Proseminar on supersymmetry:

1. Supersymmetry algebra

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The aim of this report is to present the supersymmetry algebra. After a presentation of the Coleman-Mandula theorem which is the starting point of supersymmetry, we will deduce the general form of the superextension of the Poincaré algebra with a treatment of the central charges. Then, the Casimir operators for the $N = 1$ supersymmetry algebra will be deducted and at the end we will discuss two basic properties of the supersymmetry algebra: the positivity of energy and the fact that the number of fermions and bosons should be equal. The appendix includes standard results concerning space-time symmetry groups for the reader which are not familiar with these results.

Contents

1 Coleman-Mandula theorem

Supersymmetry is a new kind of symmetry in addition of the standard Poincaré and internal symmetries, which link bosons and fermions. The bosons are the mediators of interactions and fermions are the constituents of the matter. The aim of supersymmetry is to provide a unified description of fermions and bosons, hence of matter and interactions. However we have no direct proof of the realization of supersymmetry in nature, but it is very promising in solving some of the thorny questions in modern high-energy physics. It is also an interesting approach to unify the gauge couplings of the three fundamental interactions at high energies. Moreover, the supersymmetry predicts superpartners for all standard-model particles, which however, at this stage are not observed experimentally. Through the introduction of supersymmetry in early 1970's, many supersymmetric theories was proposed, essentially due to the Coleman-Mandula theorem as we will see in the latter.

The Coleman-Mandula theorem discusses the possible symmetries of the S-matrix, under physically reasonable assumptions. More precisely, a symmetry generator G of the S-matrix is an operator mapping one-particle states into one-particle states such that

$$GS\,|\psi\rangle = SG\,|\psi\rangle \ , \tag{1}$$

or equivalently

$$[G, S] = 0 \, . \tag{2}$$

Theorem (Coleman-Mandula, 1967)**.** *Let* G *be a connected symmetry group of the* S*-matrix, and let the five conditions hold,*

1. *The group* G *contains a subgroup locally isomorphic to the Poincaré group*[1] $\mathsf{P}\,(3,1)$.
2. *For each* $m > 0$*, there are only a finite number of one-particle states with mass less than* m.
3. *The elastic scattering amplitudes are analytic functions of* s *and* t *(Mandelstam invariants), in some neighborhood of the physical region.*
4. *The scattering matrix is non-trivial in the sense that any two one-particle momentum eigenstates scatter except perhaps at isolated value of* s.
5. *The generators of* G *written as integral operator in momentum space, have distributions for their kernels.*

Then G *is locally isomorphic to the direct product of a compact Lie group* Int *and the Poincaré group* $\mathsf{P}\,(3,1)$,

$$\mathsf{G} \simeq \mathsf{P}\,(3,1) \otimes \mathsf{Int} \, . \tag{3}$$

In particular, the Lie algebra $\mathfrak{g}$ *of* G *is the direct sum of the compact Lie algebra* $\mathfrak{int}$ *of* Int *and the Poincaré algebra* $\mathfrak{p}\,(3,1)$,

$$\mathfrak{g} \simeq \mathfrak{p}\,(3,1) \oplus \mathfrak{int} \, . \tag{4}$$

The original proof is given in [1] and a different version discussed in Appendix B of Chapter 24.1 of [2] provided another proof. Since it is rather complicated it will not be discussed here. To obtain a more explicit formulation, we choose a basis $\{T_l\}$ of the internal algebra $\mathfrak{int}$, and the structure constants c_{lmn} are defined via

$$[T_l, T_m] = i c_{lmn} T_n \, . \tag{5}$$

[1]For more information about the Poincaré group and its algebra see the subsections B.5 and B.6.

