Advanced Topics in Quantum Information Theory
Solution 2

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Exercise 2.1 Three qubit bit flip code

a.) The spectral decomposition of the Pauli matrix Z is given by $Z = (+1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1|$. The eigenvectors of $Z_1 Z_2$ corresponding to the eigenvalue $+1$ are therefore $|000\rangle, |001\rangle, |110\rangle$ and $|111\rangle$. The eigenvectors of $Z_1 Z_2$ corresponding to the eigenvalue -1 are given by $|010\rangle, |011\rangle, |100\rangle$ and $|101\rangle$.

For the observable $Z_2 Z_3$ we obtain the eigenvectors $|000\rangle, |100\rangle, |011\rangle$ and $|111\rangle$ corresponding to the eigenvalue $+1$ and the eigenvectors $|010\rangle, |110\rangle, |001\rangle$ and $|101\rangle$ corresponding to the eigenvalue -1.

b.) Applying the bit flip on the first qubit gives the state $X_1|\psi\rangle = \alpha|100\rangle + \beta|011\rangle$. Measuring the observable $Z_1 Z_2$ then yields -1 with probability 1 as $X_1|\psi\rangle$ is an element of the space spanned by the eigenvectors corresponding to the eigenvalue -1 (see previous item). Furthermore, this implies that the state $X_1|\psi\rangle$ is not altered by this measurement.

Measuring $Z_2 Z_3$ yields the outcome $+1$ with probability 1 as $X_1|\psi\rangle$ is an element of the space spanned by the eigenvectors corresponding to the eigenvalue $+1$. Again, the state is not changed by this measurement.

c.) By using the same reasoning as above we can show that

- $|\psi\rangle$: measuring $Z_1 Z_2$ yields $+1$ and $Z_2 Z_3$ yields $+1$.
- $X_2|\psi\rangle$: measuring $Z_1 Z_2$ yields -1 and $Z_2 Z_3$ yields -1.
- $X_3|\psi\rangle$: measuring $Z_1 Z_2$ yields $+1$ and $Z_2 Z_3$ yields -1.

The states are not changed by any of these measurements.

d.) The previous two items imply that the following strategy corrects a single bit flip error:

- Measuring $+1$, $+1$ $\Rightarrow$ do nothing
- Measuring -1, $+1$ $\Rightarrow$ apply X_1
- Measuring -1, -1 $\Rightarrow$ apply X_2
- Measuring $+1$, -1 $\Rightarrow$ apply X_3

Exercise 2.2 Shor code

a.) First note that the faulty state is given by

$$\begin{aligned} Z_4 X_4|\psi\rangle &= \alpha \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right) \otimes \left(\frac{-|100\rangle + |011\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right) \\ &+ \beta \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right) \otimes \left(\frac{-|100\rangle - |011\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right) . \end{aligned}$$

The projector P_+^{45} which projects onto the eigenbasis of Z_4Z_5 corresponding to the eigenvalue $+1$ is given by (see Exercise 2.1)

$$P_+^{45} = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \ ,$$

where each identity operator $\mathbb{I}$ acts on a single qubit. Similarly, for the projector P_-^{45} corresponding to the eigenvalue -1 has the following expression:

$$P_-^{45} = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \ .$$

It is not hard to see that $Z_4X_4|\psi\rangle \in range(P_-^{45})$, i.e. that $Z_4X_4|\psi\rangle$ is an element of the space spanned by the eigenvectors of Z_4Z_5 corresponding to the eigenvalue -1. As $Z_4X_4|\psi\rangle \in range(P_-^{45})$ it holds that $P_-^{45}Z_4X_4|\psi\rangle = Z_4X_4|\psi\rangle$ and therefore the state is not changed by the measurement. For the measurement Z_5Z_6 we obtain the following projectors

$$\begin{aligned} P_+^{56} &= \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \\ P_-^{56} &= \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \ . \end{aligned}$$

This time we have that $Z_4X_4|\psi\rangle \in range(P_+^{56})$ and therefore obtain the measurement result $+1$ with probability 1. Again, the state is not altered by the measurement.

