Quantum Information Theory
Tips 4

FS 10
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Exercise 4.1 Bloch sphere

The Bloch sphere is a little instrument to help us visualising the effects of quantum operations in qubit states. Here we will see how to represent states in the three-dimensional ball. You are given the lovely formula

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) \tag{1}$$

and in part *a*) you only have to apply it to get a feeling for the representation of states in the Bloch sphere: you will see what we mean by "rotating" a basis, and how the purity of a state relates to its position inside the ball. Just remember that the Pauli matrices and identity matrix are represented in basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. For instance the matricial representations of pure states $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$ are

$$|\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad |\downarrow\rangle\langle\downarrow| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Part *b*) is also fairly direct. You have to check that all reasonable Bloch vectors corresponds to valid density operators. Just apply Eq. 1 and prove those properties.
Then in part *c*) you have to prove the converse, ie. that all two-level density operators can be represented as Eq. 1 proposes. For that I suggest that you expand a general density operator ρ in the basis $\mathcal{B}$ given. Remember that you can always expand an operator A in an orthonormal basis $\{e_i\}_i$ as

$$A = \sum_i (A, e_i) e_i,$$

where the inner product (A, B) is defined as $\mathrm{Tr}(A^* B)$. Do not forget that $\mathcal{B}$ is not an orthonormal basis: you have to normalise it first. You should obtain something like

$$\rho = \frac{1}{2}\left[(\rho, \mathbb{1})\mathbb{1} + \sum_{i=x,y,z} r_i \sigma_i\right].$$

Given that ρ is a density operator, what is the value of $(\rho, \mathbb{1})$?
All you need for part *d*) is to know what $\mathrm{Tr}(\rho^2)$ is like for pure states, and relate that to $\vec{r}$.

Exercise 4.2 Partial trace

Check my notes "mixed states and partial trace" for a long step-by-step introduction to partial trace. For formal definitions check pages 25-26 of the script.
Here we will prove that the partial trace of a density matrix is still a density matrix.
I suggest that you begin by expanding ρ_{AB} in some basis $\{|a_i\rangle \otimes |b_j\rangle\}_{i,j}$ in $\mathcal{H}_A \otimes \mathcal{H}_B$,

$$\begin{aligned}\rho_{AB} &= \sum_{i,j} \sum_{k,l} c_{ij}^{kl} (|a_i\rangle \otimes |b_j\rangle)(\langle a_k| \otimes \langle b_l|) \\ &= \sum_{i,k} \sum_{j,l} c_{ij}^{kl} |a_i\rangle\langle a_k| \otimes |b_j\rangle\langle b_l|.\end{aligned}$$

Verify that the reduced density matrix given by the partial trace over system $\mathcal{H}_B$ is given by

$$\rho_A = \sum_{i,k} \sum_j c_{ij}^{kj} |a_i\rangle\langle a_k|.$$

Now you are ready to solve part a). Hermicity and normalisation should be direct. I can give you a hint for positivity. Saying that the original density operator is semi-positive definite means that $\langle\psi|\rho_{AB}|\psi\rangle \geq 0$ for any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Choose $|\psi_j\rangle = |\phi\rangle \otimes |b_j\rangle$ with an arbitrary state $|\phi\rangle \in \mathcal{H}_A$ and prove that $\sum_j \langle\psi_j|\rho_{AB}|\psi_j\rangle \geq 0$ implies that ρ_A is positive.
Part b) is direct application of the partial trace. Check that although the original state is pure you obtain a fully mixed state when you trace out one of the systems.
In part c) we will treat the classical counterpart of the partial trace — marginal distributions. We have a joint probability distribution $P_{XY} = \{P_{XY}(x,y)\}_{x,y}$. You know that the marginal distribution is given by $P_X = \left\{P_X(x) = \sum_y P_{XY}(x,y)\right\}_x$, and proving positivity and normalisation should not be a problem for you. Again, applying that to the given probability distribution could not be easier.
Now you have to represent the joint probability distribution as a quantum state. Given a distribution P_X we can always represent it as a quantum state in a Hilbert space with the same dimension as the alphabet of the probability distribution:

$$\rho_{P_X} = \sum_x P_X(x)|x\rangle\langle x|$$

for some basis $\{x\}_x$. In the case of a joint distribution this becomes a state in a composed space $\mathcal{H}_X \otimes \mathcal{H}_Y$,

$$\rho_{P_XY} = \sum_{x,y} P_{XY}(x)|x\rangle\langle x| \otimes |y\rangle\langle y|.$$

Apply that to the probability distribution given in part $c.2$) and then calculate the partial trace of that state. Check that although the reduced states are the same in both cases, the original bipartite state is very different.

Exercise 4.3 Purification

Purification is explained in detail in the script (pages 32–33). In a nutshell for every mixed state $\rho_A \in \mathcal{H}A$ it is possible to find a pure state $|\psi\rangle$ in a larger system $\mathcal{H}_A \otimes \mathcal{H}_B$ such that when we trace out the purification system $\mathcal{H}B$ we recover the original state: $\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi|$.
In part a) we are given a formula for purification and have to check that it actually works. First step: any density operator may be expanded in its eigenbasis (spectral decomposition, pages 26–27 of the script) as

$$\rho_A = \sum_x \lambda_x |a_x\rangle\langle a_x|.$$

We expand the operator like that and then build a pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$ as