Consequently, the generators of the Poincaré algebra[2] and of the internal algebra commute as follow,

$$[M_{ab}, M_{cd}] = i\eta_{ad}M_{bc} - i\eta_{ac}M_{bd} + i\eta_{bc}M_{ad} - i\eta_{bd}M_{ac} \,, \tag{6a}$$
$$[P_a, P_b] = 0 \,, \tag{6b}$$
$$[T_l, T_m] = ic_{lmn}T_n \,, \tag{6c}$$
$$[M_{ab}, P_c] = i\eta_{bc}P_a - i\eta_{ac}P_b \,, \tag{6d}$$
$$[P_a, T_l] = 0 \,, \tag{6e}$$
$$[M_{ab}, T_l] = 0 \,. \tag{6f}$$

The important consequence of the Coleman-Mandula theorem, is that all the possible symmetry generators except P_a and M_{ab} should be scalars. The idea of the next section is to sidestep this theorem to obtain symmetry generators which are not scalars.

2 Superalgebra

The Coleman-Mandula does not allow a symmetry group which combine the Poincaré group and the other symmetries in a non-trivial way as well as acting non-trivially on particle spin. However these two facts are highly desirable, so the aim will be to evade the Coleman-Mandula by weakening one or more of its assumptions. Since all the five assumptions can not be physically reviewed, we will go to a more general class of symmetry group. In the theorem, the group G is a Lie group, and so its generators are in a Lie algebra. Here we will define a new class of algebra, called superalgebra and find in the next section the analogue result of the Coleman-Mandula theorem for this class of algebras.

The notion of superalgebra is a generalization of the concept of Lie algebra. To define a superalgebra, we first need the concept of graded algebra. An associative algebra $\mathfrak{g}$ is called $\mathbb{Z}_2$-graded if it admits a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \,, \tag{7}$$

such that the parity function defined on $\mathfrak{g}_0 \sqcup \mathfrak{g}_1$ as

$$\kappa\,(a) = \begin{cases} 0\,, & \text{if } a \in \mathfrak{g}_0 \,, \\ 1\,, & \text{if } a \in \mathfrak{g}_1 \,, \end{cases} \tag{8}$$

satisfies the relation

$$\kappa\,(a \cdot b) = \kappa\,(a) + \kappa\,(b) \pmod 2 \,. \tag{9}$$

The elements of $\mathfrak{g}_0$, $\mathfrak{g}_1$ and $\mathfrak{g}_0 \sqcup \mathfrak{g}_1$ are respectively called even, odd and pure.

Every $\mathbb{Z}_2$-graded algebra defines[3] a superalgebra with the supercommutator

$$[a, b\} = a \cdot b - (-1)^{\kappa(a)\kappa(b)}\, b \cdot a \,, \tag{10}$$

where a and b are pure elements. By linearity we obtain on arbitrary elements $a = a_0 \oplus a_1$ and $b = b_0 \oplus b_1$,

$$[a, b\} = [a_0, b_0] + [a_0, b_1] + [a_1, b_0] + \{a_1, b_1\} \,. \tag{11}$$

From the equation (9), we obtain

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0 \,, \qquad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1 \,, \qquad \{\mathfrak{g}_1, \mathfrak{g}_1\} \subset \mathfrak{g}_0 \,. \tag{12}$$

Moreover, we can show by direct calculation the super-Jacobi identity

$$(-1)^{\kappa(a)\kappa(c)}\,[a, [b, c\}\} + (-1)^{\kappa(b)\kappa(a)}\,[b, [c, a\}\} + (-1)^{\kappa(c)\kappa(b)}\,[c, [a, b\}\} = 0 \,. \tag{13}$$

In particular these last two properties indicate that the part $\mathfrak{g}_0$ forms a Lie algebra. Reciprocally, not every Lie algebra admits an extension to a superalgebra. In the following, our aim will be to find a superextension of the Poincaré algebra. As we will discuss in due course, the latter will not be unique but there are many inequivalent superalgebras of interest.

[2] These generators are taken Hermitian as discussed in subsection B.7.

[3] This definition is not the most general one for a superalgebra. From a mathematical point of view, we define a superalgebra without the product $\cdot$ by taking the properties of the supercommutator as axioms [3].

3 Supersymmetry algebra

In this section we will deduce the analogue result of the Coleman-Mandula theorem for the class of superalgebra. The theorem which describes all the possible symmetries of the S-matrix generated by a superalgebra is known as the Haag-Łopuszański-Sohnius theorem [4].

