1,s_1;x_2,s_2;\dots;x_n,s_n\rangle = \frac{1}{\sqrt{n!}}\,\Psi^\dagger_{s_n}(x_n)\dots\Psi_{s_2}(x_2)\Psi_{s_1}(x_1)\,|0\rangle$$

↳ meaning: 1^{st} I create a particle with spin s_1 at x_1,
then a particle with spin s_2 at x_2, ...
finally a particle with spin s_n at x_n

↳ Does the order of creation matters? (see 1-part. example in footn of page 7.

$$|x_1,s_1;x_2,s_2;\dots;x_n,s_n\rangle = \frac{1}{\sqrt{n!}}\,\Psi^\dagger_{s_n}(x_n)\dots\Psi^\dagger_{s_2}(x_2)\Psi^\dagger_{s_1}(x_1)\,|0\rangle$$

$$= \frac{1}{\sqrt{n!}}\,\Psi^\dagger_{s_n}(x_n)\dots\left([\Psi^\dagger_{s_1}(x_1),\Psi^\dagger_{s_2}(x_2)]_+ - \Psi_{s_1}(x_1)\Psi_{s_2}(x_2)\right)|0\rangle$$

*

$$= \pm\frac{1}{\sqrt{n!}}\,\Psi^\dagger_{s_n}(x_n)\dots\Psi_{s_1}(x_1)\Psi_{s_2}(x_2)\,|0\rangle$$

$$= \pm\,|x_2,s_2;x_1,s_1;\dots;x_n,s_n\rangle$$

∴ The order matters only for Fermions

* $[\Psi^\dagger_{s_0}(x),\Psi^\dagger_{z}(y)]_+ = \begin{cases} 2\,\Psi_{s_0}(x)\Psi_z(y), & \text{for Bosons} \\ 0, & \text{for Fermions}\end{cases}$

$[A,B]_- = 0 \to [A,B]_+ = AB+BA = AB-BA+2BA$
$= 2BA = 2AB$

What happens when you add another particle to an n-particle state? You get a non-normalized (n+1)-particle state!

$$\Psi_s^\dagger(x)\,|x_1,s_1;\ldots;x_n,s_n\rangle = \frac{1}{\sqrt{n!}}\,\Psi_s^\dagger(x)\,\Psi_{s_n}^\dagger(x_n)\ldots\Psi_{s_1}^\dagger(x_1)\,|0\rangle$$

$$= \sqrt{n+1}\;|x_1,s_1;\ldots;x_n,s_n;\underbrace{x,s}_{\text{goes to the end!}}\rangle$$

What happens when you try to destroy a particle?

$$\Psi_s(x)\,|x_1,s_1;\ldots;x_n,s_n\rangle = \ldots$$ you'll prove this in the exercise (Series 9, Ex. 2)

What is the inner product between two n-particle states?

$$\langle y_1,\ldots,y_m|x_1,\ldots,x_n\rangle = \frac{\delta_{nm}}{n!}\sum_{P\in S_n}(\pm 1)^{|P|}\,\delta(x_{P(1)}-y_1)\ldots\delta(x_{P(n)}-y_n)$$

notice that we're ignoring spin for now

you'll also prove this in the Exercise

$$= \frac{\delta_{nm}}{n!}\,\mathcal{P}_\pm^x\left[\delta(x_1-y_1)\ldots\delta(x_n-y_n)\right]$$

$\mathcal{P}_\pm$ acting on x (experimental notation)

meaning:

- if $\underbrace{\exists i: x_i \neq y_j\ \forall j \text{ or } y_i\neq x_j\ \forall j}_{\text{meaning } \{x_i\}_i \neq \{y_j\}_j} \rightarrow \langle y_1,\ldots,y_m|x_1,\ldots,x_n\rangle = 0$
- if $(x_1,\ldots,x_n)$ is an even permutation of $(y_1,\ldots,y_n) \rightarrow \langle\ldots|\ldots\rangle = \frac{1}{n!}$
- if $(x_1,\ldots,x_n)$ is an odd permutation of $(y_1,\ldots,y_n) \rightarrow \langle\ldots|\ldots\rangle = \frac{-1}{n!}$

