Advanced Topics in Quantum Information Theory Exercise 6

FS 12
Prof. M. Christandl
Prof. A. Imamoglu
Prof. R. Renner

The goal of this exercise is to show that, for a 2D spinless $p_x + ip_y$-superconductor modeled by the Hamiltonian

$$H = \int d^2\mathbf{r} \left(\psi^\dagger \left(-\psi^\dagger \frac{\Delta^2}{2m} - \mu \right) \psi + \frac{\Delta}{2} \left(e^{i\phi} \psi \left(\partial_x + i\partial_y \right) \psi + \text{h.c.} \right) \right),$$

there exists a topological invariant distinguishing the weak pairing phase from the strong pairing phase.
Here, $\psi^\dagger(\mathbf{r})$ creates a spinless fermion with effective mass m, the chemical potential is μ, and $\Delta \geq 0$, $\phi \in \mathbb{R}$ are assumed to be constant. Let us assume that the system is translation invariant.

a) Diagonalize the Hamiltonian by going to momentum space. That is, determine $\mathcal{H}(\mathbf{k}) = \begin{pmatrix} \epsilon(\mathbf{k}) & \tilde{\Delta}(\mathbf{k})^* \\ \tilde{\Delta}(\mathbf{k}) & -\epsilon(\mathbf{k}) \end{pmatrix}$ such that

$$H = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \mathbf{\Psi}^\dagger(\mathbf{k}) \mathcal{H}(\mathbf{k}) \mathbf{\Psi}(\mathbf{k}),$$

where $\mathbf{\Psi}^\dagger(\mathbf{k}) = (\psi^\dagger(\mathbf{k}), \psi(-\mathbf{k}))$.

b) Write $\mathcal{H}(\mathbf{k}) = \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli vector. Show that $\mathbf{h}(\mathbf{k}) \neq 0$ if $\mu \neq 0$.

c) Set $\hat{\mathbf{h}}(\mathbf{k}) = \mathbf{h}(\mathbf{k}) / \|\mathbf{h}(\mathbf{k})\|$. This defines a map from the 2D momentum space to the 2D unit sphere. Observe that as we vary $\mathbf{k}$ along a circle $\|\mathbf{k}\| \equiv$ const, $\hat{\mathbf{h}}(\mathbf{k})$ traces out a circle at constant height on the unit sphere.

d) Show that the *Chern number*

$$C = \int \frac{d^2\mathbf{k}}{4\pi} \left(\hat{\mathbf{h}} \cdot \left(\partial_{k_x} \hat{\mathbf{h}} \times \partial_{k_y} \hat{\mathbf{h}} \right) \right)$$

is an integer. In particular, it is invariant under smooth perturbations.

e) Show that $C \equiv 0$ in the (topologically trivial) *strong pairing phase* $\mu < 0$, while $C \equiv -1$ in the (topologically non-trivial) *weak pairing phase* $\mu > 0$.

Hint: This exercise follows the article *New directions in the pursuit of Majorana fermions in solid state systems* by J. Alicea, see `arXiv:1202.1293v1`.