Üb 3.1

We have for $x \in M$, (M is an oriented Riemannian manifold),

$$d\,vol = \sqrt{|g(x)|}\; dx^1 \wedge \dots \wedge dx^n$$

and

$$d\,\widetilde{vol} = \sqrt{|g(\tilde{x})|}\; d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n$$

with the change of coordinates $\varphi(x) = \tilde{x}$ such that φ is an isometry, i.e. $\varphi^* g = g$.

We have

$$d\widetilde{vol} = \sqrt{|g(\varphi(x))|}\; d(\varphi^1(x)) \wedge \dots \wedge d(\varphi^n(x))$$

$$= \sqrt{|\varphi^* g(x)|}\; \frac{\partial \varphi^1(x)}{\partial x^{i_1}} \cdots \frac{\partial \varphi^n(x)}{\partial x^{i_n}}\, dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$= \sqrt{|g(x)|}\; \underbrace{\det\left(\frac{\partial \varphi(x)}{\partial x}\right)}_{}\; dx^1 \wedge \dots \wedge dx^n.$$

$\neq 0$ if $\varphi(x)$ are good local coordinates

Since M is oriented we can choose a basis of T^*M such that $d\,vol > 0$.

Then we have that $d\widetilde{vol} \neq 0 \;\; \forall \varphi$ (isometry).

$\Rightarrow$ $d\,vol$ is a smooth n-form that is everywhere non-zero, we call it a volume form.