U 3.2

Helmholtz - Hodge decomposition

We want first to show that a vector field $\vec{V}$ can be decomposed in a sum of an irrotational part (vanishing curl) and a solenoidal part (vanishing divergence), i.e.:

$$\vec{V} = -\vec{\nabla}\phi + \vec{\nabla}\times\vec{A}, \qquad (1)$$

under the condition that $\vec{\nabla}\cdot\vec{V} =: S(\vec{r})$ and $\vec{\nabla}\times\vec{V} =: \vec{C}(\vec{r})$ vanish at infinity. (Helmholtz' theorem)

Rem: It easy to show that $\vec{\nabla}\varphi$ is irrotational:

In locale coordinates we have: $\left(\vec{\nabla}\times\vec{\nabla}\phi\right)_i = \varepsilon_{ikl}\dfrac{\partial}{\partial x^k}\left(\dfrac{\partial}{\partial x^l}\phi\right)$

$$= \frac{1}{2}\left(\varepsilon_{ikl}\frac{\partial^2\phi}{\partial x^k\partial x^l} + \varepsilon_{ilk}\frac{\partial^2\phi}{\partial x^l\partial x^k}\right)$$

$$= 0 \quad \text{since } \varepsilon_{ikl} = -\varepsilon_{ilk} \text{ and } \frac{\partial^2\phi}{\partial x^k\partial x^l} = \frac{\partial^2\phi}{\partial x^l\partial x^k}.$$

We also show easily that $\vec{\nabla}\times\vec{A}$ is solenoidal:

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = \partial_i\ \varepsilon_{ikl}\partial_k A_l = \varepsilon_{ikl}\,\partial_i\partial_k A_l$$

$$= \frac{1}{2}\left(\varepsilon_{ikl}\,\partial_i\partial_k A_l + \varepsilon_{kil}\,\partial_k\partial_i A_l\right)$$

$$= \frac{1}{2}\left(\varepsilon_{ikl}\,\partial_i\partial_k A_l - \varepsilon_{ikl}\,\partial_i\partial_k A_l\right) = 0$$

<u>Proof of the Helmholtz theorem</u> :

We can write for $\vec{r} \in \Omega \subset E^3$

$$\vec{V}(\vec{r}) = \int_\Omega \vec{V}(\vec{r}\,') \, \delta(\vec{r}-\vec{r}\,') \, d\vec{r}^{3\prime}$$

and with $\vec{\nabla}^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}\,')$

$$= -\frac{1}{4\pi}\int_\Omega \vec{V}(\vec{r}\,') \, \vec{\nabla}^2\left(\frac{1}{R}\right) d\vec{r}^{3\prime} \qquad (R = |\vec{r}-\vec{r}\,'|)$$

$$= -\frac{1}{4\pi} \nabla^2 \int_\Omega \frac{\vec{V}(\vec{r}\,')}{R} \, d\vec{r}^{3\prime}$$

Then with $\vec{\nabla}\times\vec{\nabla}\times\vec{a} = \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2\vec{a}$ we can write

$$\vec{V}(\vec{r}) = \frac{1}{4\pi} \nabla\times\nabla\times\int_\Omega \frac{\vec{V}(\vec{r}\,')}{R} \, d\vec{r}^{3\prime} - \frac{1}{4\pi}\nabla\nabla\cdot\int\frac{\vec{V}(\vec{r}\,')}{R} d\vec{r}^{3\prime} \quad (*)$$

In the second term we have

$$(I_2) \equiv -\frac{1}{4\pi}\nabla\cdot\int_\Omega \frac{\vec{V}(\vec{r}\,')}{R} d\vec{r}^{3\prime} = -\frac{1}{4\pi}\int_\Omega \vec{V}(\vec{r}\,') \, \nabla\left(\frac{1}{R}\right) d\vec{r}^{3\prime}$$

where $\vec{V}(\vec{r}\,')\,\nabla_r\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -\nabla_{r'}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\vec{V}(\vec{r}\,')$

$$= -\nabla_{r'}\left(\frac{\vec{V}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) + \frac{\vec{\nabla}_{r'}\cdot\vec{V}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

then

$$(I_2) = -\frac{1}{4\pi}\int_\Omega \frac{\nabla_{r'}\cdot\vec{V}(\vec{r}\,')}{R} d\vec{r}^{3\prime} + \frac{1}{4\pi}\int_{\partial\Omega} \frac{\vec{V}(\vec{r}\,')\cdot\hat{n}}{R} \, d^2r'$$

$$\equiv \phi(\vec{r}) \qquad (2)$$

(we used the Gauss theorem in the last term)