As we have outcomes -1 and $+1$ we can conclude, by using Exercise 2.1, that we have to apply X_4 in order to correct the bit flip on the fourth qubit.

b.) Applying the bit flip operation X_4 on the faulty state yields the bit flip corrected state

$$X_4(Z_4X_4|\psi\rangle) = -Z_4X_4X_4|\psi\rangle = -Z_4|\psi\rangle \ , \tag{1}$$

where we used that X_4 and Z_4 anti-commute and that $X_4X_4 = \mathbb{I}$.

Let $|+\rangle := 1/\sqrt{2}(|0\rangle + |1\rangle)$ and $|-\rangle := 1/\sqrt{2}(|0\rangle - |1\rangle)$. Note that $X|+\rangle = (+1)|+\rangle$ and $X|-\rangle = (-1)|-\rangle$. The projector P_+ corresponding to the eigenbasis of the observable $X_1X_2X_3$ belonging to the eigenvalue $+1$ is then given by

$$P_+ = |+++\rangle\langle +++| + |+--\rangle\langle +--| + |-+-\rangle\langle -+-| + |--+\rangle\langle --+| \ .$$

And similarly, for the projector belonging to the eigenvalue -1 we obtain

$$P_- = |---\rangle\langle ---| + |++-\rangle\langle ++-| + |-++\rangle\langle -++| + |+-+\rangle\langle +-+| \ .$$

The corresponding projectors for the measurement $X_1X_2X_3X_4X_5X_6$ are then given by

$$\begin{aligned} P_+^{1..6} &= P_+ \otimes P_+ \otimes \mathbb{I}^{\otimes 3} + P_- \otimes P_- \otimes \mathbb{I}^{\otimes 3} \\ P_-^{1..6} &= P_+ \otimes P_- \otimes \mathbb{I}^{\otimes 3} + P_- \otimes P_+ \otimes \mathbb{I}^{\otimes 3} \ . \end{aligned}$$

By using that $1/\sqrt{2}(|000\rangle + |111\rangle) \in range(P_+)$ and $1/\sqrt{2}(|000\rangle - |111\rangle) \in range(P_-)$ we can conclude that with probability 1 the measurement outcome -1 is obtained, and therefore the state is not changed by the measurement.

For the measurement $X_4X_5X_6X_7X_8X_9$ we obtain the projectors

$$\begin{aligned} P_+^{4..9} &= \mathbb{I}^{\otimes 3} \otimes P_+ \otimes P_+ + \mathbb{I}^{\otimes 3} \otimes P_- \otimes P_- \\ P_-^{4..9} &= \mathbb{I}^{\otimes 3} \otimes P_+ \otimes P_- + \mathbb{I}^{\otimes 3} \otimes P_- \otimes P_+ \ , \end{aligned}$$

and therefore, we obtain the outcome -1 with probability 1. Again, the state is not changed.

As we have the measurement outcomes -1 and -1 we can conclude, by using Exercise 2.1 and the fact that a phase flip in the $\{|0\rangle, |1\rangle\}$ basis is a bit flip in the $\{|+\rangle, |-\rangle\}$ basis, that a phase flip error has occurred in the second block of three qubits.

c.) Note that $(Z \otimes Z \otimes \mathbb{I})|000\rangle = |000\rangle$ and $(Z \otimes Z \otimes \mathbb{I})|111\rangle = |111\rangle$. Applying $Z_4Z_5Z_6$ on the state given in (1) then yields

$$(Z_4Z_5Z_6)(-Z_4|\psi\rangle) = -Z_5Z_6|\psi\rangle = -|\psi\rangle \ .$$

Hence, we have recovered the initial state $|\psi\rangle$ (with a global phase).

