Advanced Topics in Quantum Information Theory
Exercise 1

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Exercise 1.1 Three qubit bit flip code

Let $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$, be an encoding of the qubit $\alpha|0\rangle + \beta|1\rangle$.

a.) Compute the eigenvalues and eigenvectors of the observables $Z_1Z_2 := Z \otimes Z \otimes \mathbb{I}$ and $Z_2Z_3 := \mathbb{I} \otimes Z \otimes Z$.

b.) Perform the measurement of the observable Z_1Z_2 followed by the observable Z_2Z_3 on the faulty state $X_1|\psi\rangle$ with $X_1 := X \otimes \mathbb{I} \otimes \mathbb{I}$. What are the corresponding outcomes, measurements probabilities and post-measurement states?

c.) Do the same calculations for the states $|\psi\rangle$, $X_2|\psi\rangle$ and $X_3|\psi\rangle$.

d.) How can a single bit-flip error in $|\psi\rangle$ be corrected by using the information obtained by the measurements of Z_1Z_2 and Z_2Z_3?

Exercise 1.2 Coding and Decoupling

Let us now consider the three qubit bit flip code from a different perspective by considering the error process in the Choi-Jamiolkowski picture. The noise channel $\mathcal{E} : \mathcal{S}(\mathcal{H}_C) \to \mathcal{S}(\mathcal{H}_C)$ on the encoded state is then represented by the state

$$|\phi\rangle_{A'CE} := (\mathbb{I}_{A'} \otimes U_{\mathcal{E}})|\psi\rangle$$

with $U_{\mathcal{E}} : \mathcal{H}_C \to \mathcal{H}_C \otimes \mathcal{H}_E$ an isometric purification of $\mathcal{E}$ (Stinespring Dilation) and

$$|\psi\rangle := \frac{1}{\sqrt{2}}|0\rangle_{A'} \otimes |000\rangle_C + \frac{1}{\sqrt{2}}|1\rangle_{A'} \otimes |111\rangle_C \ .$$

a.) The error model is such that at most a single bit flip error can occur with probability p. Furthermore, assume that if a bit flip occurs all three qubits on the encoding space $\mathcal{H}_C$ are affected equally often. Hence, the probability that a bit flip happens on the first qubit is $p/3$. Represent this error process by a unitary $U_{\mathcal{E}}$ and compute $|\phi\rangle_{A'CE}$ and $\rho_{A'E} := tr_C(|\phi\rangle\langle\phi|_{A'CE})$.

b.) Consider now an error model where zero, one or two bit flips can occur. Each error has equal probability $p/6$ and, hence, the probability that no error happens is $1-p$. Represent this error process by an unitary $U_{\mathcal{E}}$ and compute that states $|\phi\rangle_{A'CE}$ and $\rho_{A'E} := tr_C(|\phi\rangle\langle\phi|_{A'CE})$. What can you say about the differences/similarities of the states $\rho_{A'E}$ in this and the previous item?

Exercise 1.3 Shor code

Let $|\psi\rangle$ be the nine qubit Shor-encoding of the qubit $\alpha|0\rangle + \beta|1\rangle$. Assume that $|\psi\rangle$ is exposed to a noise process which introduces a bit and a phase flip error on the fourth qubit yielding the faulty state $Z_4X_4|\psi\rangle$.

a.) Perform the measurement Z_4Z_5 followed by Z_5Z_6 on $Z_4X_4|\psi\rangle$. What are the corresponding outcomes, measurements probabilities and post-measurement states? Infer from the measurement results where the bit flip operation has to be applied in order to correct the error.

b.) Measure the observables $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$ on the bit-flip corrected state of the previous item. What are the corresponding outcomes, measurements probabilities and post-measurement states? What can be inferred from the measurement results?

c.) Apply the operator $Z_4Z_5Z_6$ on the resulting state of the previous item. What is the final state?