Advanced Topics in Quantum Information Theory
Exercise 2

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Exercise 2.1 Shor code and arbitrary single qubit errors

We have seen in the previous exercise that the Shor code protects against bit flips X, phase flips Z and the combined error XZ on a single qubit. Let us now show that this is sufficient in order to protect against arbitrary single qubit errors.

a.) Let E be an arbitrary (complex) 2×2 matrix. Show that there exist complex numbers $e_1, e_2, e_3, e_4 \in \mathbb{C}$ such that

$$E = e_1 \cdot \mathbb{I} + e_2 \cdot X + e_3 \cdot Z + e_4 \cdot X \cdot Z \ .$$

b.) Every noise process $\mathcal{N} : \mathcal{S}(\mathcal{H}_C) \to \mathcal{S}(\mathcal{H}_C)$ acting only on the j'th qubit of the Shor code can be written in the operator-sum representation as

$$\mathcal{N}(|\psi\rangle\langle\psi|) = \sum_i E_j^i |\psi\rangle\langle\psi| (E_j^i)^\dagger \ ,$$

with $|\psi\rangle = \alpha|0_L\rangle + \beta|1_L\rangle \in \mathcal{H}_C$ the Shor-encoding of a qubit and $E_j^i = \mathbb{I} \otimes ... \otimes E^i \otimes ... \otimes \mathbb{I}$, i.e., E_j^i is acting with the operation element E^i on the j'th qubit and the identity is acting on the remaining 8 qubits. Show that the Shor code protects against any such process $\mathcal{N}$.

Exercise 2.2 Error analysis and concatenation of codes

Let $|\psi\rangle = \alpha|0_L\rangle + \beta|1_L\rangle \in \mathcal{H}_C$ be a Shor-encoded qubit. Assume that the depolarization channel $\mathcal{N}$, which is given by $\mathcal{N}(\rho) = (1-p)\rho + p/3(X\rho X + Y\rho Y + Z\rho Z)$, is acting simultaneously, but *independently*, on each qubit of $|\psi\rangle$. Hence, the noise process is formally described by $\mathcal{N}^{\otimes 9} : \mathcal{S}(\mathcal{H}_C) \to \mathcal{S}(\mathcal{H}_C)$.

a.) Show that the depolarization channel can also be written as

$$\mathcal{N}(\rho) = (1 - \frac{4}{3}p)\rho + \frac{4}{3}p\frac{\mathbb{I}}{2} \ .$$

b.) What is the probability that an error occurs which cannot be corrected by the Shor code? Neglect higher order terms in the calculation, i.e., do not take into consideration when three or more errors occur simultaneously on different qubits.

c.) How large can p maximally be such that concatenation of the Shor code reduces the error probability?