3.1 Generators of the supersymmetry algebra[4]

The Coleman-Mandula theorem together with the Jacobi identity will impose many restrictions on the possible extensions of the Poincaré algebra. More precisely, we search for the more general superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that

$$\mathfrak{p}\,(3,1) \subset \mathfrak{g}_0 \,. \tag{14}$$

The equation (4) of the Coleman-Mandula theorem implies the following form of the even part

$$\mathfrak{g}_0 \simeq \mathfrak{p}\,(3,1) \oplus \mathfrak{int}\,, \tag{15}$$

where $\mathfrak{int}$ is the Lie algebra of the internal symmetry group Int. Consequently, it remains to find the odd part $\mathfrak{g}_1$.

Any odd generator should be an element of an irreducible representation of the Lorentz group[5]:

$$Q_{\alpha_1\cdots\alpha_m\dot{\alpha}_1\cdots\dot{\alpha}_n} \in (m/2, n/2) \ . \tag{16}$$

The commutator

$$\left\{Q_{\alpha_1\cdots\alpha_m\dot{\alpha}_1\cdots\dot{\alpha}_n}, \bar{Q}_{\dot{\beta}_1\cdots\dot{\beta}_m\beta_1\cdots\beta_n}\right\} , \tag{17}$$

is a tensor of rank $m+n$, and equation (12) implies that this is an even element of the algebra. By the Coleman-Mandula theorem, the generators of the even part can be scalars or vectors, so the preceding commutator is null unless $m+n=1$. On a Hilbert space with positive definite norm, the considered commutator for $\beta_j = \dot{\alpha}_j$ and $\dot{\beta}_i = \alpha_i$ is positive definite because[6]

$$0 \leq \left\| Q_{\alpha_1\cdots\alpha_m\dot{\alpha}_1\cdots\dot{\alpha}_n} \left|\psi\right\rangle \right\|^2 + \left\| \bar{Q}_{\dot{\beta}_1\cdots\dot{\beta}_m\beta_1\cdots\beta_n} \left|\psi\right\rangle \right\|^2 = \left\langle\psi\right| \left\{Q_{\alpha_1\cdots\alpha_m\dot{\alpha}_1\cdots\dot{\alpha}_n}, \bar{Q}_{\dot{\beta}_1\cdots\dot{\beta}_m\beta_1\cdots\beta_n}\right\} \left|\psi\right\rangle \ , \tag{18}$$

this implies

$$Q_{\alpha_1\cdots\alpha_m\dot{\alpha}_1\cdots\dot{\alpha}_n} = 0\,, \tag{19}$$

for $m+n>1$. Consequently, we have shown that the odd generators should be spinors and therefore we can choose N spinors Q^I_α such that they form together with $\bar{Q}^I_{\dot{\alpha}}$ a complete set of generators for the odd part of the algebra.

For the latter, it will pleasant to switch to spinor notation for the tensors. For the vector like P_a, the link is done by equation (143),

$$P_{\alpha\dot{\alpha}} = \left(\sigma^a\right)_{\alpha\dot{\alpha}} P_a\,, \tag{20}$$

and for the antisymmetric tensor M_{ab} by (151) and (152),

$$M_{\alpha\beta} = \frac{1}{2}\left(\sigma^{ab}\right)_{\alpha\beta} M_{ab}\,, \qquad \bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}\left(\bar{\sigma}^{ab}\right)_{\dot{\alpha}\dot{\beta}} M_{ab}\,. \tag{21}$$

The following table summarizes the generators of the supersymmetric extension of the Poincaré algebra,

$$T_l \in (0,0) \ , \qquad P_{\alpha\dot{\alpha}} \in (1/2, 1/2) \ , \tag{22a}$$

$$M_{\alpha\beta} \in (1,0) \ , \qquad \bar{M}_{\dot{\alpha}\dot{\beta}} \in (0,1) \ , \tag{22b}$$

$$Q^I_\alpha \in (1/2, 0) \ , \qquad \bar{Q}^I_{\dot{\alpha}} \in (0, 1/2) \ . \tag{22c}$$

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[4] The facts concerning the odd generators is inspired by [5].

