Quantum Information Theory
Tips 5

FS 11
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Some properties of trace before we get started

Definition: $\text{Tr}(A) = \sum_x \lambda_x$, $\{\lambda_x\}_x$ eigenvalues of A.

Linearity: $\text{Tr}(\alpha A + \beta B) = \alpha \text{Tr}(A) + \beta \text{Tr}(B)$.

Chain-*ity*: $\text{Tr}(\rho_{AB}) = \text{Tr}(\text{Tr}_B \rho_{AB}) = \text{Tr}(\text{Tr}_A(\rho_{AB}))$, $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$.

This too: $\text{Tr}(f(A)) = \sum_x f(\lambda_x)$, $\{\lambda_x\}_x$ eigenvalues of A, f complex function. Examples:

$$\text{Tr}(|A|) = \sum_x |\lambda_x|; \qquad \text{Tr}(\sqrt{A}) = \sum_x \sqrt{\lambda_x}$$

Exercise 5.1 Measurements on bipartite systems

All you need to know to solve this exercise are the properties of trace given above.
That and that measurements may be represented by observables, such as $O = \sum_y y P_y$, where $\{y\}_y$ are the outcomes of the measurement (see the FAQ section of the website).
What do the projectors $\{P_y\}_y$ look like? Well, in the case of a non-degenerate eigenvalue y, $P_y = |y\rangle\langle y|$, where $|y\rangle$ is the eigenstate associated with y. If there are several eigenvectors $\{|y^\alpha\rangle\}_\alpha$ for a single eigenvalue y we have $P_y = \sum_\alpha |y^\alpha\rangle\langle y^\alpha|$.
The probability of getting the outcome y in a measurement on state ρ represented by the observable O is given by $\text{Pr}_{O,\rho}(y) = \text{Tr}(P_y \rho)$.

Exercise 5.2 Distinguishing two quantum states

This exercise is similar to the one where we knew the behaviour of two dice and after throwing one of them we had to guess which one that was. The difference now is that we are dealing with quantum states with density matrices instead of classical probability distributions. Back then our strategy was to choose the die that was more likely to outcome the event we measured. Here we will do the same. However, we have to decide in which basis to perform the measurement. In the case of two density matrices that share the same eigenstates it is easy – we can simply perform a measurement in that basis. Take the example of

$$\rho = 0.7\ |0\rangle\langle 0| + 0.3\ |1\rangle\langle 1|, \qquad \sigma = 0.1\ |0\rangle\langle 0| + 0.9\ |1\rangle\langle 1|.$$

The probability of measuring eg. $|0\rangle$ when looking at state ρ is given by $\text{Tr}(|0\rangle\langle 0|\ \rho)$. We could perform a measurement in their common eigenbasis $\{|0\rangle, |1\rangle\}$. If we obtained 0 we should guess we had measured state ρ and vice-versa. Things get more complicated when the two states are not diagonalised in the same basis. Consider for instance

$$\rho = 0.7\ |0\rangle\langle 0| + 0.3\ |1\rangle\langle 1|, \qquad \sigma = \frac{\mathbb{1}}{2}, \quad \tau = 0.7\ |+\rangle\langle +| + 0.3.\ |-\rangle\langle -|$$

Let us see what happens if we perform a measurement in three different bases. The probabilities of obtaining the different outcomes for these states are

outcome:	$\|0\rangle$	$\|1\rangle$	$\|+\rangle$	$\|-\rangle$	$\|\circlearrowright\rangle$	$\|\circlearrowleft\rangle$
ρ	0.7	0.3	0.5	0.5	0.5	0.5
σ	0.5	0.5	0.5	0.5	0.5	0.5
τ	0.5	0.5	0.7	0.3	0.5	0.5

Even though $\{|+\rangle, |-\rangle\}$ is an eigenbasis of σ, a measurement in this basis would not help distinguish σ from ρ. A measurement in basis $\{|\circlearrowright\rangle, |\circlearrowleft\rangle\}$ would be even worse: all three states have the same measurement

statistics in this basis. And although the statistics for σ and ρ are different in the computational basis, one may wonder if there is a better choice of basis where they're even more apart.
In general, we are looking for a measurement O that maximises our probability of distinguishing two states, ρ and σ. Let us see if we can find some correlation with the classical case. For each state (say ρ) the probabilities of measuring any of the eigenvalues $\{y\}_y$ of the operator $O = \sum_y y P_y$ that represents the measurement define a classical probability distribution $\Pr_{O,\rho}(y) = \mathrm{Tr}(P_y \rho)$. Naturally, if we obtain y after measuring our unknown state and obtain we should say it was ρ if $\Pr_{O,\rho}(y) \geq \Pr_{O,\sigma}(y)$ and vice-versa. We know that using this strategy the probability of guessing correctly is proportional to the trace distance of those two probability distributions, so we want to maximise it. In particular, we want to maximise

$$\begin{aligned} 2\Pr_{\checkmark} &= \sum_{y \in G} \Pr_{O,\rho}(y) + \sum_{y \in \bar{G}} \Pr_{O,\sigma}(y), && G = \{y : \Pr_{O,\rho}(y) \geq \Pr_{O,\sigma}(y)\} \\ &= \sum_{y \in G} \mathrm{Tr}(P_y \rho) + \sum_{y \in \bar{G}} \mathrm{Tr}(P_y \sigma), && G = \{y : \mathrm{Tr}(P_y \rho) \geq \mathrm{Tr}(P_y \sigma)\} \\ &= \mathrm{Tr}\left(\left[\sum_{y \in G} P_y\right] \rho\right) + \mathrm{Tr}\left(\left[\sum_{y \in \bar{G}} P_y\right] \sigma\right), && G = \{y : \mathrm{Tr}(P_y(\rho - \sigma)) \geq 0\}. \end{aligned}$$