$$|\psi\rangle = \sum_x \sqrt{\lambda_x}|a_x\rangle_A \otimes |b_x\rangle_B,$$

where $\{b_x\}_x$ forms an orthonormal basis of $\mathcal{H}_B$. Note that this implies that the dimension of the purification space $\mathcal{H}_B$ is the same as the dimension of the original space $\mathcal{H}_A$. Here you only have to check that $|\psi\rangle$ is indeed a purification of ρ_A, ie. that $\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi|$.
In part b) you have to prove that any two purifications of the form given above are equivalent up to a unitary operation. for instance suppose you had a purification $|\psi'\rangle = \sum_x \sqrt{\lambda_x}|a_x\rangle_A \otimes |b'_x\rangle_B$ that was performed using another basis $\{b'_x\}_x$ of $\mathcal{H}_B$. We have to show that $|\psi\rangle = U|\psi'\rangle$ for some unitary operator U. If look careful the only real difference between $|\psi\rangle$ and $|\psi'\rangle$ is the basis in which they are expressed in system $\mathcal{H}_B$, so a change of basis operation should move us from one to the other. Check that $U = \mathbb{1}_A \otimes \sum_x |b_x\rangle\langle b'_x|$ is unitary ($UU^* = U^*U = \mathbb{1}_B$) and actually does the job here.
Let us now approach part c) of the exercise. Suppose you want to create a certain mixed state ρ'. It is relatively easy to create pure states because you know exactly what the state should be like — things like a spin up or a bunch of photons with a certain polarisation — but mixed states are more tricky as they are states about which we do not have full information, i.e. we are not sure about their exact states. One way to do it is to diagonalise $\rho' = \sum_z \alpha_z |z\rangle\langle z|$ and then get a machine that produces the pure state $|z\rangle$ with probability α_z. Of course you need to be sure that the machine is genuinely random and that you have no

access to the information "which state has been created". The first condition in particular is hard to achieve classically, and one may think that there has to be a neater way to do it, maybe using quantum mechanics. Well, there is, and it involves purification.

The idea is that you prepare a special pure state in a bipartite system (easy) and then perform a measurement in a part of that system (also easy). What is left is a mixed state in the other subsystem, and you know which mixed state it is according to the result of your measurement. What we will now see is what pure state and measurements we should prepare for a desired set of mixed states $\{\rho_A^x\}_x$.

The first step is to write down a mixed state that is a convex combination of $\{\rho_A^x\}_x$, $\rho_A = \sum_x \lambda_x \rho_A^x$ (ie. the state ρ_A is decomposed in the possibly mixed states $\{\rho_A^x\}_x$). We will purify ρ_A, but instead of first mixing the components $\{\rho_A^x\}_x$ and then purify the resulting state ρ_A, we will do it the other way around, purifying the components, then mixing the resulting pure states and finally purifying the state we get from there. We will see that the two processes are equivalent.

We can decompose every $\rho_A^x \in \mathcal{H}_A$ in its eigenbasis $\left\{|a_y^x\rangle\right\}_y$ as $\rho_A^x = \sum_y \alpha_y^x |a_y^x\rangle\langle a_y^x|$ and then purify it as usual using the extra Hilbert space $\mathcal{H}_C$,

$$|\phi^x\rangle = \sum_y \sqrt{\alpha_y^x} |a_y^x\rangle_A \otimes |c_y^x\rangle_C.$$

We have now a set of pure states $\{|\phi^x\rangle\}_x$, and we will combine them to make a mixed state using the coefficients $\{\lambda_x\}_x$ used to make ρ_A. The state we will obtain belongs to the composed space $\mathcal{H}_A \otimes \mathcal{H}_C$,

$$\rho_{AC} = \sum_x \lambda_x |\phi^x\rangle\langle\phi^x|.$$

Now we purify this state using a purifying system $\mathcal{H}_D$,

$$|\phi\rangle = \sum_x \sqrt{\lambda_x} |\phi^x\rangle_{AC} \otimes |d_x\rangle_D.$$

This state lives in $\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ and you will see that $|\phi\rangle$ is a purification of ρ_A in system $\mathcal{H}_B = \mathcal{H}_C \otimes \mathcal{H}_D$. For this you have to check that $\mathrm{Tr}_B(|\phi\rangle\langle\phi|) = \rho_A$.

So now we have a pure state that is a purification of ρ_A. The next step is to choose a measurement in the purifying system $\mathcal{H}_B$ such that after measuring and tracing $\mathcal{H}_B$ out we obtain a mixed state from the set $\{\rho_A^x\}_x$, as we wanted. Remember what we saw last week about how measurements can be represented by a set of operators out of which one is chosen at random? You may want to test the following measurements:

$$M_B = \left\{M_B^x = \mathbb{1}_C \otimes |d_x\rangle\langle d_x|_D\right\}_x.$$

In particular, check that $\rho_A^x = \frac{\mathrm{Tr}_B[|\Phi\rangle\langle\Phi|(\mathbb{1}_A \otimes M_B^x)]}{\lambda_x}$ and $\lambda_x = \mathrm{Tr}\left[|\Phi\rangle\langle\Phi|(\mathbb{1}_A \otimes M_B^x)\right]$.

Summary of the mixed state recipe: you have a set of mixed states $\{\rho_A^x\}_x$ you would like to create (or alternatively you have one mixed state and you would like to obtain a certain decomposition). Mix the states in a convex combination ρ_A with coefficients $\{\lambda_x\}_x$. Go back to your initial set of states. Purify them. Now mix the purified states just like you did with the mixed ones before. Purify that global mixed state. Now apply a measurement in the purifying systems that measures the element of basis of this last purifying system and acts as the identity in the first one. Trace out both the purifying systems (the state had collapsed there anyway). Now with probability λ_x you measured $|d_x\rangle$ in the last purifying system and are left with ρ_A^x. You always know which state you have because you know the outcome of your measurement. Simple, right?