[5] For more information about the spinorial representations of the Lorentz group, see subsection B.4.

[6] To be more exact, we have to choose a unitary representation T of the algebra for which Q and $\bar{Q}$ are related by

$$T\,(\bar{Q}) = T\,(Q)^\dagger \ ,$$

where † means the adjoint operator. Here, by abuse of notation, we identify $T\,(Q)$ with Q in order to save writing. For more details on this point, see the paragraph 2.2 of [3].

3.2 Commutation and anticommutation relations[7]

Since the odd generators are in the representation $(1/2,0)$ or $(0,1/2)$, we have[8]

$$\left[Q^I_\alpha, M_{ab}\right] = i\left(\sigma_{ab}\right)_\alpha{}^\beta Q^I_\beta\,, \qquad\qquad \left[\bar{Q}^I_{\dot\alpha}, M_{ab}\right] = i\left(\bar\sigma_{ab}\right)_{\dot\alpha}{}^{\dot\beta} \bar{Q}^I_{\dot\beta}\,. \tag{23}$$

The first aim is to find an explicit form for the commutator

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\right\} \in (1/2,1/2)\ , \tag{24}$$

which by equation (12) is an element of the odd part of the algebra. By the Coleman-Mandula theorem the most general form should be

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\right\} = C^{IJ} P_{\alpha\dot\alpha}\,, \tag{25}$$

where C^{IJ} are some complex coefficients. By taking the adjoint,

$$\left\{Q^J_\alpha, \bar{Q}^I_{\dot\alpha}\right\} = \bar{C}^{IJ} P_{\alpha\dot\alpha}\,, \tag{26}$$

which proofs that the matrix C is Hermitian. We can choose a basis where C is diagonal and moreover since the commutator is a positive definite operator, we can rescale the generators to have

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\right\} = \delta^{IJ} P_{\alpha\dot\alpha}\,. \tag{27}$$

The most general possible form for the commutator

$$\left\{Q^I_\alpha, Q^J_\beta\right\} \in (0,0)\oplus(1,0)\ , \tag{28}$$

is

$$\left\{Q^I_\alpha, Q^J_\beta\right\} = X^{IJ}\varepsilon_{\alpha\beta} + Y^{IJ} M_{\alpha\beta}\,, \tag{29}$$

where X^{IJ} are some scalar constants and Y^{IJ} some complex constants.

Now we consider the commutator

$$\left[Q^I_\alpha, P_a\right] \in (1,1/2)\oplus(0,1/2)\ . \tag{30}$$

Since generators in the representation $(1,1/2)$ are not allowed, the most general possible form is

$$\left[Q^I_\alpha, P_{\beta\dot\beta}\right] = Z^I{}_J \varepsilon_{\alpha\beta}\bar{Q}^J_{\dot\beta}\,, \tag{31}$$

where $Z^I{}_J$ are some complex constants. The super-Jacobi identity (13) for (P,P,Q)

$$\left[P_{\alpha\dot\alpha}, \left[P_{\beta\dot\beta}, Q^I_\gamma\right]\right] + \left[P_{\beta\dot\beta}, \left[Q^I_\gamma, P_{\alpha\dot\alpha}\right]\right] + \left[Q^I_\gamma, \left[P_{\alpha\dot\alpha}, P_{\beta\dot\beta}\right]\right] = 0\,, \tag{32}$$

becomes

$$Z^I{}_J\varepsilon_{\beta\gamma}\left[P_{\alpha\dot\alpha}, \bar{Q}^J_{\dot\beta}\right] - Z^I{}_J\varepsilon_{\alpha\gamma}\left[P_{\beta\dot\beta}, \bar{Q}^J_{\dot\alpha}\right] = 0\,. \tag{33}$$