In the first term in (*) we have

$$\frac{1}{4\pi} \nabla \times \int_\Omega \frac{\vec{V}(\vec{r}\,')}{R}\, d^3r' = \frac{1}{4\pi} \int_\Omega \nabla_{\vec{r}}\left(\frac{1}{R}\right) \times \vec{V}(\vec{r}\,')\, d^3r'$$

$$= -\frac{1}{4\pi} \int_\Omega \nabla_{\vec{r}\,'}\left(\frac{1}{R}\right) \times \vec{V}(\vec{r}\,')\, d^3r'$$

$$= \frac{1}{4\pi} \int_\Omega \vec{V}(\vec{r}\,') \times \nabla_{\vec{r}\,'}\left(\frac{1}{R}\right) d^3r'$$

$$= \frac{1}{4\pi} \int_\Omega \frac{\nabla_{\vec{r}\,'} \times \vec{V}(\vec{r}\,')}{R}\, d^3r' - \frac{1}{4\pi} \int_\Omega \nabla_{\vec{r}\,'} \times \left(\frac{\vec{V}(\vec{r}\,')}{R}\right) d^3r'$$

$$= \frac{1}{4\pi} \int_\Omega \frac{\nabla_{\vec{r}\,'} \times \vec{V}(\vec{r}\,')}{R}\, d^3r' + \frac{1}{4\pi} \int_{\partial\Omega} \frac{\vec{V}(\vec{r}\,') \times \hat{n}}{R}\, d^2r' \qquad \text{(Stoke's theorem)}$$

$$\equiv \vec{A}(\vec{r}) \,. \qquad (3)$$

From (1), (2) and (3) we have

$$\vec{V}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) + \vec{\nabla} \times \vec{A}(\vec{r})$$

If we take $\Omega \to \mathbb{E}^3$ and $\vec{V}(\vec{r})$ is regular at infinity, then (2) and (3) reduce to

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int \frac{\nabla_{\vec{r}'} \cdot \vec{V}(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3r'$$

$$\vec{A}(\vec{r}) = \frac{1}{4\pi}\int \frac{\nabla_{\vec{r}'} \times \vec{V}(\vec{r}')}{R}\, d^3r'$$

For given $\vec{\nabla}\cdot\vec{V} =: S(\vec{r})$ and $\vec{\nabla}\times\vec{V} =: \vec{C}(\vec{r})$, we can show that $\vec{V}$ is uniquely defined (see appendix).

Appendix

We show that a vector field is uniquely defined through its divergence and its curl (on a simply connected region) and its normal component over the boundary.

Let be the vector field $\vec{V_1}$ defined on Ω such that

$$\begin{cases} \vec{\nabla} \cdot \vec{V_1} = s \\ \vec{\nabla} \times \vec{V_1} = \vec{C} \end{cases} \quad \text{with the normal component on } \partial\Omega : V_{1\perp}$$

Let us assume that the vector $\vec{V_2}$ also satisfies those equations and $V_{2\perp} = V_{1\perp}$ on $\partial\Omega$.
We can define the vector $\vec{W} = \vec{V_1} - \vec{V_2}$. Taking the divergence and the curl, we have:

$$\vec{\nabla} \cdot \vec{W} = \vec{\nabla} \cdot \vec{V_1} - \vec{\nabla} \cdot \vec{V_2} = s - s = 0 \qquad (1)$$

and

$$\vec{\nabla} \times \vec{W} = \vec{C} - \vec{C} = 0 \qquad (2)$$

From (2), it follows that $\vec{W}$ can be given through the gradient of a function: $\vec{W} = -\vec{\nabla}\phi$,

inserted in (1) we have

$$-\vec{\nabla} \cdot \vec{\nabla}\phi = 0 \Leftrightarrow \Delta\phi = 0 \quad \text{(Laplace's equation)}.$$

By the Green's theorem we have

$$\int_\Omega \vec{\nabla}\phi \cdot \vec{\nabla}\phi \, dv = -\int_\Omega \phi \overbrace{\vec{\nabla} \cdot \vec{\nabla}\phi}^{\overset{!}{=}0} \, dv + \int_{\partial\Omega} \phi \underbrace{\vec{\nabla}\phi \cdot \vec{ds}}_{W_\perp \, ds}$$

which gives, with $W_\perp = V_{1\perp} - V_{2\perp} = 0$,

$$\int_\Omega \vec{W} \cdot \vec{W} \, dv = \int_{\partial\Omega} \phi \, W_\perp \, ds = 0 .$$

Since $\vec{W}^2 = W^2 > 0$ we conclude that $\vec{W} = \vec{0}$!

$\Rightarrow$ $\vec{V}_1$ is uniquelly defined.