FS 12
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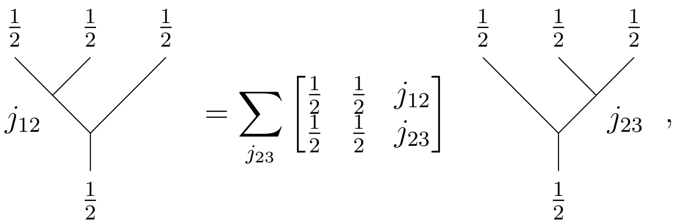
Advanced Topics in Quantum Information Theory
Exercise 7

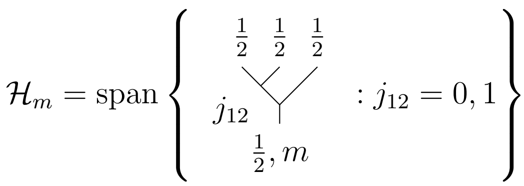
Exercise 7.1 Schur–Weyl Duality

The goal of this exercise is to compute the *F-matrix* elements in



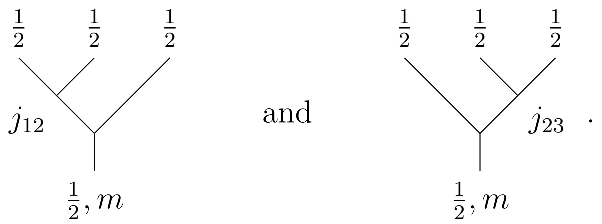
(1)

and thereby the action of the permutation group S_3 on the two-dimensional subspaces



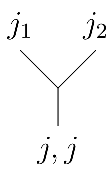
(with $m = \pm\frac{1}{2}$ arbitrary but fixed).

a) Compute the basis vectors

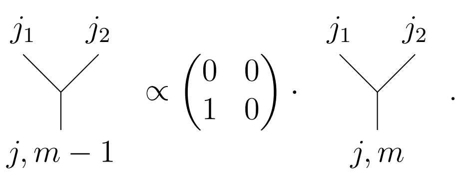


explicitely as vectors in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

Convention: The highest weight vector

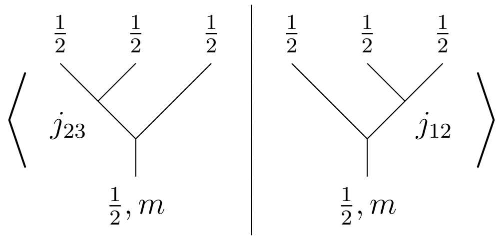


in the Clebsch–Gordan decomposition can be chosen as the unique unit vector with positive coefficient in front of $|j_1, j_1\rangle \otimes |j_2, j - j_2\rangle$, and the other weight vectors are obtained by applying lowering operators and normalizing,



b) Compute the F-matrix elements

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & j_{12} \\ \frac{1}{2} & \frac{1}{2} & j_{23} \end{bmatrix} =$$



c) Determine the action of the permutation $(2\ 3) \in S_3$ on each of the two-dimensional subspaces $\mathcal{H}_m$ $(m = \pm\frac{1}{2})$. Conclude that both representations are isomorphic to the representation of S_3 on

$$\mathcal{K} = \big\{(x, y, z) \in \mathbb{C}^3 : x + y + z = 0\big\}$$

by permuting coordinates.

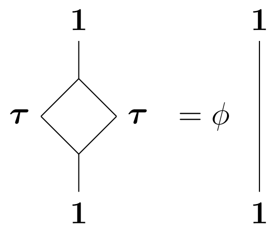
Exercise 7.2 Fibonacci Anyons

Knots and more general *links*, which are finite collections of closed curves in $\mathbb{R}^3$, can be represented by diagrams in the plane (Figure 1). It is well-known that two links are *isotopic*, i.e., one can be continuously deformed into the other, if and only if their link diagrams are related by a sequence of *Reidemeister moves* (Figure 2). The analog statement is true if the links are equipped with an orientation. However, checking this condition is in general a difficult problem. Similar to the case of distinguishing topological phases, one can approach this problem by finding *invariants*, i.e., functions which do not change under continuous deformation of a knot. A well-known such invariant is the *Jones polynomial* $V_{\vec{L}}(t)$ of an oriented link $\vec{L}$.
The goal of this exercise is to show that a quantum computer using Fibonacci anyons can be naturally used to evaluate the Jones polynomial at the fifth root of unity $t = e^{2\pi i/5}$. In particular, it can distinguish the trefoil knot from its mirror image.