By taking the adjoint of (31),

$$\left[\bar{Q}^I_{\dot\alpha}, P_{\beta\dot\beta}\right] = \bar{Z}^I{}_J \varepsilon_{\dot\alpha\dot\beta} Q^J_\beta\,, \tag{34}$$

and therefore

$$Z^I{}_J \bar{Z}^J{}_K \varepsilon_{\dot\alpha\dot\beta}\left(\varepsilon_{\beta\gamma}Q^K_\alpha + \varepsilon_{\alpha\gamma}Q^K_\beta\right) = 0\,, \tag{35}$$

which implies that the matrix Z satisfies

$$Z\bar{Z} = 0\,. \tag{36}$$

The super-Jacobi identity (13) for (P,Q,Q) is

$$\left[P_{\alpha\dot\alpha}, \left\{Q^I_\beta, Q^J_\gamma\right\}\right] + \left\{Q^I_\beta, \left[Q^J_\gamma, P_{\alpha\dot\alpha}\right]\right\} - \left\{Q^J_\gamma, \left[P_{\alpha\dot\alpha}, Q^I_\beta\right]\right\} = 0\,, \tag{37}$$

[7] The results given in this section is some synthesis of [6, 7, 5, 8].

[8] The subsection B.3 proofs that σ_{ab} are the generators of $\mathsf{SL}(2,\mathbb{C})$ and consequently spinors with one undotted index transform with σ_{ab}.

and since scalars commute with P_a it becomes

$$Y^{IJ}\left[P_{\alpha\dot{\alpha}}, M_{\beta\gamma}\right] + \varepsilon_{\gamma\alpha} Z^J{}_K \left\{Q^I_\beta, \bar{Q}^K_{\dot{\alpha}}\right\} + \varepsilon_{\beta\alpha} Z^I{}_K \left\{Q^J_\gamma, \bar{Q}^K_{\dot{\alpha}}\right\} = 0\,. \tag{38}$$

Since $M_{\beta\gamma}$ is symmetric the contraction with $\varepsilon^{\beta\gamma}$ becomes by using (27)

$$\left(Z^{JI} - Z^{IJ}\right) P_{\alpha\dot{\alpha}} = 0\,, \tag{39}$$

which proofs that the matrix Z is symmetric. By combining with equation (36), we obtain $ZZ^\dagger = 0$ which implies that Z vanishes. Consequently the odd generators are invariant under translations

$$\left[Q^I_\alpha, P_a\right] = 0\,. \tag{40}$$

Now the equation (38) is simply

$$Y^{IJ}\left[P_{\alpha\dot{\alpha}}, M_{\beta\gamma}\right] = 0\,, \tag{41}$$

which is only possible for $Y = 0$, entailing

$$\left\{Q^I_\alpha, Q^J_\beta\right\} = X^{IJ}\varepsilon_{\alpha\beta}\,. \tag{42}$$

The scalars X^{IJ} are called the central charges and trivially satisfy

$$X^{IJ} + X^{JI} = 0\,. \tag{43}$$

The commutator

$$\left[Q^I_\alpha, T_l\right] \in (1/2, 0)\,, \tag{44}$$

is by (12) an element of the odd part of the algebra and consequently should take the form

$$\left[Q^I_\alpha, T_l\right] = \left(S_l\right)^I{}_J Q^J_\alpha\,, \tag{45}$$

where $\left(S_l\right)^I{}_J$ are some complex constants. The super-Jacobi identity (13) for (T, T, Q) is

$$\left[T_l, \left[T_m, Q^I_\alpha\right]\right] + \left[T_m, \left[Q^I_\alpha, T_l\right]\right] + \left[Q_\alpha, \left[T_l, T_m\right]\right] = 0\,, \tag{46}$$

and becomes with (5)

$$\left(S_m\right)^I{}_J \left(S_l\right)^J{}_K Q^K_\alpha - \left(S_l\right)^I{}_J \left(S_m\right)^J{}_K Q^K_\alpha + i c_{lmn} \left(S_n\right)^I{}_K Q^K_\alpha = 0\,. \tag{47}$$

Consequently, we have proved that the S_l form a representation of the internal algebra $\mathfrak{int}$,

$$\left[S_l, S_m\right] = i c_{lmn} S_n\,. \tag{48}$$

The super-Jacobi identity (13) for $(T, Q, \bar{Q})$ gives us

$$\left[T_l, \left\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\right\}\right] + \left\{Q^I_\alpha, \left[\bar{Q}^J_{\dot{\alpha}}, T_l\right]\right\} - \left\{\bar{Q}^J_{\dot{\alpha}}, \left[T_l, Q^I_\alpha\right]\right\} = 0\,. \tag{49}$$