Now, since $\{\psi_i\}$ are a basis for $L^2(\mathbb{R}^3)$, the $\{|x_1, \dots x_n\rangle\}_{x_1,\dots x_n}$ form a basis for the n-particle space.

sticking to ket notation for convenience

we can expand any other wave function ϕ as

$$|\phi\rangle = \int d^3\vec{x}_1\, d^3\vec{x}_2 \dots d^3\vec{x}_n\, \langle x_1, \dots, x_n|\phi\rangle\, |x_1, \dots, x_n\rangle$$

where $\langle x_1, \dots, x_n|\phi\rangle = \frac{1}{n!} \sum_{P\in S_n} (\pm 1)^{|P|} \langle x_{P(1)} \dots x_{P(n)}|\phi\rangle$

$$= \frac{1}{n!} \mathcal{P}_\pm^x \left[\langle x_1, \dots, x_n|\phi\rangle\right]$$

$$= \begin{cases} 1, \text{ if } |\phi\rangle \text{ even perm. of } |x_1, \dots, x_n\rangle \\ -1, \text{ if } |\phi\rangle \text{ odd perm. of } |x_1, \dots x_n\rangle \\ 0, \text{ otherwise} \end{cases}$$

$\lvert x_1, \dots x_n\rangle$ has the same role as $\lvert\delta_x\rangle$, for many particles:
$\langle x_1, \dots, x_n\lvert\phi\rangle := \phi(x_1, \dots, x_n)$ *

4.2.1 Fockspace

(there should be a space between Fock and space

Let $\mathcal{H}_n$ be the Hilbert space for n particles (Bosons or Fermions). Then

$$\Psi_s^\dagger(\vec{x}) : \mathcal{H}_n \to \mathcal{H}_{n+1}, \quad \Psi_s(\vec{x}) : \mathcal{H}_n \to \mathcal{H}_{n-1}$$

We can define the Fock space,

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \quad , \mathcal{H}_0 = \{|0\rangle\}$$

, $\oplus$ means direct sum:

$\mathcal{H}_n \cap \mathcal{H}_m = \emptyset, \; m \neq n$

and together they "span" $\mathcal{F}$

* in fact $|x, s\rangle = \Psi_s^\dagger(x)\,|0\rangle = \sum_i \langle\psi_i|\delta_x\rangle a_i^\dagger|0\rangle = \sum_i \langle\psi_i|\delta_x\rangle\, |\psi_i\rangle \otimes |s\rangle = \left(\sum_i |\psi_i\rangle\langle\psi_i|\right)|\delta_x\rangle \otimes |s\rangle$

$= \mathbb{1} \cdot |\delta_x\rangle \otimes |s\rangle$

The elements of $\mathcal{F}$ have the form

$$|\Psi\rangle = \{ |\psi_0\rangle_{\mathcal{H}_0}, |\psi_1\rangle_{\mathcal{H}_1}, \dots, |\psi_n\rangle_{\mathcal{H}_n}, 0, 0 \dots \}$$

$\hookrightarrow$ stops here (somewhere)

With inner product

$$\langle \Psi' | \Psi \rangle = \sum_{n=0}^{\infty} \langle \psi_n' | \psi_n \rangle_{\mathcal{H}_n} < \infty$$

4.3 Observables

4.3.1 Density of particles

just like in the 1-particle case, we define the operator

$$\hat{\rho}(\vec{x}) = \sum_{i=1}^{n} \delta(\vec{x} - \hat{x}_i)$$

$n \to$ max # particles (may as well set to ∞)

