Ex sheet 3 - PPP II

i) light-cone gauge $d^{\mu\nu}(p) = -g^{\mu\nu} + \left(\frac{p^\mu n^\nu + p^\nu n^\mu}{p\cdot n}\right)$ $\quad n^2 = 0$

ii) Sudakow parametrisation

$$k^\mu = z p^\mu + k_T^\mu + \frac{k^2 - k_T^2}{2z\, p\cdot n} n^\mu \quad \text{with} \quad 0<z<1 \quad k_T^2 < 0$$
$$p\cdot k_T = 0 \quad n\cdot k_T = 0$$

$$(p-k)^\mu = (1-z)p^\mu - k_T^\mu - \frac{k^2-k_T^2}{2z\,p\cdot n} n^\mu$$

$$(p-k)^2 = (1-z)^2 \overset{0}{p^2} + \underline{k_T^2} + \left(\frac{k^2-k_T^2}{2z\,p\cdot n}\right)\overset{=0}{n^2} - 2(1-z)\overset{=0}{p\cdot k_T} - 2(1-z)\frac{k_T^2-k_T^2}{2z\cancel{p\cdot n}}\cancel{p\cdot n}$$
$$+ 2\underbrace{k_T\cdot n}_{0}\frac{k^2-k_T^2}{2z\,p\cdot n} = \boxed{k_T^2 - \cancel{2}(1-z)\frac{k^2-k_T^2}{\cancel{2}z} \equiv 0} \quad \text{on-shell condition}$$

from which $k_T^2 = \frac{1-z}{z}(k^2 - k_T^2) \Rightarrow k_T^2\left(1+\frac{1-z}{z}\right) = \frac{1-z}{z}k^2$

$$k_T^2\left(\frac{1}{\cancel{z}}\right) = \frac{(1-z)}{\cancel{z}}k^2$$

$$\boxed{k^2 = \left(\frac{k_T^2}{1-z}\right)}$$

• $p^\mu d_{\mu\nu}(p-k) = k^\mu d_{\mu\nu}(p-k)$ by definition transverse (physical) gauge

$$k^\mu d_{\mu\nu} = -k_\nu + \frac{k\cdot p\, n^\nu + p^\nu\, k\cdot n}{p\cdot n} = \frac{-k_\nu\, p\cdot n + k\cdot p\, n^\nu + p^\nu k\cdot n}{p\cdot n}$$

Use $k^\mu = z p^\mu + k_T^\mu + \frac{k^2-k_T^2}{2z\,p\cdot n}n^\mu = z p^\mu + k_T^\mu + \frac{k_T^2}{2(1-z)\,p\cdot n} n^\mu$

$$k^\mu d_{\mu\nu}(p-k) = z\, p^\mu d_{\mu\nu}(p-k) + k_T^\mu\, d_{\mu\nu}(p-k)$$

$$+ \frac{k^2 - k_T^2}{2z\, p\cdot n}\, n^\mu d_{\mu\nu}(p-k)$$

Using $\boxed{n^\mu d_{\mu\nu}(p-k) = 0}$

$$k^\mu d_{\mu\nu}(p-k) = z\, p^\nu\, d_{\mu\nu}(p-k) + k_T^\mu d_{\mu\nu}(p-k)$$

$$= z\, k^\nu\, d_{\mu\nu}(p-k) + k_T^\mu\, d_{\mu\nu}(p-k)$$

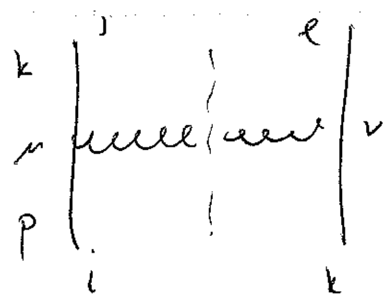
$$\Rightarrow \boxed{k^\mu d_{\mu\nu}(p-k) = \frac{k_T^\mu}{(1-z)}\, d_{\mu\nu}(p-k)}$$