Since P_a commutes with T_l, the first term vanishes, and by using (45) and its conjugate,

$$0 = \left\{Q^I_\alpha, \bar{Q}^K_{\dot{\alpha}}\right\} \left(\bar{S}_l\right)^J{}_K - \left\{\bar{Q}^J_{\dot{\alpha}}, Q^K_\alpha\right\} \left(S_l\right)^I{}_K = P_{\alpha\dot{\alpha}} \left(\left(\bar{S}_l\right)^{JI} - \left(S_l\right)^{IJ}\right)\,. \tag{50}$$

This equation is true only if S_l is Hermitian,

$$S_l^\dagger = S_l\,. \tag{51}$$

In the following table we summarize the supersymmetric extension of the Poincaré algebra:

$$\left[Q^I_\alpha, T_l\right] = \left(S_l\right)^I{}_J Q^J_\alpha\,, \tag{52a}$$

$$\left[Q^I_\alpha, P_a\right] = 0\,, \tag{52b}$$

$$\left[Q^I_\alpha, M_{ab}\right] = i\left(\sigma_{ab}\right)_\alpha{}^\beta Q^I_\beta\,, \tag{52c}$$

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\right\} = \delta^{IJ} P_{\alpha\dot{\alpha}}\,, \tag{52d}$$

$$\left\{Q^I_\alpha, Q^J_\beta\right\} = X^{IJ}\varepsilon_{\alpha\beta}\,, \tag{52e}$$

where the matrices S_l are Hermitian and form a representation of the internal algebra $\mathfrak{int}$.

3.3 Anticommutation relations in four-component spinor notation[9]

We can group together the odd generators Q^I_α and $\bar{Q}^I_{\dot\alpha}$ in a four-component Majorana spinor,

$$Q^I_r = \begin{pmatrix} Q^I_\alpha \\ \bar{Q}^{\dot\alpha}_I \end{pmatrix}_r , \qquad \bar{Q}^I_r = \begin{pmatrix} Q^\alpha_I & \bar{Q}^I_{\dot\alpha} \end{pmatrix}_r , \tag{53}$$

where the spinors indices are raised and lowered with the totally antisymmetric tensor and the place of the I has no importance, i.e is raised and lowered with δ. In this notation, we obtain by (52d) and (52e)

$$\{Q^I_r, \bar{Q}^J_s\} = \begin{pmatrix} \{Q^I_\alpha, Q^\beta_J\} & \{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} \\ \{\bar{Q}^{\dot\alpha}_I, Q^\beta_J\} & \{\bar{Q}^{\dot\alpha}_I, \bar{Q}^J_{\dot\beta}\} \end{pmatrix}_{rs} = \begin{pmatrix} X^I{}_J & \delta^{IJ}\sigma^a P_a \\ \delta^{IJ}\bar\sigma^a P_a & \bar{X}_I{}^J \end{pmatrix}_{rs} . \tag{54}$$

If the central charges vanish we obtain

$$\{Q^I_r, \bar{Q}^J_s\} = \delta^{IJ} \left(\gamma^a\right)_{rs} P_a , \tag{55}$$

where the matrices γ^a are the in the Weyl representation of the Clifford algebra,

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar\sigma^a & 0 \end{pmatrix} . \tag{56}$$

3.4 Central charges[10]

In this section we will prove that the central charges X^{IJ} and $\bar{X}^{IJ}$ commute with everything, including with themselves. Since these central charges are scalars, we can express them in terms of the generators of the internal symmetries,

$$X^{IJ} = \left(a^l\right)^{IJ} T_l , \tag{57}$$

and this proves that the central charges commute with the generators of the Poincaré algebra P_a and M_{ab}.

