Quantum Information Theory
Series 8

Exercise 8.1 Measurements as unitary evolutions

Consider a measurement on a system $\mathcal{H}_A$, whose output is in $\mathcal{H}_B$ that is described by the observable $O = \sum_{x\in\mathcal{X}} xP_x$, where $\{P_x\}_x$ are projectors. Suppose we want to apply the measurement to the state ρ_A. We can represent the measurement as a unitary evolution with output on a larger system, $\mathcal{H}_B \otimes \mathcal{H}_R$, followed by a partial trace over R.

a) Show that $\mathcal{E}(\rho_A)$ can be written as unitary followed by a partial trace over R. This task can be broken down into the following steps:

i) What is the operator-sum representation of the measurement of the operator O?

ii) If we write the projectors as $P_x = \sum_i |\phi_i^x\rangle\langle\phi_i^x|$, what is the Choi-Jamiołkowski matrix?

iii) Give an expression for a purification of the Choi-Jamiołkowski matrix. Note that since the CJ matrix is positive semi-definite and hermitian, you can treat it like an unnormalized density operator.

iv) Apply the inverse of the CJ isomorphism to the purified state in (iii), and show that it is of the form $U\rho_A U^*$, where U is a unitary. The inverse CJ isomorphism is the map that takes a state $\rho_{A'BR}$ as input, and outputs a map $\mathcal{F}$. Specifically:

$$\mathcal{F}(\rho_A) = |\mathcal{H}_A|\mathrm{Tr}_{A'}\left(\left(\sum_{i,j} |i\rangle_{A'}\langle j|_A\rho_A|i\rangle_A\langle j|_{A'}\right)\otimes \mathbb{1}_{BR}\cdot\rho_{A'BR}\right),$$

where $\{|i\rangle\}_i$ is an orthonormal basis for A and A' (similarly for $\{|j\rangle\}_j$), and $\rho_{A'BR}$ is the CJ matrix purified.

v) Finally, show that $\mathrm{Tr}_R(\mathcal{F}(\rho_A))$ has the same output as the measurement in (i).

b) Give an explicit expression for the map $\mathcal{E}$ for two different measurement on a qubit state described by the POVMs:

1. $\mathcal{M}_1 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.
2. $\mathcal{M}_2 = \{p|0\rangle\langle 0|, p|1\rangle\langle 1|, (1-p)\mathbb{1}_2\}$. What is the physical interpretation of this POVM?

Exercise 8.2 Unambiguous State Discrimination

Suppose you are given two states, ρ, σ, of the same space $\mathcal{H}$ and want to distinguish them with a single measurement. We have seen that, unless the states are orthogonal ($\delta(\rho,\sigma) = 1$), it is impossible to always distinguish them with certainty.
If you are willing to have a probability of making an error, you can follow the strategy that gives you the greatest possibility of guessing correctly by measuring one of the states in the eigenbasis of $\rho - \sigma$. You will be right with probability $\Pr_{\checkmark} = \frac{1}{2}(1 + \delta(\rho,\sigma))$ and wrong in $1 - \Pr_{\checkmark}$ of the cases.
You may want to never make an error in your distinguishing procedure: you want to perform a measurement such that sometimes you are certain whether the state was ρ or σ and sometimes will not know which state you have. You will be correct less often than if you took the risky strategy, but you will never make a mistake in distinguishing the states.
In this exercise we will see how to construct such a measurement in the case where the two states are pure, $\rho = |\psi\rangle\langle\psi|, \sigma = |\phi\rangle\langle\phi|$.
Consider a measurement described by the POVM $\{M_\psi, M_\phi, M_?\}$, with

$$M_\psi = \alpha|\phi^\perp\rangle\langle\phi^\perp|, \qquad M_\phi = \alpha|\psi^\perp\rangle\langle\psi^\perp|, \qquad M_? = \mathbb{1} - M_\psi - M_\phi, \tag{1}$$

where $|x^\perp\rangle$ denotes a state orthogonal to $|x\rangle$ (ie. $\langle x|x^\perp\rangle = 0$) and α depends on the trace distance between the states. For for pure sates, the trace distance is given by $\delta(|\psi\rangle, |\phi\rangle) = \sqrt{1 - |\langle\psi|\phi\rangle|^2}$.

a) *i)* Compute the probabilities of obtaining the outcomes M_ψ, M_ϕ and $M_?$ when measuring states $|\psi\rangle$ and $|\phi\rangle$. Remember that you can expand one of the states in terms of the other and a vector orthogonal to it, for instance

$$|\psi\rangle = a|\phi\rangle + b|\phi^\perp\rangle, \quad |\psi^\perp\rangle = -b|\phi\rangle + a|\phi^\perp\rangle, \qquad a = \langle\psi|\phi\rangle, \quad |a|^2 + |b|^2 = 1.$$

ii) Verify that if the outcome of the measurement is M_ϕ the state measured could not have been $|\psi\rangle$, when it is M_ψ it could not have been $|\phi\rangle$ and when it is $M_?$ it could have been either with equal probability.

b) We want to maximise the probability of guessing correctly. This is equivalent to minimising the probability of obtaining $M_?$ (when we do not know which state we measured).

i) Give an expression for α that minimises the probability of obtaining $M_?$ when measuring one of the states at random, while still assuring that $M_?$ is a positive operator.

