# Quantum Information Theory
# Series 11



## Exercise 11.1 Entanglement and Teleportation

**Double points for this exercise! Solving two parts gives you 2 points, and you get 4 if you solve all.**

Imagine that Alice ($A$) has a pure state $|\psi\rangle_S$ of a system $S$ in her lab. She wants to send that state to Bob, that lives, of course, on the Moon, but she does not trust the postwoman Eve to carry it there personally. We will see that if Alice and Bob share an entangled state, Alice can "teleport" the state $|\psi\rangle$ to the system $B$ that Bob controls.
Formally, we have three systems $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. In this exercise we will assume all three are qubits. The initial state is

$$|\psi\rangle_S \otimes \frac{1}{\sqrt{2}} \left( |0_A 0_B\rangle + |1_A 1_B\rangle \right), \tag{1}$$

i.e. $S$ is decoupled from $A$ and $B$ and these two are fully entangled in a Bell state. We may write $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

*a*) In a first step, Alice will measure systems $S$ and $A$ jointly in the Bell basis,

$$\left\{ \begin{array}{ll} \frac{1}{\sqrt{2}} \left( |0_S 0_A\rangle + |1_S 1_A\rangle \right), & \frac{1}{\sqrt{2}} \left( |0_S 0_A\rangle - |1_S 1_A\rangle \right), \\ \frac{1}{\sqrt{2}} \left( |0_S 1_A\rangle + |1_S 0_A\rangle \right), & \frac{1}{\sqrt{2}} \left( |0_S 1_A\rangle - |1_S 0_A\rangle \right) \end{array} \right\}. \tag{2}$$

Then Alice communicates (classicaly) the result of her measurement to Bob. What is the reduced state of Bob's system ($B$) for each of the possible outcomes?

*b*) Depending on the outcome of the measurement by Alice, Bob may have to perform certain unitary operations on his qubit so that he recovers $|\psi\rangle$. Which operations are these?

*c*) Suppose that Alice does not manage to tell Bob the outcome of her measurement. Show that in this case he does not have any information about his reduced state and therefore does not know which operation to apply in order to obtain $|\psi\rangle$.

*d*) Show that this method of quantum teleportation also works for mixed states $\rho_S$.

*e*) There is no reason why the state $\rho_S$ cannot be entangled with some other system that Alice and Bob do not control. Consider a purification of $\rho_S$ on a reference system $R$,

$$\rho_S = \mathrm{Tr}_R |\phi\rangle\langle\phi|_{SR}. \tag{3}$$

Show that if you apply the quantum teleportation protocol on $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, not touching the reference system, the final state on $\mathcal{H}_B \otimes \mathcal{H}_R$ is $|\phi\rangle$.

This implies that quantum teleportation preserves entanglement — it simply transfers it from $S$ and $R$ to $B$ and $R$.

## Exercise 11.2 One-time Pad

Consider three random variables: a message $M$, a secret key $K$ and a ciphertext $C$. We want to encode $M$ as a ciphertext $C$ using $K$ with perfect secrecy, so that no one can guess the message from the cipher: $I(C : M) = 0$.
After the transmission, we want to be able to decode the ciphertext: someone that knows the key and the cipher should be able to obtain the message perfectly, i.e. $H(M|C, K) = 0$.
Show that this is only possible if the key contains at least as much randomness as the message, namely $H(K) \geq H(M)$. Give an optimal algorithm for encoding and decoding.

**Exercise 11.3 Extra points!**

**These are optional exercises, just in case you are struggling to get the Testat.**

*a*) **(0.5 points)** Show that to have $\begin{matrix} \overset{m}{\rightarrow} \\ \underset{\infty}{\rightsquigarrow} \end{matrix} \geq \begin{matrix} \overset{n}{\rightarrow} \\ \underset{\infty}{\rightsquigarrow} \end{matrix}$ we need $m \geq n$.

*b*) **(1 point)** Make a nice LaTeX package to draw resource inequalities.

For instance, the lines of $\overset{n}{\rightsquigarrow}$ and $\underset{n}{\rightsquigarrow}$ should both look more or less like the latter. They should also have the sime size, be at the same height and extend in case of $\overset{\text{long text here}}{\rightsquigarrow}$, like this: $\xrightarrow{\text{long text here}}$.

The macros for resource inequalities used in the slides are available here:
`http://www.itp.phys.ethz.ch/education/lectures_fs11/qit/ugly`

*c*) **(0.5 to 3 points, depending on how well you do it)**

Pick one of the "talk of the week" and say something interesting about it. Contact us if you have questions.