Then the super-Jacobi identity for $(Q, Q, \bar{Q})$ is

$$\left[Q^I_\alpha, \{Q^J_\beta, \bar{Q}^K_{\dot\gamma}\}\right] + \left\{Q^J_\beta, \{\bar{Q}^K_{\dot\gamma}, Q^I_\alpha\}\right\} + \left[\bar{Q}^K_{\dot\gamma}, \{Q^I_\alpha, Q^J_\beta\}\right] = 0 , \tag{58}$$

and becomes with the previous results

$$\varepsilon_{\alpha\beta} \left[\bar{Q}^K_{\dot\gamma}, X^{IJ}\right] = 0 . \tag{59}$$

On the other hand, the super-Jacobi identity for $(X, Q, \bar{Q})$ is

$$\left[X^{IJ}, \{Q^K_\alpha, \bar{Q}^L_{\dot\alpha}\}\right] + \left\{Q^K_\alpha, [\bar{Q}^L_{\dot\alpha}, X^{IJ}]\right\} - \left\{\bar{Q}^K_{\dot\alpha}, [X^{IJ}, Q^L_\alpha]\right\} = 0 , \tag{60}$$

and becomes since P_a commutes with T_l,

$$\left\{\bar{Q}^K_{\dot\alpha}, [X^{IJ}, Q^L_\alpha]\right\} = 0 . \tag{61}$$

By using (57) and (52a),

$$\left[X^{IJ}, Q^L_\alpha\right] = \left(a^l\right)^{IJ} \left(S_l\right)^L{}_M Q^M_\alpha , \tag{62}$$

and in view of (27) we should have

$$\left(a^l\right)^{IJ} \left(S_l\right)^L{}_M = 0 . \tag{63}$$

Consequently, we have proved

$$\left[X^{IJ}, Q^K_\alpha\right] = 0 , \qquad \left[X^{IJ}, \bar{Q}^K_{\dot\alpha}\right] = 0 . \tag{64}$$

[9] This part is some summary of section 4.1 of [9].
[10] Most of the properties shown here concerning the central charges can be found in [5].

The last two equations together with (52e) imply directly

$$\left[X^{IJ}, X^{KL}\right] = 0\,, \qquad\qquad \left[X^{IJ}, \bar{X}^{KL}\right] = 0\,, \tag{65}$$

which shows that the central charges X^{IJ} form an abelian subalgebra of $\mathfrak{int}$.

The super-Jacobi identity for (T, Q, Q) gives us

$$\left[T_l, \{Q^I_\alpha, Q^J_\beta\}\right] + \left\{Q^I_\alpha, [Q^J_\beta, T_l]\right\} - \left\{Q^J_\beta, [T_l, Q^I_\alpha]\right\} = 0\,, \tag{66}$$

and so with (52a),

$$\varepsilon_{\alpha\beta}\left[T_l, X^{IJ}\right] = -\left\{Q^J_\beta, Q^K_\alpha\right\}\left(S_l\right)^I{}_K - \left\{Q^I_\alpha, Q^K_\beta\right\}\left(S_l\right)^J{}_K = \varepsilon_{\alpha\beta}\left(X^{JK}\left(S_l\right)^I{}_K - X^{IK}\left(S_l\right)^J{}_K\right)\,. \tag{67}$$

This proofs that the central charges X^{IJ} and $\bar{X}^{IJ}$ form an ideal of the algebra $\mathfrak{int}$, which we will now proof to be abelian.

By the Coleman-Mandula theorem, the internal algebra $\mathfrak{int}$ is a compact Lie algebra, and consequently admits a decomposition

$$\mathfrak{int} = \mathcal{S} \oplus \mathcal{A}\,, \tag{68}$$

where $\mathcal{S}$ is a semisimple algebra and $\mathcal{A}$ an abelian algebra. Since a semisimple algebra admits no abelian ideal, the central charges are part of the algebra $\mathcal{A}$, and consequently commute with all the elements of $\mathfrak{int}$,

$$\left[X^{IJ}, T_l\right] = 0\,, \qquad\qquad \left[\bar{X}^{IJ}, T_l\right] = 0\,. \tag{69}$$

We have proved that the central charges X^{IJ} and $\bar{X}^{IJ}$ commutes with everything, namely are in the center of the supersymmetry algebra.