$\hookrightarrow$ operator $\hat{x}$ for $\mathcal{H}_i$

$\hookrightarrow$ 3D delta function, aka δ^3

Let $|\phi\rangle, |\chi\rangle$ be two n-particle states, ie

$$|\phi\rangle = \int d^3x_1 \dots d^3x_n \; \langle x_1, \dots, x_n | \phi \rangle \, |x_1, \dots, x_n\rangle$$

(same for $|\chi\rangle$)

$$\langle \chi | \hat{\rho}(x) | \phi \rangle = \langle \chi | \mathbb{1} \, \hat{\rho}(x) \, \phi \rangle$$

$$= \langle \chi | \left[\int d^3x_1 \dots d^3x_n \, |x_1, \dots, x_n\rangle \langle x_1, \dots, x_n| \right] \hat{\rho}(x) \, |\phi\rangle$$

$$= \int d^3x_1 \dots d^3x_n \; \langle \chi | x_1, \dots, x_n \rangle \langle x_1, \dots, x_n | \sum_i \delta(x - \hat{x}_i) \, |\phi\rangle$$

$$= \sum_i^n \int d^3x_1 \dots d^3x_n \; \delta(x - x_i) \, \langle \chi | x_1, \dots, x_n \rangle \langle x_1, \dots, x_n | \phi \rangle$$

Identical particles: $F(x_i) = F(x_n)$

$\boxed{?}$ $\delta(x - x_i) = \delta(x - x_n)$

(-1 phase?)
(absorbed later?)

$$\langle \chi | \hat{\rho}(x) | \phi \rangle = \sum_{i=1}^{n} \int d^3x_1 \ldots d^3x_n \; \delta(x - x_n) \langle \chi | x_1, \ldots, x_n \rangle \langle x_1, \ldots, x_n | \phi \rangle$$

$$= n \int d^3x_1 \ldots d^3x_{n-1} \langle \chi | x_1, \ldots, x_{n-1}, x \rangle \langle x_1, \ldots, x_{n-1}, x | \phi \rangle$$

$$\cancel{= n \int d^3x_1 \ldots d^3x_{n-1} \langle \chi | x_1, \ldots, x_{n-1}, x \rangle \langle x_1, \ldots, x_{n-1}, x | \phi \rangle}$$

↳ this "counts" the number of particles, n

Just like before we had

$$\hat{n} = a^\dagger a,$$

now we have

$$\hat{\rho}(\vec{x}) = \Psi^\dagger(\vec{x}) \Psi(\vec{x}) \quad \ldots \text{ and we'll prove it!}$$

$$\langle \chi | \Psi^\dagger(x) \Psi(x) | \phi \rangle = \langle \chi | \Psi^\dagger(x) \; \mathbb{1} \; \Psi(x) | \phi \rangle$$

$$= \langle \chi | \Psi^\dagger(x) \left[\int d^3x_1 \ldots d^3x_{n-1} | x_1, \ldots, x_{n-1} \rangle \langle x_1, \ldots, x_{n-1} | \right] \Psi(x) | \phi \rangle$$

$$= \int d^3x_1 \ldots d^3x_{n-1} \langle \chi | \left(\Psi^\dagger(x) | x_1, \ldots, x_{n-1} \rangle \right) \left(\langle x_1, \ldots, x_{n-1} | \Psi(x) \right) | \phi \rangle$$

$$= \int d^3x_1 \ldots d^3x_{n-1} \langle \chi | \left(\sqrt{n} \, | x_1, \ldots, x_{n-1}, x \rangle \right) \left(\sqrt{n} \, \langle x_1, \ldots, x_{n-1}, x | \right) | \phi \rangle$$

$$= n \int d^3x_1 \ldots d^3x_{n-1} \langle \chi | x_1, \ldots x_{n-1}, x \rangle \langle x_1, \ldots, x_{n-1}, x | \phi \rangle$$

$$= \langle \chi | \hat{\rho}(x) | \phi \rangle$$

$\cancel{\ldots}$ $\langle x | x_1 \ldots$

$$a_i^{+} \psi_i^{*}(x) \quad \xi \quad \psi_i(x)\, a_j$$

$$= a_i^{+} \langle \psi_i | x \rangle \quad \xi \langle x | \psi_i \rangle\, a_j$$

$$= \langle \psi_i | x \rangle \quad \xi \langle x | \psi_i \rangle\, a_i^{+} a_j$$

