

Advanced Topics in Quantum Information Theory

Exercise 2

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Exercise 2.1 Three qubit bit flip code

Let $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$, be an encoding of the qubit $\alpha|0\rangle + \beta|1\rangle$.

- Compute the eigenvalues and eigenvectors of the observables $Z_1Z_2 := Z \otimes Z \otimes \mathbb{I}$ and $Z_2Z_3 := \mathbb{I} \otimes Z \otimes Z$.
- Perform the measurement of the observable Z_1Z_2 followed by the observable Z_2Z_3 on the faulty state $X_1|\psi\rangle$ with $X_1 := X \otimes \mathbb{I} \otimes \mathbb{I}$. What are the corresponding outcomes, measurements probabilities and post-measurement states?
- Do the same calculations for the states $|\psi\rangle$, $X_2|\psi\rangle$ and $X_3|\psi\rangle$.
- How can a single bit-flip error in $|\psi\rangle$ be corrected by using the information obtained by the measurements of Z_1Z_2 and Z_2Z_3 ?

Exercise 2.2 Shor code

Let $|\psi\rangle$ be the nine qubit Shor-encoding of the qubit $\alpha|0\rangle + \beta|1\rangle$. Assume that $|\psi\rangle$ is exposed to a noise process which introduces a bit and a phase flip error on the fourth qubit yielding the faulty state $Z_4X_4|\psi\rangle$.

- Perform the measurement Z_4Z_5 followed by Z_5Z_6 on $Z_4X_4|\psi\rangle$. What are the corresponding outcomes, measurement probabilities, and post-measurement states? Infer from the measurement results where the bit flip operation has to be applied in order to correct one of the errors.
- Measure the observables $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$ on the bit-flip corrected state of part a.). What are the corresponding outcomes, measurements probabilities and post-measurement states? What can be inferred about the error(s) left in the state from the measurement results?
- Apply the operator $Z_4Z_5Z_6$ to the resulting state of the previous part. What is the final state?
- How would you correct the error $Z_iX_i|\psi\rangle$, where the position i of the error is not known?

Exercise 2.3 Coding and Decoupling

Let us now consider the three qubit bit flip code from a different perspective by considering the error process in the Choi-Jamiolkowski picture. The noise channel $\mathcal{E} : \mathcal{S}(\mathcal{H}_C) \rightarrow \mathcal{S}(\mathcal{H}_C)$ on the encoded state is then represented by the state

$$|\phi\rangle_{A'CE} := (\mathbb{I}_{A'} \otimes U_{\mathcal{E}})|\psi\rangle$$

with $U_{\mathcal{E}} : \mathcal{H}_C \rightarrow \mathcal{H}_C \otimes \mathcal{H}_E$ an isometric purification of \mathcal{E} (i.e. the Stinespring Dilation) and

$$|\psi\rangle := \frac{1}{\sqrt{2}}|0\rangle_{A'} \otimes |000\rangle_C + \frac{1}{\sqrt{2}}|1\rangle_{A'} \otimes |111\rangle_C .$$

- a.) The error model is that at most a single bit flip error occurs with probability p . Furthermore, assume that if a bit flip occurs, then all three qubits on the encoding space \mathcal{H}_C are affected equally often. Hence, the probability that a bit flip happens on the first qubit is $p/3$. Represent this error process by an isometry $U_{\mathcal{E}}$ and compute $|\phi\rangle_{A'CE}$ and $\rho_{A'E} := \text{tr}_C(|\phi\rangle\langle\phi|_{A'CE})$. Is it possible to determine where the error is and correct it?
- b.) Consider now an error model where zero, one or two bit flips can occur. Each error has equal probability $p/6$ and, hence, the probability that no error happens is $1 - p$. Represent this error process by an isometry $U_{\mathcal{E}}$ and compute that states $|\phi\rangle_{A'CE}$ and $\rho_{A'E} := \text{tr}_C(|\phi\rangle\langle\phi|_{A'CE})$. Is it possible to determine where the errors are and correct them? What can you say about the differences/similarities of the state $\rho_{A'E}$ here and in part a.)?