d.) The same procedure as above can be used.

i.) Measure Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9. This leaves the state unchanged, and then given the measurement outcomes (syndrome), we can correct the bit flip error. More specifically, we have the four cases in part b.) and c.) of Exercise 2.1 in either block 123, 456, or 789, and can determine where to apply an X operator.

ii.) For the phase flip we can measure $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$. This determines which block the Z error occurs in. Specifically, -1 +1 eigenvalues mean the Z error is in block 123, -1 -1 eigenvalues mean the Z error is in block 456, +1 -1 eigenvalues mean the Z error is in block 789. By applying a Z operation to each qubit in the block with an error $-|\psi\rangle$ is left.

Exercise 2.3 Coding and Decoupling

a.) The noise channel $\mathcal{E}$ can be written as

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}X_1\rho X_1 + \frac{p}{3}X_2\rho X_2 + \frac{p}{3}X_3\rho X_3 \ .$$

The isometric purification of this channel is given by

$$U_{\mathcal{E}} = \sqrt{1-p}\mathbb{I} \otimes |0\rangle_E + \sqrt{\frac{p}{3}}X_1 \otimes |1\rangle_E + \sqrt{\frac{p}{3}}X_2 \otimes |2\rangle_E + \sqrt{\frac{p}{3}}X_3 \otimes |3\rangle_E \ .$$

Applying $\mathbb{I}_{A'} \otimes U_{\mathcal{E}}$ on the state $1/\sqrt{2}|0\rangle_{A'} \otimes |000\rangle_C + 1/\sqrt{2}|1\rangle_{A'} \otimes |111\rangle_C$ then yields

$$\begin{aligned}
|\phi\rangle_{A'CE} &= \frac{1}{\sqrt{2}}|0\rangle_{A'} \otimes \Big(\sqrt{1-p}|000\rangle_C \otimes |0\rangle_E + \sqrt{\frac{p}{3}}|100\rangle_C \otimes |1\rangle_E \\
&+ \sqrt{\frac{p}{3}}|010\rangle_C \otimes |2\rangle_E + \sqrt{\frac{p}{3}}|001\rangle_C \otimes |3\rangle_E\Big) \\
&+ \frac{1}{\sqrt{2}}|1\rangle_{A'} \otimes \Big(\sqrt{1-p}|111\rangle_C \otimes |0\rangle_E + \sqrt{\frac{p}{3}}|011\rangle_C \otimes |1\rangle_E \\
&+ \sqrt{\frac{p}{3}}|101\rangle_C \otimes |2\rangle_E + \sqrt{\frac{p}{3}}|110\rangle_C \otimes |3\rangle_E\Big) \ .
\end{aligned}$$

Taking the partial trace over the system C results in

$$\begin{aligned}
tr_C(|\phi\rangle\langle\phi|_{A'CE}) &= \left(\frac{1}{2}|0\rangle\langle 0|_{A'} + \frac{1}{2}|1\rangle\langle 1|_{A'}\right) \\
&\otimes \left((1-p)|0\rangle\langle 0|_E + \frac{p}{3}|1\rangle\langle 1|_E + \frac{p}{3}|2\rangle\langle 2|_E + \frac{p}{3}|3\rangle\langle 3|_E\right) \ .
\end{aligned}$$

b.) The channel representing this noise process can be written as

$$\begin{aligned}
\mathcal{E}(\rho) &= (1-p)\rho + \frac{p}{6}X_1\rho X_1 + \frac{p}{6}X_2\rho X_2 + \frac{p}{6}X_3\rho X_3 \\
&+ \frac{p}{6}X_1X_2\rho X_1X_2 + \frac{p}{6}X_2X_3\rho X_2X_3 + \frac{p}{6}X_1X_3\rho X_1X_3 \ . \qquad (2)
\end{aligned}$$