finally $\quad p^\mu p^\nu d_{\mu\nu}(p-k) = \dfrac{k_T^\mu k_T^\nu}{(1-z)^2}\, d_{\mu\nu}(p-k)$

$$= -\frac{k_T^2}{(1-z)^2} + \frac{1}{(1-z)^2}\, k_T^\mu k_T^\nu \left(\frac{(p-k)^\mu n^\nu + (p-k)^\nu \cdot n^\mu}{(p-k)\cdot n} \right)$$

$$= -\frac{k_T^2}{(1-z)^2} + \frac{1}{(1-z)^2} \left[\frac{n\cdot k_T\, (\ldots)}{(p-k)\cdot n} \right] \qquad \boxed{n\cdot k_T = 0}$$

$$\Rightarrow \boxed{p_\mu p_\nu\, d^{\mu\nu}(p-k) = -\frac{k_T^2}{(1-z)^2}}$$

iii) 

(sum over spins only on-shell part)

Diagram gives $\delta_{ik}\, t^a_{ji}\, t^a_{kl} \dfrac{\not k \gamma^\mu \not p \gamma^\nu \not k}{(k^2)^2} d_{\mu\nu}(p-k)$

$$= C_F \delta_{jl} \frac{1}{2} \not k \left(\gamma^\mu \not p \gamma^\nu + \gamma^\nu \not p \gamma^\mu \right) \not k \frac{1}{(k^2)^2} d_{\mu\nu}(p-k)$$

= anticommuting gamma matrices

$$= C_F \delta_{jl} \not k \left[p^\mu \gamma^\nu + p^\nu \gamma^\mu - g^{\mu\nu} \not p \right] \not k \frac{1}{(k^2)^2} d_{\mu\nu}(p-k)$$

Use now • $\not k \not p \not k = -k^2 \not p + 2 \not k (k\cdot p)$

• $\not k \gamma^\mu \not k = -k^2 \gamma^\mu + 2 \not k k^\mu$

$$= \frac{C_F \delta_{jl}}{(k^2)^2} \Big\{ g_{\mu\nu} \left(k^2 \not p - 2 \not k (k\cdot p) \right) - k^2 \gamma^\nu p^\mu + 2 \not k \underbrace{k^\nu}_{\to p^\nu} p^\mu - k^2 \gamma^\mu p^\nu + 2 \not k \underbrace{k^\mu}_{p^\mu} p^\nu \Big\} d^{\mu\nu}$$

$\boxed{2k\cdot p = k^2}$

$$= \frac{C_F \delta_{jl}}{(k^2)^2} \Big\{ g^{\mu\nu} k^2 (\not p - \not k) - k^2 (p^\nu \gamma^\mu + p^\mu \gamma^\nu) + 4 \not k p^\mu p^\nu \Big\} d_{\mu\nu}(p-k)$$

notice now

- $g_{\mu\nu} d^{\mu\nu}(p-k) = -2$
- $p_\mu d^{\mu\nu}(p-k) = k_\mu d^{\mu\nu}(p-k) \sim k_{T\mu}$
- $p_\mu p_\nu d^{\mu\nu}(p-k) \sim k_T^2$

$$= \frac{C_F \delta_{je}}{(k^2)^2} \Big\{ \underbrace{-2k^2(\not{p}-\not{k})}_{\sim O(k_T^2)} \underbrace{- k^2 (p^\nu \gamma^\mu + p^\mu \gamma^\nu) d_{\mu\nu}(p-k)}_{\sim O(k_T^3)\ \text{neglected}} \underbrace{- 4\not{k} \frac{k_T^2}{(1-z)^2}}_{\sim O(k_T^2)} \Big\}$$

send $k_T^2 \to 0$ $\boxed{\not{k} \sim z\not{p}}$ and use $\left(k^2 = \frac{k_T^2}{1-z}\right)$

