1 e^- scattering off a static Coulomb potential (H&M 8.1)

$$T_{fi} = -i\int d^4x'\, j^\mu(x') A_\mu(x')$$

$$= -ie\, \bar{u}_{s'}(p')\gamma^0 u_s(p) \times$$

$$\times \int d^4x'\, e^{i(p-p')\cdot x'} \int d^3\vec{y}\,' \frac{\rho(\vec{y}\,')}{|\vec{x}\,'-\vec{y}\,'|}$$

$q = p - p'$

$$\stackrel{\int dx^0}{=} -ie\, u^\dagger_{s'}(p') u_s(p) (2\pi)\, \delta(E-E')$$

$$\underbrace{\int d^3\vec{y}\,' \rho(\vec{y}\,')\, e^{i\vec{q}\cdot\vec{y}\,'}}_{= Ze F(\vec{q})} \underbrace{\int d^3\vec{x}\, \frac{e^{i\vec{q}\cdot\vec{x}}}{|\vec{x}|}}_{= \frac{4\pi}{|\vec{q}|^2}}$$

$$= -8\pi^2 i Ze^2\, u^\dagger_{s'}(p')\, u_s(p)\, \delta(E-E')\, \frac{F(\vec{q})}{|\vec{q}|^2}$$

$$\overline{|T_{fi}|^2} = \sum_s \sum_{s'} |T_{fi}|^2$$

$$= 2^8 \pi^4 Z^2 e^4 |\delta(E-E')|^2 E^2 \left(1-\beta^2 \sin^2\frac{\vartheta}{2}\right) \frac{|F(\vec{q})|^2}{|\vec{q}|^4}$$

$$\lim_{T\to\infty} \frac{1}{T} |\delta(E-E')|^2 = \frac{1}{2\pi} \lim_{T\to\infty} \frac{2}{\pi T} \frac{\sin^2\left((E-E')\frac{T}{2}\right)}{(E-E')^2}$$

$$= \frac{1}{2\pi}\, \delta(E-E')$$

$s = \sin\vartheta/2$

$$\Rightarrow \quad w = \lim_{T\to\infty} \frac{\overline{|T_{fi}|^2}}{T} = 2^7\pi^3 Z^2 e^4 E^2(1-\beta^2 s^2)\frac{|F(\vec{q})|^2}{|\vec{q}|^4}$$

$$d^3\vec{p}\,'\,\delta(E-E') = |\vec{p}\,'|^2 d|\vec{p}\,'|\, d\Omega\, \delta(E-E')$$

$$= |\vec{p}\,'|\,|\vec{p}\,'|\, d|\vec{p}\,'|\, d\Omega\, \delta(E-E')$$

$$= |\vec{p}\,'|\, E' dE'\, d\Omega\, \delta(E-E')$$

$$= |\vec{p}\,'|\, E\, d\Omega$$

$$\Rightarrow d\sigma = \frac{d^3\vec{p}\,'}{(2\pi)^3 2E'}\frac{w}{2\beta E} = 2^2 Z^2 e^4 \frac{E|\vec{p}\,'|}{\cancel{E^2}\beta} d\Omega\, E^2(1-\beta^2 s^2)\frac{|F(\vec{q})|^2}{|\vec{q}|^4}$$

$$|\vec{q}|^2 = |\vec{p}-\vec{p}\,'|^2 = |\vec{p}|^2 + |\vec{p}\,'|^2 - 2|\vec{p}|\,|\vec{p}\,'|\cos\vartheta$$

$$E = E' \Rightarrow |\vec{p}| = |\vec{p}\,'| \Rightarrow \quad |\vec{q}|^2 = 2|\vec{p}|^2(1-\cos\vartheta)$$

$$\cos\vartheta = \cos 2\cdot\frac{\vartheta}{2} = \cos^2\vartheta/2 - s^2 = 1-2s^2$$

$$\Rightarrow 1-\cos\vartheta = 2s^2 \Rightarrow |\vec{q}|^2 = 4|\vec{p}|^2 s^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \underbrace{\frac{(Z\alpha)^2 E^2}{4|\vec{p}|^4 s^4}(1-\beta^2 s^2)}_{\left(\frac{d\sigma}{d\Omega}\right)_{Mott}} |F(\vec{q})|^2$$