Advanced Topics in Quantum Information Theory
Exercise 3

FS 2011
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Exercise 3.1 The role of initial correlations: beyond CP maps and the Kraus representation

In assessing the dynamics of an open quantum system, the system of interest and its environment are often assumed to be initially in a separable tensor product state. In this case the reduced evolution of the system is known to be described by a completely positive (CP) map. However, as was pointed out by Pechukas[1] and others, this needs not be the case if one takes into account *initial correlations* between the system and its environment. The purpose of this exercise is to convince you with a simple example that such initial correlations play an important role in the reduced dynamics of a quantum system.

Consider two qubit systems interacting via an exchange-type Hamiltonian

$$H = H_0 + H_1$$

with

$$H_0 = \hbar\omega\sigma_{ee}^{(1)} + \hbar\omega\sigma_{ee}^{(2)}$$
$$H_1 = \hbar g(\sigma_{eg}^{(1)}\sigma_{ge}^{(2)} + \sigma_{ge}^{(1)}\sigma_{eg}^{(2)})$$

where we have denoted $\sigma_{ab}^{(i)} \equiv |a\rangle\langle b|_i$ with $a, b \in \{e, g\}$ and $|e\rangle_i$, $|g\rangle_i$ are respectively the ground and the excited states of the i^{th} qubit. In what follows, we shall always refer to the first qubit as the *system* and to the second as the *reservoir*.

a.) Assume that the two qubits are initially in an arbitrary state $\rho(0)$ and find the state $\rho(t)$ at time t. (Hint: To simplify your calculation, solve the Schrödinger equation in the interaction picture with respect to H_0, and transform the state back into the Schrödinger picture in the end).

b.) Express your result in terms of the three following quantities: the initial reduced density matrices $\rho^{(1)}(0)$, $\rho^{(2)}(0)$ of the system (1) and the reservoir (2), and the initial correlations between the two (embedded in the remaining components of the initial density matrix $\rho(0)$ of the complete system (1+2)).

The reduced dynamics of the system after a time t can always be expressed as a linear, homogeneous map

$$\rho_{ab}^{(1)}(t) = \sum_{cd} A_{ab;cd}(t)\rho_{cd}^{(1)}(0)$$

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[1]P. Pechukas, Phys. Rev. Lett. 73, 1060 (1994).

where here A is a 4×4 matrix. One can show[2] that the action of the map A on the initial density matrix can be written as

$$\rho^{(1)}(t) = \sum_i \lambda_i M_i \rho^{(1)}(0) M_i^\dagger$$

with

$$\sum_i \lambda_i M_i^\dagger M_i = I$$

where I denotes the identity matrix, λ_i are the (real) eigenvalues of the matrix A defined above, and M_i are operators obtained from the spectral decomposition of the map A (their explicit form is not important here).
We note that the above expression for the map A is very similar to the Kraus representation of CP maps. In fact, in the case where $\lambda_i \geq 0 \ \forall i$, one may define new operators $M_i' = \sqrt{\lambda_i} M_i$ and recover the Kraus representation, thereby showing the map A is a CP map. However, in general some of the λ_i could be *negative*, in which case the map is *not completely positive* (NCP).

c.) Find the eigenvalues λ_i of the map A defined above.

d.) Examine the two cases where the two qubits are: (i) in an initial product state with $\rho^{(2)}(0) = \frac{1}{2} I$ and (ii) in an initial entangled state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e, g\rangle + e^{i\varphi}|g, e\rangle)$. In both cases, argue on the complete positiveness of the map A.

[2]See for example A. R. Usha Devi, A. K. Rajagopal, Sudha, Phys. Rev. A 70, 052110 (2004), sect. II.