$$\simeq \frac{C_F \delta_{je}}{(k^2)^2} \Big\{ -2 \frac{k_T^2}{\cancel{(1-z)}} \not{p} \cancel{(1-z)} - 4z\not{p} \frac{k_T^2}{(1-z)^2} \Big\} \big(1 + O(k_T)\big)$$

$$= (-1) 2 \frac{C_F \delta_{je}}{(k^2)^2} (k_T^2) \left[1 + \frac{2z}{(1-z)^2} \right] \not{p} \big(1 + O(k_T)\big)$$

or

$$= (-1)\, 2 \frac{C_F \delta_{je}}{(k^2)^{\cancel{2}}} \cancel{(k^2)} \left[(1-z) + \frac{2z}{1-z} \right] \not{p} \big(1 + O(k_T)\big)$$

$$= (-1)\, 2 C_F \delta_{je} \frac{1}{(k^2)} \left[\frac{1+z^2}{1-z} \right] \not{p} \big(1 + O(k_T)\big)$$

the real diagram gives

$$(-1)\, 2 C_F\, \delta_{je}\, \frac{1}{(k^2)} \left[\frac{1+z^2}{1-z} \right] \cancel{\phi} \left(1 + O(k_T)\right)$$

this is exactly factorization formula with

$$P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z} \right)$$ notice divergence (not integrable) in $\underline{\underline{z=1}}$.

- $\boxed{\text{VIRTUAL CONTRIB.}}$

i) $P_{qq}^{tot}(z) = P_{qq}^{real}(z) + P_{qq}^{virt}(z)$

$$\int_0^1 P_{qq}^{tot}(z)\, dz = 0$$ Conservation number of quarks.

ii) put $P_{qq}^{virt}(z) = A\, \delta(1-z)$ (quark doesn't split!)

$$\int_0^1 \left\{ C_F \left(\frac{1+z^2}{1-z} \right) + A\, \delta(1-z) \right\} dz = 0$$ gives

$$A = -C_F \int_0^1 \left(\frac{2}{1-z} - (1+z) \right) dz$$

$$= \frac{3}{2} C_F - 2 C_F \underbrace{\int_0^1 \frac{1}{1-z}\, dz}$$ ill defined, we need to put a regulator.

$$P_{qq}^{virt}(z) = A\delta(1-z) \stackrel{def.}{=} C_F\left\{\frac{3}{2}\delta(1-z) - 2\delta(1-z)\int_0^{1-\varepsilon} dt\, \frac{1}{1-t}\right\}$$

$$= C_F\left\{\frac{3}{2}\delta(1-z) - 2\delta(1-z)\int_0^1 dt\, \frac{1}{1-t}\,\theta(1-t-\varepsilon)\right\}$$

Use now plus distribution

$$-\delta(1-z)\int_0^1 dt\, \frac{1}{(1-t)}\,\theta(1-t-\varepsilon) = \left(\frac{1}{(1-z)_+} - \lim_{\varepsilon\to 0}\frac{1}{1-z}\,\theta(1-z-\varepsilon)\right)$$

$$P_{qq}^{virt}(z) = C_F\left\{\frac{3}{2}\delta(1-z) + \frac{2}{(1-z)_+} - 2\frac{1}{(1-z)}\right\}$$

cancels divergence

<u>in $z=1$</u>

this means that

$$P_{qq}^{tot}(z) = C_F\left\{\cancel{\frac{2}{(1-z)}} - (1+z) + \frac{2}{(1-z)_+} - \cancel{\frac{2}{(1-z)}} + \frac{3}{2}\delta(1-z)\right\}$$

$$= C_F\left\{\frac{3}{2}\delta(1-z) + \frac{2}{(1-z)_+} - (1+z)\right\}$$

this can also be written as

$$P_{qq}^{tot}(z) = C_F\left\{\frac{3}{2}\delta(1-z) + \frac{1+z^2}{(1-z)_+}\right\}$$

(plus distribution not needed at numerator)