1 particle : $-\frac{\hbar^2}{2m} \langle \beta_x | x \rangle \nabla^2 \langle x |$

?

$[B, C] = BC - CB$

$BC = [B, C] + CB$

$$[AB, C] = ABC - CAB$$

$$= A(CB + [B, C]) - CAB$$

$$= A[B, C] + [A, C]B$$

$$\psi^{+}(y) \underbrace{[\nabla^2 \psi(y), \psi(x)]}_{0} + [\psi^{+}(y), \psi(x)] \nabla^2 \psi(x)$$

Now $\hat{\rho}(\vec{x})$ gives you the density of particles at position $\vec{x}$. For the total number of particles, simply integrate over $\vec{x}$:

$$\hat{N} = \int d^3\vec{x}\ \rho(\vec{x})$$

$$= \int d^3x\ \Psi^{+}(x)\,\Psi(x)$$

$$= \sum_{ij} \int d^3x\ \langle \varphi_i | x\rangle \langle x | \psi_j\rangle\ a_i^{+} a_j$$

$$= \sum_{ij} \langle \varphi_i | \left[\int d^3x\ |x\rangle\langle x| \right] |\psi_j\rangle\ a_i^{+} a_j$$

$$= \sum_{ij} \langle \varphi_i | \psi_j\rangle\ a_i^{+} a_j$$

$$= \sum_{i} a_i^{+} a_i$$ → looks familiar?

4.3.2 Other operators

We want to relate this field formalism to quantities we know (and can measure). Take for instance the kinetic energy operator $\hat{T}$

1 particle: $$\hat{t} = \frac{-\hbar^2}{2m} \int d^3x\ |x\rangle\ \nabla^2 \langle x|$$

So that $$\langle \chi | \hat{t} | \phi\rangle = \frac{-\hbar^2}{2m} \int d^3x\ \langle \chi | x\rangle\ \nabla^2 \langle x | \phi\rangle$$

$$= \frac{-\hbar}{2m} \int d^3x\ \chi^{*}(x)\ \nabla^2 \phi(x)$$

Field operator: $$\hat{T} = \sum_{ij} a_i^{+} \langle \varphi_i | \hat{t} | \psi_j\rangle\ a_j = \sum_{ij} \underbrace{a_i^{+} a_j}_{②} \underbrace{\langle \varphi_i | \hat{t} | \psi_j\rangle}_{①}$$

② destroys $|\psi_j\rangle$ creates $|\psi_i\rangle$

① if $\hat{t}$ takes $|\psi_j\rangle$ to $|\psi_i\rangle$

So that

$$\langle x|\hat{T}|\phi\rangle = \sum_{i,j} \langle x| a_i^{\dagger} \langle \psi_i | \hat{t} | \psi_j \rangle a_j |\phi\rangle$$

$$\cancel{\sum_{ij} \langle \psi_i|\hat{t}|\psi_j\rangle \langle x| a_i^{\dagger} a_j |\phi\rangle}$$

$$\cancel{\sum_i \langle x| a_i^{\dagger}}$$

$$= \int \langle x| \left(\sum_i a_i^{\dagger} \langle \psi_i | x\rangle\right) \left(\frac{-\hbar^2}{2m} \nabla^2\right) \left(\sum_j a_j \langle x|\psi_j\rangle\right) |\phi\rangle$$

$$= \int \langle x| \Psi^{\dagger}(x) \left(\frac{-\hbar^2}{2m} \nabla^2\right) \underline{\underline{\Psi(x)}} |\phi\rangle$$

where we had $|\vec{x}\rangle$, now we have $\Psi(\vec{x})$

$$\hat{T} = \frac{-\hbar^2}{2m} \int d^3x \, \Psi^{\dagger}(x) \, \nabla^2 \, \Psi(x)$$

