Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Quantum Information Theory
Tips 7

FS 11
TA Lídia del Rio

Quick recap on maps

The maps we care about in quantum mechanics are TPCPMs (pages 40–45 of the script), trace preserving completely positive maps. We like them because they map density operators to density operators and thus can be used to describe the physical evolution of the state of a system. Formally, a map $\mathcal{E}_A : \mathcal{H}_A \mapsto \mathcal{H}_C$ is:

a) *trace preserving* if $\mathrm{Tr}(\rho_A) = \mathrm{Tr}(\mathcal{E}_A(\rho_A))$,

b) *positive* if $\mathcal{E}_A(\rho_A) \geq 0$ for $\rho_A \geq 0$,

c) *completely positive* if $(\mathcal{E}_A \otimes \mathcal{I}_B)(\rho_{AB}) \geq 0$ for $\rho_{AB} \geq 0$.

At first glance one may think that *b)* and *c)* are equivalent. However, if ρ_{AB} is entangled then applying a positive operator on $\mathcal{H}_A$ may result in a non-positive operator, as we will see in exercise 7.2.

Exercise 7.1 Quantum operations can only decrease distance

You have to prove that the trace distance between two quantum states,

$$\begin{aligned}\delta(\rho,\sigma) &= \frac{1}{2}\mathrm{Tr}\,|\rho-\sigma| \\ &= \max_P \mathrm{Tr}\,|P(\rho-\sigma)|\,,\end{aligned}$$

may only decrease after applying a TPM $\mathcal{E}$. In the second definition the maximum is taken over projectors. You will need to use the fact that for any two density operators ρ and σ, $\rho - \sigma = R - S$, with R, S positive operators with orthogonal support (with $\mathrm{Tr}R = \mathrm{Tr}S$). Also handy is the fact that if P is a projector, then $\mathrm{Tr}(PA) \leq \mathrm{Tr}(A)$ (think for instance of what a projector does to a matrix that is diagonalised in the projector's eigenbasis: it keeps only a few of the eigenvalues).
One example of a TPM is the partial trace. This exercise implies that if you have two bipartite states, ρ_{AB} and σ_{AB}, and dismiss system B, the distance between the two states will not increase.

Exercise 7.2 A sufficient entanglement criterion

We have seen that two systems may share a quantum state in many different ways, from product states, when local operations do not affect the other system, to a maximally entangled state, when applying an operation on one of the systems is equivalent to applying its transpose on the other.

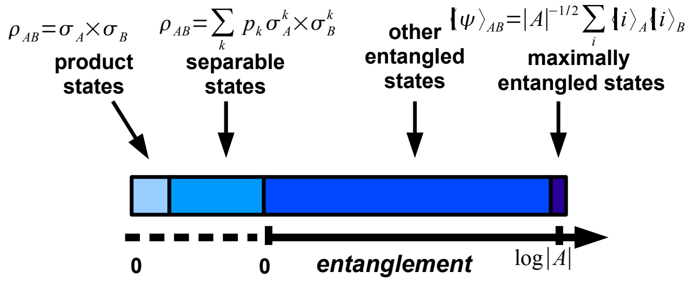

Figure 1: Some types of bipartite states of composed system $\mathcal{H}_A \otimes \mathcal{H}_B$. Here $\{p_k\}_k$ is a convex set (all p_k are non negative and sum up to one) and $\{|i\rangle_A\}_i$ and $\{|i\rangle_B\}_i$ are bases of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

Entangled states are useful in cryptography and computation, so it is important to find simple tests that quantify the entanglement of a shared state. Finding those tests, sometimes called *entanglement witnesses* is no easy task though—in fact it feeds a whole area of research. In this exercise we will tackle a very specific

and small bit of that field: distinguishing between separable states and entangled states, for low dimensions (apparently the problem is NP hard for systems with arbitrary dimensions).
Separable states have the form $\rho_{AB} = \sum_k p_k \sigma_A^k \otimes \sigma_B^k$, where $\{p_k\}_k$ is a classical probability distribution. This kind of states may be interpreted as "product state $\sigma_A^k \otimes \sigma_B^k$ was prepared with probability p_k". Just like in the case of product states, local operations on one of the systems do not change the reduced state of the other—there is no entanglement.
The test we will use is simple: we will apply the (local) transpose operation T on one of the systems. If the state we obtain is still a valid density operator (all eigenvalues are positive and sum up to one) then we had a separable state. If some of the eigenvalues of the new state turn out to be negative—and we are no more in the presence of a valid quantum state—then the original state was entangled,

$$(T_A \otimes \mathbb{1}_B)\rho_{AB} \begin{cases} \text{positive operator} & \Rightarrow \rho_{AB} \text{ separable,} \\ \text{non positive} & \Rightarrow \rho_{AB} \text{ entangled.} \end{cases}$$

To prove that this method works we should show the following:

I . If ρ_{AB} is separable then $(T_A \otimes \mathbb{1}_B)\rho_{AB}$ is positive:

a) for positive local operators Λ and separable states ρ_{AB}, $(\Lambda_A \otimes \mathbb{1}_B)\rho_{AB}$ is positive;

b) the transpose is positive.

