Phenomenology of
Particle Physics II

Exercise Sheet 3

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Lecturers: Dr. M. Dittmar, Dr. M. Grazzini — Due : 13.03.2012

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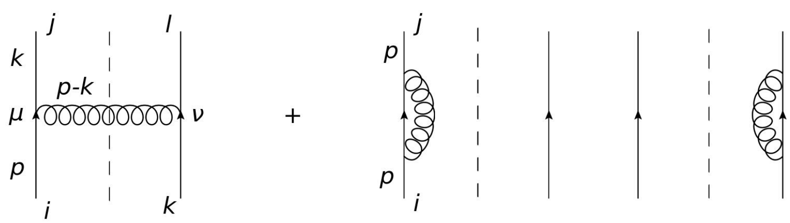
Exercise 3 [*The quark splitting-function*]

In the following exercise we compute the leading order quark splitting function $P_{qq}^{(LO)}(z)$, which represents the probability of a quark to emit a gluon, and so to become a quark with momentum reduced by a fraction z.

At leading order there are both real and virtual contributions to the amplitude squared:



- We start analizing the real contribution.

  In the first diagram above an on-shell massless quark with momentum p^μ emits an on-shell gluon of momentum $q^\mu = (p-k)^\mu$ and becomes an off-shell quark of momentum k^μ, so that

  $$p^2 = 0, \quad q^2 = 0, \quad k^2 \neq 0.$$

  We study the process in the limit of collinear emission, i.e. $k \approx$ parallel to p.

  (i) To get rid of the unphysical polarizations we work in a light-cone gauge, i.e. the numerator of the gluon propagator is:

  $$\sum_{spins} \epsilon^{*\mu}(q)\epsilon^{\nu}(q) = d^{\mu\nu}(q) = -g^{\mu\nu} + \frac{q^\mu n^\nu + q^\nu n^\mu}{q \cdot n}$$

  where

  $$n \cdot A = 0 \qquad \text{and} \qquad n^2 = 0.$$

(ii) To parametrize the off-shellness of the quark we use the *Sudakov parametrization* for the momentum k:

$$k^\mu = z\, p^\mu + k_T^\mu + \frac{k^2 - k_T^2}{2\, z\, p\cdot n}\, n^\mu, \quad \text{with} \quad 0 < z < 1,\;\; k_T^2 < 0,\;\; p\cdot k_T = 0,\;\; n\cdot k_T = 0.$$

With this choice the gluon momentum becomes:

$$(p-k)^\mu = (1-z)\, p^\mu - k_T^\mu - \frac{k^2 - k_T^2}{2\, z\, p\cdot n}\, n^\mu.$$

Show that:

$$k^2 = 2p\cdot k = \frac{k^2 - k_T^2}{z} \quad \longrightarrow \quad k^2 = \frac{k_T^2}{1-z},$$

i.e. the quark is off-shell of an amount proportional to k_T^2. In the collinear limit $k_T^2 \to 0$ and $k^2 \to 0$.

Other useful relations to prove are:

$$p^\mu d_{\mu\nu}(p-k) = k^\mu d_{\mu\nu}(p-k) = \frac{k_T^\mu}{1-z} d_{\mu\nu}(p-k),$$

$$p^\mu p^\nu d_{\mu\nu} = -\frac{k_T^2}{(1-z)^2} = -\frac{k^2}{1-z}.$$

From this parametrization it is clear that z represents the fraction of momentum of the quark after the splitting, while the gluon carries a fraction $(1-z)$.

(iii) Using these relations, compute the real contribution to the amplitude squared. Show that in the collinear limit ($k_T \to 0$) it reads:

$$C_F\, \delta_{jl}\, \frac{(-1)}{k^4}\, (2k^2) \left(\frac{1+z^2}{1-z}\right) \not{p} \Big(1 + O(k_T)\Big)$$
$$= (-1)\, \delta_{jl}\, \frac{2}{k^2}\, \not{p}\, P_{qq}(z) \Big(1 + O(k_T)\Big).$$

Where C_F is the usual colour factor $\sum_{a,i} t^a_{ji} t^a_{il} = C_F \delta_{il}$.

Hint: Start from the following expression for the diagram (modulus charge factors):

$$\delta_{ik}\, t^a_{ji}\, t^a_{kl}\, \frac{\not{k}\, \gamma^\mu \not{p}\, \gamma^\nu\, \not{k}}{(k^2)^2}\, d_{\mu\nu}(p-k).$$

Before using the gamma algebra to simplify this expression, notice that since $d_{\mu\nu}$ is symmetric, the string of gamma matrices can be symmetrized:

$$\not{k}\, (\gamma^\mu \not{p}\, \gamma^\nu)\, \not{k} = \frac{1}{2}\, \not{k} \left(\gamma^\mu \not{p}\, \gamma^\nu + \gamma^\nu \not{p}\, \gamma^\mu\right) \not{k}.$$

(iv) From this relation one can read that

$$P_{qq}^{real}(z) = C_F \left(\frac{1+z^2}{1-z}\right) = C_F \left(\frac{2}{1-z} - (1+z)\right), \tag{1}$$

which exhibits a IR divergence for $z \to 1$.

- Let us focus now on the virtual contribution to the splitting function $P_{qq}^{virt}(z)$ from diagrams (2) and (3). We'll show that it can be obtained without computing explicitly any loop diagrams.

  (i) Define
  $$P_{qq}^{tot}(z) = P_{qq}^{real}(z) + P_{qq}^{virt}(z).$$
  We know that the following relation has to hold:
  $$\int_0^1 P_{qq}^{tot}(z)\, dz = 0. \tag{2}$$
  Can you give a physical justification for that?

  (ii) We can parametrize the virtual contribution as:
  $$P_{qq}^{virt}(z) = A\,\delta(1-z), \tag{3}$$
  where A is a constant. Can you justify this expected functional form?

  (iii) Use (2) with (1) and the parametrization (3) to prove that:
  $$P_{qq}^{tot}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right]. \tag{4}$$
  Where we introduced the *plus distribution* defined as:
  $$\int_0^1 f(z)\, \frac{1}{(1-z)_+}\, dz \equiv \int_0^1 \frac{f(z)-f(1)}{1-z}\, dz. \tag{5}$$
  Hint: Notice that since (1) is divergent in $z=1$, to perform the integral (2) we need to introduce a regulator, i.e.:
  $$\int_0^1 g(z)\, dz \longrightarrow \int_0^{1-\epsilon} g(z)\, dz \equiv \int_0^1 g(z)\, \theta(1-z-\epsilon) dz.$$
  To connect this with the plus distribution, notice that definition (5) is satisfied for:
  $$\frac{1}{(1-z)_+} \equiv \lim_{\epsilon\to 0} \left[\frac{1}{1-z}\theta(1-z-\epsilon) - \delta(1-z) \int_0^1 \frac{1}{1-t}\theta(1-t-\epsilon)\, dt \right].$$

Informations relative to the exercises

Testat condition : 60% of the exercise sheets worked out and solve one exercise at the blackboard.

Exercises may be solved in groups of up to 3 people.

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