Sheet 2

Due: 22/03/11

Question 1 [*The four point function in $\lambda\phi^4$ theory*]:
Consider a real scalar field ϕ of mass m with a ϕ^4-self-interaction whose dynamics is described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\mathcal{I} \ , \qquad \mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(m^2 - i\epsilon)\phi^2 \ , \qquad \mathcal{L}_\mathcal{I} = -\frac{1}{4!}\lambda\phi^4 \ ,$$

where $\lambda \ll 1$. The generating functional is defined as

$$Z[J] = \frac{\exp[i\int d^4x \mathcal{L}_\mathcal{I}(i\frac{\delta}{\delta J(x)})]Z_0[J]}{\exp[i\int d^4x \mathcal{L}_\mathcal{I}(i\frac{\delta}{\delta J(x)})]Z_0[J]|_{J=0}} \ ,$$

where $Z_0[J]$ is the generating functional for the free field

$$Z_0[J] = Z_0[0]\exp[-\frac{1}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)] \ ,$$

and in $\mathcal{L}_\mathcal{I}$ we have replaced $\phi(x)$ by the functional derivative, *i.e.*

$$\mathcal{L}_\mathcal{I}\left(i\frac{\delta}{\delta J(x)}\right) = -\frac{\lambda}{4!}\frac{\delta^4}{\delta J(x)^4}.$$

(i) Compute the four point function

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \frac{1}{i^4}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}Z[J]\bigg|_{J=0}$$

to order $\mathcal{O}(\lambda)$ and draw the corresponding Feynman diagrams.

(ii) Compute the *connected* four point function

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle_{\text{connected}} = \frac{i}{i^4}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}W[J]\bigg|_{J=0} \ ,$$

where

$$W[J] = -i\log Z[J] \ ,$$

and check that the corresponding diagrams are indeed connected.

(iii) *(Optional)* Compute the connected four point function at $\mathcal{O}(\lambda^2)$.

Question 2 [*Integration with Grassmann variables*]:
Consider a set of N Grassmann numbers $\{\theta_i\}$ (*i.e.* $\theta_i\theta_j = -\theta_j\theta_i$) and a $N \times N$ matrix B_{ij} of commuting numbers.

(i) Using the integration properties of Grassmann variables show that

$$\int \prod_i d\theta_i^* d\theta_i \, e^{-\theta_i^* B_{ij}\theta_j} = \det(B)$$

and

$$\int \prod_i d\theta_i^* d\theta_i \, \theta_l^* \theta_k \, e^{-\theta_i^* B_{ij}\theta_j} = (B^{-1})_{lk} \det(B) \ .$$

Hints:

- Expand the exponential function in a Taylor series, and set to zero terms containing at least twice the same Grassmann variable. Then compute the first integral using the properties

$$\int d\theta_i 1 = 0 \ , \qquad \int d\theta_i \theta_i = 1 \ .$$

- The second integral can be computed by introducing a new matrix $A_{ij}(l,k)$ defined by

$$\begin{aligned} A_{ij}(l,k) &= B_{ij} && \text{if } i \neq l, j \neq k \\ A_{ij}(l,k) &= \delta_{il}\delta_{jk} && \text{if } i = l \text{ or } j = k, \end{aligned}$$

and rewriting

$$\theta_l^* \theta_k \, e^{-\theta_i^* B_{ij}\theta_j} = \sum_{n=1}^{N} A_{ln}(l,k)\theta_l^*\theta_n.$$

- Finally one uses the formula for the inverse of a matrix C (with $\det C \neq 0$)

$$(C^{-1})_{lk} = \frac{(-1)^{l+k}}{\det C} C(l,k) \ ,$$

where $C(l,k)$ is the determinant of the matrix that is obtained from C by removing the l'th row and k'th column.

(ii) Following the same approach, compute the two integrals

$$\int \prod_i d\theta_i^* d\theta_i \, \theta_k \theta_l \, e^{-\theta_i^* B_{ij}\theta_j}$$

and

$$\int \prod_i d\theta_i^* d\theta_i \, \theta_l^* \theta_k \, \theta_n^* \theta_m e^{-\theta_i^* B_{ij}\theta_j} \ .$$