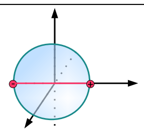
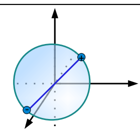
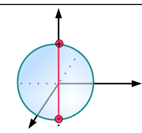
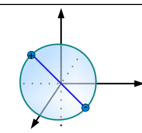
ii) What is the probability that you successfully distinguish which state you have?

Exercise 8.3 Bell inequalities and hidden variable models

Consider a setting like in exercise 6.2. Alice and Bob share a two-qubit state and are allowed to perform local measurements in their qubit.
The POVM corresponding to a measurement can be written in function of the angle α that the measurement basis makes with the $\{|0\rangle, 1\}$ basis,

$$M^\alpha = \left\{|\alpha\rangle\langle\alpha|, |\alpha^\perp\rangle\langle\alpha^\perp|\right\}, \qquad |\alpha\rangle = \cos\frac{\alpha}{2}\;|0\rangle, \sin\frac{\alpha}{2}\;|1\rangle, \qquad |\alpha^\perp\rangle = \sin\frac{\alpha}{2}\;|0\rangle, \cos\frac{\alpha}{2}\;|1\rangle,$$

where the $1/2$ factor comes from the Bloch sphere notation. We label the outcomes $+$ for $|\alpha\rangle$ and $-$ for $|\alpha^\perp\rangle$.
Alice and Bob can choose two different bases each:

meas.	0 	1 	2 	3 
Alice	$\alpha = 0, \begin{cases} \vert 0\rangle & \to + \\ \vert \pi\rangle & \to - \end{cases}$		$\alpha = \frac{\pi}{2}, \begin{cases} \vert \frac{\pi}{2}\rangle & \to + \\ \vert \frac{-\pi}{2}\rangle & \to - \end{cases}$	
Bob		$\alpha = \frac{\pi}{4}, \begin{cases} \vert \frac{\pi}{4}\rangle & \to + \\ \vert \frac{5\pi}{4}\rangle & \to - \end{cases}$		$\alpha = \frac{3\pi}{4}, \begin{cases} \vert \frac{3\pi}{4}\rangle & \to + \\ \vert \frac{-\pi}{4}\rangle & \to - \end{cases}$

a) Suppose that Alice and Bob share the state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The joint probabilities $P_{XY|ab}(x,y)$ of them obtaining outcomes x and y when they measure $A = a$ and $B = b$ are given by

Alice / Bob		A =0 +	A =0 −	A =2 +	A =2 −
B=1	+	$\frac{1}{2}-\epsilon$	ϵ	$\frac{1}{2}-\epsilon$	ϵ
	−	ϵ	$\frac{1}{2}-\epsilon$	ϵ	$\frac{1}{2}-\epsilon$
B=3	+	ϵ	$\frac{1}{2}-\epsilon$	$\frac{1}{2}-\epsilon$	ϵ
	−	$\frac{1}{2}-\epsilon$	ϵ	ϵ	$\frac{1}{2}-\epsilon$

with $\epsilon = \frac{1}{2}\sin^2(\pi/8) \approx 0.07$.

Compute

$$I_N(P_{XY|AB}) = P(X = Y|A = 0, B = 3) + \sum_{|a-b|=1} P(X \neq Y|A = a, B = b).$$

b) Now we introduce a PR-box, which is a joint probability distribution that cannot be created by measurements on a quantum state:

Alice / Bob		A =0 +	A =0 −	A =2 +	A =2 −
B=1	+	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	−	0	$\frac{1}{2}$	0	$\frac{1}{2}$
B=3	+	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	−	$\frac{1}{2}$	0	0	$\frac{1}{2}$

Show that the PR-box

(i) is non-signalling: $P(X|a, b_1) = P(X|a, b_2), \forall a$;

(ii) is non-local: $P_{XY|ab} \neq P_{X|a} P_{X|b}$;

(iii) yields $I_N(P_{XY|AB}) = 0$.

c) Consider now that Alice and Bob get their qubits and measurement devices from Eve. Eve will try to trick them into thinking that they share a singlet and perform quantum measurements. In fact, she will give them a device that allows her to guess the results of their "measurements" with some probability.

Eve is a post-quantum adversary, limited only by non-signaling. She will give them:

- with probability $1 - p$ a PR-box;
- with probability $p/4$, one of four deterministic boxes, that always outcome $++$, $+-$, $-+$ and $--$ respectively.

Find p so that the final joint probability distribution equals the one of the singlet state. What is the probability that Eve can guess the outcomes of their measurements?

Extra: Imagine that Alice and Bob are allowed to perform more measurements, with closer angles (so that the overlap between two consecutive bases is larger). What happens to p?