This applies to other operators too. Eg potential energy:

$$\hat{u} = \int d^3\vec{x} \; U(\vec{x}) \, |x\rangle\langle x|$$

becomes

$$\hat{U} = \sum_{ij} \langle \psi_i | \hat{u} | \psi_j \rangle \, a_i^{\dagger} a_j$$

$$= \int d^3\vec{x} \; U(\vec{x}) \; \Psi^{\dagger}(\vec{x}) \, \Psi(\vec{x})$$

(more examples in the script)

Eg 2-particle interaction described by potential $V(x-y)$

$$\hat{H}_{ww} = \underbrace{\frac{1}{2}}_{\text{normalization}} \underbrace{\sum_{s_1, s_2}}_{\text{just in case you had forgotten about spin!}} \int d^3x \, d^3y \; \Psi_{s_1}^{\dagger}(x) \Psi_{s_2}^{\dagger}(y) \, V(x-y) \, \Psi_{s_2}(y) \Psi_{s_1}(x)$$

4.3.3 Field equations

Heisenberg picture: operators evolve in time

$$\Psi(x,t) = e^{i\hat{H}t/\hbar}\, \Psi(x)\, e^{-i\hat{H}t/\hbar}$$

Equations of motion

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = -[\hat{H}, \Psi(x,t)]$$
$$= -e^{i\hat{H}t/\hbar} \underbrace{[\hat{H}, \Psi(x)]}_{\text{we have to compute this commutator!}} e^{-i\hat{H}t/\hbar}$$

Bosons

Kinetic energy:

$$[\hat{T}, \Psi(x)] = \frac{-\hbar^2}{2m} \int d^3y\, [\Psi^\dagger(y)\, \nabla^2 \Psi(y), \Psi(x)]$$

$$= \frac{-\hbar^2}{2m} \int d^3y\, \Psi^\dagger(y) [\nabla^2 \Psi(y), \Psi(x)] + [\Psi^\dagger(y), \Psi(x)]\, \nabla^2 \Psi(y)$$

$$= \frac{-\hbar^2}{2m} \int d^3y\, \Psi^\dagger(y) \cdot 0 - \delta(x-y)\, \nabla^2 \Psi(y)$$

$$= \frac{\hbar^2}{2m} \nabla^2 \Psi(x)$$

$$[\hat{H}_{ww}, \Psi(x)] = \frac{1}{2} \int d^3y_1 d^3y_2\, V(y_1-y_2) [\Psi^\dagger(y_1)\Psi^\dagger(y_2)\Psi(y_2)\Psi(y_1), \Psi(x)]$$

$$= \frac{1}{2} \int d^3y_1 d^3y_2\, V(y_1-y_2) \underbrace{[\Psi^\dagger(y_1)\Psi^\dagger(y_2), \Psi(x)]}_{\text{let's just look at this}} \Psi(y_2)\Psi(y_1) + \underbrace{0}_{\text{other term}}$$

$$[\psi^+(y_1)\,\psi^+(y_2),\ \psi(x)] = \psi^+(y_1)\,[\psi^+(y_2),\ \psi(x)] + [\psi^+(y_1),\ \psi(x)]\,\psi^+(y_2)$$

$$= -\psi^+(y_1)\ \delta(x-y_2) - \delta(x-y_1)\ \psi(y_2)$$

$$[H_{ww},\ \psi(x)] = -\int d^3y\ \psi^+(y)\ V(x-y)\ \psi(y)\ \psi(x)$$