Sheet 3

Due: 29/03/11

Question 1 [*Gauge Invariance of QED*]:

Consider the QED Lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad (1)$$

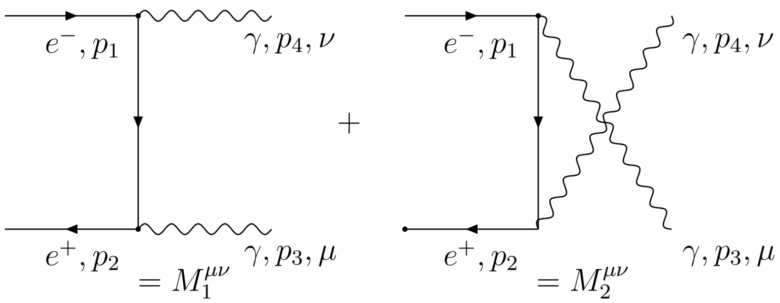
where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $D^\mu = \partial^\mu - igA^\mu$.

(i) Using the transformation rules of the photon field A_μ under the gauge group $U(1)$, show that the electromagnetic strength tensor $F_{\mu\nu}$ is gauge invariant.

(ii) Using Noether's theorem, find the conserved current and charge due to the invariance of the above Lagrangian under $U(1)$.

(iii) Gauge invariance manifests itself through Ward identities. Consider the QED process $e^+e^- \to \gamma\gamma$, whose leading order Feynman diagrams are shown below

$M^{\mu\nu}_{e^+e^-\to\gamma\gamma} =$



Using QED Feynman rules show that

$$\begin{aligned} M_1^{\mu\nu} &= (-ie^2)\frac{\bar{v}_2\gamma^\mu(\not{p}_3 - \not{p}_2)\gamma^\nu u_1}{(p_2 - p_3)^2}, \\ M_2^{\mu\nu} &= (-ie^2)\frac{\bar{v}_2\gamma^\nu(\not{p}_1 - \not{p}_3)\gamma^\mu u_1}{(p_1 - p_3)^2}, \end{aligned} \qquad (2)$$

and verify that the QED Ward identity

$$(p_3)_\mu M^{\mu\nu}_{e^+e^-\to\gamma\gamma} = 0 = (p_4)_\nu M^{\mu\nu}_{e^+e^-\to\gamma\gamma}$$

is fulfilled. Conclude that QED amplitudes are purely transverse.

Question 2 [*SU(N) structure constants*]:

The Lie group $SU(N)$ has the following unitary representation

$$U(\theta) = e^{ig\theta^a T_a}, \tag{3}$$

where T_a are the generators of the group.

(i) Show that the T_a matrices fulfil the following algebra

$$\left[T^a, T^b\right] = i f^{abc} T_c, \tag{4}$$

where f^{abc} are the structure constants of $SU(N)$.

Hint
Consider two independent group elements U, U' and compute the quantity $U'^{-1}U^{-1}U'U$.

(ii) Show that the structure constants f^{abc} are fully antisymmetric and real.

(iii) Show that the structure constants f^{abc} fulfil the Jacobi identity

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0. \tag{5}$$

Hint
Compute the quantity

$$\left[\left[T^a, T^b\right], T^c\right] + \left[\left[T^b, T^c\right], T^a\right] + \left[T^b, \left[T^a, T^c\right]\right] = 0 \tag{6}$$

and obtain Equation (**??**).

Question 3 [*SU(N) Lagrangian*]:

The $SU(N)$ Lagrangian is given by

$$\mathcal{L}_{SU(N)} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a, \tag{7}$$

where $G^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g f_{abc} A^{b\mu} A^{c\nu}$ and $D^\mu = \partial^\mu - igA^\mu_a T^a$.

(i) Starting from the commutation relation of the covariant derivative with itself

$$\left[D_\mu, D_\nu\right] = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]), \tag{8}$$

expand the gauge field in the basis of $SU(N)$ generators

$$A_\mu = A^a_\mu T_a, \tag{9}$$

$$A^a_\mu = 2\mathrm{tr}(A_\mu T^a), \tag{10}$$

and derive the expression of the gauge field strength $G^a_{\mu\nu}$.

(ii) Show that $G^a_{\mu\nu}$ tranforms according to the adjoint representation of $SU(N)$ under an infinitesimal gauge transformation of the gauge field $A^a_\mu \to A^a_\mu + \frac{1}{g}\partial_\mu \theta^a + f^{abc} A_{\mu b} \theta_c$.