Notice that if all the P_y are projectors then $P_G := \sum_{y \in G} P_y$ and $P_{\bar{G}} := \sum_{y \in \bar{G}} P_y$ are projectors too.
Show that this quantity is maximised if we choose the P_y to be projectors in the eigenbasis $\{y\}_y$ of $\rho - \sigma$.
In another example, suppose that $\rho - \sigma$ looks like

$$\rho - \sigma = \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & -0.3 \end{pmatrix}.$$

when diagonalised. Then the projectors, in the same basis, are given by

$$P_G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P_{\bar{G}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This solution corresponds to the following strategy. We measure our state (ρ or σ) in the eigenbasis of $\rho - \sigma$. If we obtain a state that corresponds to a positive eigenvalue of $\rho - \sigma$ (ie. $y \in G$) then it is more likely that we have measured ρ. If we get a negative eigenvalue of $\rho - \sigma$ (ie. $y \in \bar{G}$) we should say the state was σ.
In the particular case where the two density operators share the same eigenbasis this corresponds to following the classical strategy after measuring the state in their common eigenbasis.
For part *b*) try using $P = P_G$ and play with properties of trace until you get the desired result.
You can also use Lemma 4.3.5 (page 35 of the script).

Exercise 5.3 Fidelity and Uhlmann's Theorem

Fidelity, like the trace distance, is a measure of how similar two states are. It is given by

$$F(\rho, \sigma) = ||\sqrt{\rho}\sqrt{\sigma}||_1,$$

where to calculate $\sqrt{\rho}$ one has to diagonalise ρ, apply the square root to its eigenvalues and bring the resulting diagonal matrix back to the original basis,

$$D = U^* \rho U = \sum_i \lambda_i |i\rangle\langle i| \quad \Rightarrow \quad D' = \sum_i \sqrt{\lambda_i} |i\rangle\langle i| \quad \Rightarrow \quad \sqrt{\rho} = U D' U^*,$$

and $||A||_1 = \mathrm{Tr}(\sqrt{A^* A})$ (in particular $||\rho|| = 1$ for density operators). For pure states, the fidelity becomes simply

$$F(|\psi\rangle|\phi\rangle) = |\langle\psi|\phi\rangle|,$$

which is much easier to compute.
Uhlmann's theorem gives us a way to calculate the fidelity between two states using *purifications* of them, so that all we need to know is to compute the fidelity on all possible purifications of both states and maximise it,

$$F(\rho,\sigma) = \max_{|\psi\rangle,|\phi\rangle} |\langle\psi|\phi\rangle| \,,$$

which is easier than it sounds.
Here $|\psi\rangle$ and $|\phi\rangle$ are purifications of ρ and σ respectively. The original states ρ and σ belong to space $\mathcal{H}_A$ and the purification system, $\mathcal{H}_B$, is just a copy of $\mathcal{H}_A$. Also, since all purifications are equivalent (up to a unitary), you can fix one of the purifications and maximise only over the other.
All you need to know about fidelity and the Uhlmann's theorem is on pages 37–40 of the script. The idea of the exercise is not to prove the theorem (that is done in the script) but rather to test some particular cases without applying the theorem directly so that you get a feeling of how it works.
First, in part *a*), you are given a state and have to show it is a purification of ρ. The state is

$$|\psi\rangle = (\sqrt{\rho} \otimes U_B)\,|\gamma\rangle,$$

where $|\gamma\rangle$ is the fully entangled state of $\mathcal{H}_A \otimes \mathcal{H}_B$ (up to a normalisation factor) and U_B an arbitrary unitary on $\mathcal{H}_B$. All you need to do is to check that $|\psi\rangle$ is normalised and that you recover ρ once you trace out system $\mathcal{H}_B$.
In part *b*) you are asked to work out why all purifications of ρ can be written in that form. Use the fact that all purifications are equivalent up to a unitary transformation on the purification system.
Now comes the first example. You are given the fully mixed state and a classical state,

$$\sigma = \frac{\mathbb{1}_A}{2}, \qquad \rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|,$$

and have to compute the fidelity between them using purifications. Since the states are already diagonalised in the same basis, I suggest that before doing it you calculate the fidelity directly. You should obtain

$$F(\rho,\sigma) = \frac{1}{\sqrt{2}}(\sqrt{p} + \sqrt{1-p}).$$

Now we will do it with the purifications. Since you are allowed to fix one of the purifications I suggest you use $U_B = \mathbb{1}_B$ in the purification of σ, so that $|\phi\rangle$ is the (normalised) fully entangled state,

$$|\psi\rangle = (\sqrt{\rho} \otimes U_B)\,|\gamma\rangle, \qquad |\phi\rangle = \frac{1}{\sqrt{2}}|\gamma\rangle,$$

with $|\gamma\rangle = |00\rangle + |11\rangle$. You have to calculate $\langle\psi|\psi\rangle$, obtaining $\frac{1}{\sqrt{2}}|\mathrm{Tr}\left(\sqrt{\rho}{U_B}^T\right)|$, where A^T is the transpose. Now you have to choose the unitary U_B that maximises that quantity. Hint: it should be one that recovers the previous result for the fidelity. Lemma 4.12 may help (page 28 of the script).
In the second example you have to compute the fidelity between any pure state $\rho = |\psi\rangle\langle\psi|$ and the fully mixed state $\sigma = \mathbb{1}_n/n$ for arbitrary n-dimensional spaces. You do not need to use purifications but merely the original definition of fidelity. Hint: what does $\sqrt{\rho}$ look like for pure states? In the end you should get $F(\rho,\sigma) = 1/\sqrt{n}$.