II . If ρ_{AB} is an entangled state of a two- or three-dimentional system then $(T_A \otimes \mathbb{1}_B)\rho_{AB}$ is non positive (*in this exercise you will not have to prove this, only to apply it to an example.*)

As a curiosity you will also see that although the partial transpose is basis-dependent the eigenvalues of the final state are invariant under local basis transformations.
Note that the transpose acts as $T\sum_{i,j} a_{ij}|i\rangle\langle j| = \sum_{i,j} a_{ij}|j\rangle\langle i|$. In part *a*) you have to show that the transpose is positive and basis dependent. The latter should be trivial to get from the definition of transpose. As for positivity, you know that density operators (for instance) may be diagonalisable as $\rho = U^* D U$, where U is a unitary and and D a diagonal matrix. You know what diagonal matrices look like under transpose transformations, so try applying $T\rho$ and see what happens to the eigenvalues of ρ.
In part *b*) you are asked to prove *I. a*). Well, applying the operator directly and reasoning a little should do it. In part *c*) you have to aplly the method to a particular state, the Werner state. I suggest that before doing that you expand the state and check for what values of x it becomes separable. Then apply the partial transpose and compute the eigenvalues of he resulting operator. You should get $(1+x)/4$ (three times degenerate) and $(1-3x)/4$. The latter is positive only when $x < 1/3$, which has to be consistent with when the Werner state is separable.
To solve part *d*) you have to apply a local change of basis to $(T_A \otimes \mathbb{1}_B)\rho_{AB}$ and think about what happens to the eigenvalues of the resulting state. Note that local changes of basis act like $(U_A^* \otimes U_B^*)\rho_{AB}(U_A \otimes U_B)$.

Exercise 7.3 The Choi-Jamiołkowski Isomorphism

The CJ isomorphism gives a necessary and sufficient condition for CP: if

$$(\mathcal{E}_A \otimes \mathcal{I}_{A'})(|\Psi\rangle_{AA'}) \geq 0, \qquad |\Psi\rangle = \frac{1}{|A|}\sum_i |i\rangle_A |i\rangle_A,$$

then $\mathcal{E}_A$ is completely positive. The CJ isomorphism, τ, is the map from $\mathcal{E}_A$ to the state above. We will find an alternative way to derive τ that may clarify why it works.

First things first: notation. We use $|x\rangle\rangle$ to represent a non-normalised vector of a Hilbert space, just to keep in mind that it is not a quantum state. We do the same for non-normalised bras, $\langle\langle x|$. Now we represent operators as vectors. We can do this arbitrarly, as long as we know how to go back to the matrix representation. In this case we map the operators $C = \sum_{ij} c_{ij} |i\rangle_2 \langle j|_1$ as $|C\rangle\rangle = \sum_{ij} c_{ij} |i\rangle_2 |j\rangle_1$.

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}_{\mathcal{H}_1\to\mathcal{H}_2} \longrightarrow \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \end{pmatrix}_{\mathcal{H}_2\otimes\mathcal{H}_1}$$

Parts $a)$ and $b)$ are just to get tools that allows us to show why this would be interesting. It's just linear algebra. ☺

In part $a)$ we see how multiplying operators works in this picture. Here goes a little picture to keep track of all the different operators and Hilbert spaces.

$$\text{A: } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}_{\mathcal{H}_2\to\mathcal{H}_3} \quad \text{B: } \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}_{\mathcal{H}_1\to\mathcal{H}_4} \quad \text{C: } \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}_{\mathcal{H}_1\to\mathcal{H}_2}$$

$$(ACB^T)_{\mathcal{H}_4\to\mathcal{H}_3}\text{: } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}_{\mathcal{H}_2\to\mathcal{H}_3} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}_{\mathcal{H}_1\to\mathcal{H}_2} \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \end{pmatrix}_{\mathcal{H}_4\to\mathcal{H}_1}$$

In part $b)$ we have only two operators, A and B, both from $\mathcal{H}_1$ to $\mathcal{H}_2$, and you have to show that $\mathrm{Tr}_1(|A\rangle\rangle\langle\langle B|) = AB^*$. In case you are not tired of them yet, have another picture:

$$\text{A: } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{\mathcal{H}_1\to\mathcal{H}_2} \quad \text{B: } \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}_{\mathcal{H}_1\to\mathcal{H}_2} \quad \text{B*: } \begin{pmatrix} b^*_{11} & b^*_{21} \\ b^*_{12} & b^*_{22} \\ b^*_{13} & b^*_{23} \end{pmatrix}_{\mathcal{H}_2\to\mathcal{H}_1}$$

$$(|A\rangle\rangle\langle\langle B|)_{\mathcal{H}_2\otimes\mathcal{H}_1\to\mathcal{H}_2\otimes\mathcal{H}_1}\text{: } \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{pmatrix}_{\mathbb{C}\to\mathcal{H}_2\otimes\mathcal{H}_1} \begin{pmatrix} b^*_{11} & b^*_{12} & b^*_{13} & b^*_{21} & b^*_{22} & b^*_{23} \end{pmatrix}_{\mathcal{H}_2\otimes\mathcal{H}_1\to\mathbb{C}}$$

Note: *the pictures are really just to give an idea of what is happening. Please stick to the nice ket/bra notation when solving the exercises; if you try to draw the matrices you will need a lot of paper and patience.*
Now we will see how one can obtain the CJ isomorphism directly from the operator-sum representation of an operator. First, in part c), you use the results from b) and a) to show that we can obtain a matrix, τ, from that representation. You should get $\tau = \sum_k |E_k\rangle\rangle\langle\langle E_k|$.
For part c) we have to prove that the τ we obtained is the same from the CJ isomorphism. Hint: show that $|\Psi\rangle = |\mathbb{1}\rangle\rangle$.
Finally, check how to get the TP and CP conditions of a map directly from τ. The script may help.
Another note: sorry for the notation mess. In the script τ is the CJ isomorphism itself, while in this exercise τ is the outcome of that isomorphism.