For this, recall the Fibonacci model from the lecture: There are two anyon types, $\mathbf{1}$ and $\boldsymbol{\tau}$, subject to $\boldsymbol{\tau} \otimes \boldsymbol{\tau} = \mathbf{1} + \boldsymbol{\tau}$, and the braiding phases are given by

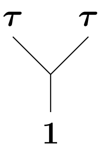
$$R_{\mathbf{1}}^{\boldsymbol{\tau},\boldsymbol{\tau}} = e^{4\pi i/5}, \quad R_{\boldsymbol{\tau}}^{\boldsymbol{\tau},\boldsymbol{\tau}} = e^{-3\pi i/5} = -e^{2\pi i/5}.$$

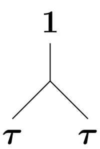
Moreover,



(2)

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio (ϕ is called the *quantum dimension* of $\boldsymbol{\tau}$).
Let us also introduce the following notation for creation and fusion of a pair of $\boldsymbol{\tau}$ anyons:

$\cup =$ 

, $\cap =$ 

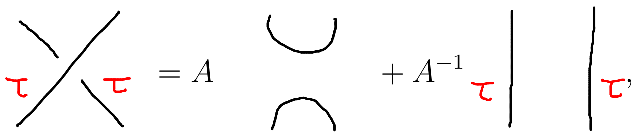
This makes sense since $\mathbf{1}$ is the trivial anyon satisfying $\mathbf{1} \otimes x = x \otimes \mathbf{1} = x$ (cf. the exercise class).

Now consider a braid B of an even number $2n$ of strands, and denote by $\overline{B}$ its *plat closure*, i.e., the link diagram formed from B by connecting adjacent strand endings (Figure 3). Using the notation fixed above, the link diagram $\overline{B}$ also defines a diagram in the Fibonacci model, which we shall denote by $\langle \overline{B} \rangle$. Since $\langle \overline{B} \rangle$ is a multiple of the identity morphism, it can be identified with a scalar. Let us call $\langle \overline{B} \rangle$ the *bracket* of the link $\overline{B}$.

a) Observe that the bracket is invariant under applying Reidemeister moves of type II and III.

b) Show that the bracket of a disjoint union of N circles is equal to ϕ^N.

c) Show that we have the following relation in the Fibonacci model:



where $A = e^{3\pi i/5}$. Therefore, by successively eliminate crossings and applying b), we can evaluate $\langle \overline{B} \rangle$ for any braid B.

d) Show that the effect of applying a Reidemeister move of type I to a diagram in the Fibonacci model amounts to multiplication by $-A^{\pm 3}$.

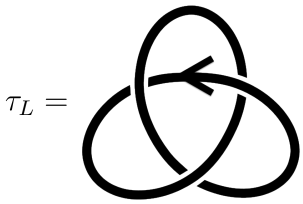
e) Conclude that

$$(-A^{-3})^{-w(\overline{B})} \langle \overline{B} \rangle$$

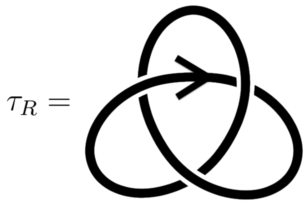
is an invariant of *oriented* knots and links. Here, $w(\overline{B})$ denotes the *writhe* of an oriented link diagram, which is given as the number of positive crossings minus the number of negative crossings (Figure 4). This invariant is (up to a factor ϕ) equal to the value of the *Jones polynomial* $V_{\overline{B}}(t)$ evaluated at $t = A^{-4} = e^{-2\pi i/5}$.

Why did we need to choose an orientation?

f) Evaluate the Jones polynomial of the left-handed and right-handed trefoil knots (Figure 1) at $t = e^{-2\pi i/5}$ and show that the two knots are not isotopic.

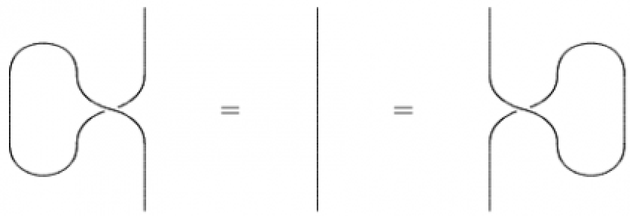


(a) The left-handed trefoil knot (with a choice of orientation.)

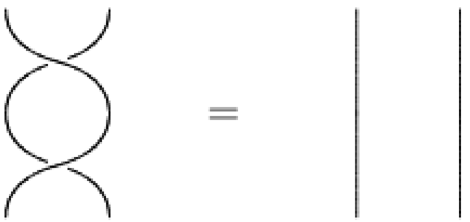


(b) Its mirror image, the right-handed trefoil knot.

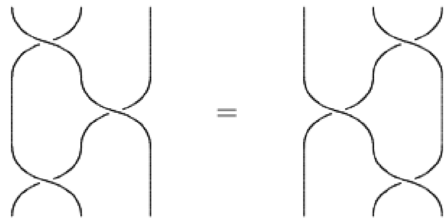
Figure 1: Trefoil knots (images from Wikipedia).



(a) Type I



(b) Type II



(c) Type III

Figure 2: The Reidemeister moves (images from Wikipedia).

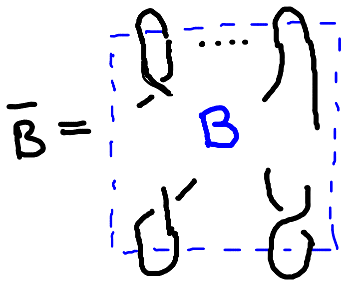


Figure 3: Plat closure of a braid B.

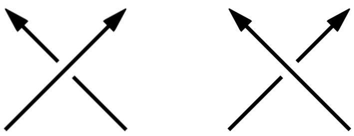


Figure 4: Positive and negative crossings in an oriented link diagram.