The corresponding isometric purification is

$$\begin{aligned}
U_{\mathcal{E}} &= \sqrt{1-p}\mathbb{I} \otimes |0\rangle_E + \sqrt{\frac{p}{6}}X_1 \otimes |1\rangle_E + \sqrt{\frac{p}{6}}X_2 \otimes |2\rangle_E + \sqrt{\frac{p}{6}}X_3 \otimes |3\rangle_E \\
&+ \sqrt{\frac{p}{6}}X_1X_2 \otimes |4\rangle_E + \sqrt{\frac{p}{6}}X_2X_3 \otimes |5\rangle_E + \sqrt{\frac{p}{6}}X_1X_3 \otimes |6\rangle_E \ .
\end{aligned}$$

The state $|\phi\rangle_{A'CE}$ is then given by

$$\begin{aligned}
|\phi\rangle_{A'CE} &= \frac{1}{\sqrt{2}}|0\rangle_{A'} \otimes \Big(\sqrt{1-p}|000\rangle_C \otimes |0\rangle_E + \sqrt{\frac{p}{6}}|100\rangle_C \otimes |1\rangle_E \\
&+ \sqrt{\frac{p}{6}}|010\rangle_C \otimes |2\rangle_E + \sqrt{\frac{p}{6}}|001\rangle_C \otimes |3\rangle_E + \sqrt{\frac{p}{6}}|110\rangle_C \otimes |4\rangle_E \\
&+ \sqrt{\frac{p}{6}}|011\rangle_C \otimes |5\rangle_E + \sqrt{\frac{p}{6}}|101\rangle_C \otimes |6\rangle_E\Big) \\
&+ \frac{1}{\sqrt{2}}|1\rangle_{A'} \otimes \Big(\sqrt{1-p}|111\rangle_C \otimes |0\rangle_E + \sqrt{\frac{p}{6}}|011\rangle_C \otimes |1\rangle_E \\
&+ \sqrt{\frac{p}{6}}|101\rangle_C \otimes |2\rangle_E + \sqrt{\frac{p}{6}}|110\rangle_C \otimes |3\rangle_E + \sqrt{\frac{p}{6}}|001\rangle_C \otimes |4\rangle_E \\
&+ \sqrt{\frac{p}{6}}|100\rangle_C \otimes |5\rangle_E + \sqrt{\frac{p}{6}}|010\rangle_C \otimes |6\rangle_E\Big) \ .
\end{aligned}$$

Taking the partial trace over the system C results in

$$\begin{aligned}
tr_C(|\phi\rangle\langle\phi|_{A'CE}) &= \left(\frac{1}{2}|0\rangle\langle 0|_{A'} + \frac{1}{2}|1\rangle\langle 1|_{A'}\right) \otimes \Big((1-p)|0\rangle\langle 0|_E + \frac{p}{6}|1\rangle\langle 1|_E \\
&+ \frac{p}{6}|2\rangle\langle 2|_E + \frac{p}{6}|3\rangle\langle 3|_E + \frac{p}{6}|4\rangle\langle 4|_E + \frac{p}{6}|5\rangle\langle 5|_E + \frac{p}{6}|6\rangle\langle 6|_E\Big) \\
&+ \left(\frac{1}{2}|0\rangle\langle 1|_{A'} + \frac{1}{2}|1\rangle\langle 0|_{A'}\right) \otimes \Big(\frac{p}{6}|1\rangle\langle 5|_E + \frac{p}{6}|2\rangle\langle 6|_E \\
&+ \frac{p}{6}|3\rangle\langle 4|_E + \frac{p}{6}|4\rangle\langle 3|_E + \frac{p}{6}|5\rangle\langle 1|_E + \frac{p}{6}|6\rangle\langle 2|_E\Big) \ .
\end{aligned}$$

This state is not a product state between systems A' and E as was the state in the previous item. Consequently, error correction does not work for the three qubit bit flip code if two bit flips can occur.