The last two equations have an important consequence: together with (67) we obtain

$$X^{JK}\left(S_l\right)^I{}_K - X^{IK}\left(S_l\right)^J{}_K = 0\,, \tag{70}$$

and by using (57) we find

$$\left(S_l\right)^I{}_K\left(a^m\right)^{JK} - \left(S_l\right)^J{}_K\left(a^m\right)^{IK} = 0\,. \tag{71}$$

Since the central charges are antisymmetric and the S_l are Hermitian matrices (51),

$$S_l a^m + a^m \bar{S}_l = 0\,. \tag{72}$$

We have proved that the S_l form a representation of $\mathfrak{int}$, and consequently the last equation tells us that the matrices a^m intertwine the representation S_l with its complex conjugate $-\bar{S}_l$. Consequently, the central charges exists if and only if the algebra $\mathfrak{int}$ admits such an intertwiner. An example with non-zero central charges is given by orthonormal groups,

$$S_l = -S_l^{\mathsf{T}}\,, \qquad\qquad [S_l, a^m] = 0\,. \tag{73}$$

4 Properties of the supersymmetry algebra

4.1 Casimir operators[11]

The Casimir operators for the Poincaré algebra are P^2 and W^2 as discussed in subsection B.8. Since P_a commutes with the odd generators Q^I_α, the first Casimir operator of the supersymmetry algebra is still P^2. But the second Casimir operator W^2 for the Poincaré algebra is no more a Casimir operator of the superalgebra extension, because M_{ab} does not commute with Q^I_α. For simplicity, we consider the $N = 1$ case, because in this case the central charges vanishes. A generalization to supersymmetry algebra with central charges can be found in [10]. In spinors notations, we obtain

$$[W_{\alpha\dot\alpha}, Q_\beta] = \frac{1}{2} Q_\beta P_{\alpha\dot\alpha} - Q_\alpha P_{\beta\dot\alpha}\,. \tag{74}$$

[11] This part is highly inspired by the subsection 2.3 of [3].

Since

$$\left[\left[Q_\alpha, \bar{Q}_{\dot\alpha}\right], Q_\beta\right] = \left\{Q_\alpha, \left\{Q_\beta, \bar{Q}_{\dot\alpha}\right\}\right\} - \left\{\left\{Q_\alpha, Q_\beta\right\}, \bar{Q}_{\dot\alpha}\right\} = 2 Q_\alpha P_{\beta\dot\alpha} , \tag{75}$$

by defining

$$Z_{\alpha\dot\alpha} = W_{\alpha\dot\alpha} - \frac{1}{2}\left[Q_\alpha, \bar{Q}_{\dot\alpha}\right] , \tag{76}$$

we obtain

$$[Z_a, Q_\beta] = \frac{1}{2} Q_\beta P_a , \qquad \left[Z_{[a} P_{b]}, Q_\alpha\right] = 0 . \tag{77}$$

On the other hand, we can proof that

$$[Z_a, P_b] = 0 , \qquad [Z_a, Z_b] = i\varepsilon_{abcd} P^c Z^d . \tag{78}$$

Consequently, the operator

$$C = \frac{-1}{2} Z_{[a} P_{b]} Z^{[a} P^{b]} = (Z_a P^a)^2 - Z^2 P^2 , \tag{79}$$

commutes with P_a, Q_α and we can prove that is also the case with M_{ab}. This proves that C is the second Casimir operator of the supersymmetry algebra.

Like for the Poincaré algebra, we are interested in unitary representations acting on a Hilbert space. We can choose a basis of $\mathcal{H}$ of eigenvectors $\left\{\left|p_a, m^2, z^2\right\rangle\right\}$ with

$$P_a \left|p_a, m^2, z^2\right\rangle = p_a \left|p_a, m^2, z^2\right\rangle , \tag{80a}$$

$$P^2 \left|p_a, m^2, z^2\right\rangle = -m^2 \left|p_a, m^2, z^2\right\rangle , \tag{80b}$$

$$C \left|p_a, m^2, z^2\right\rangle = z^2 \left|p_a, m^2, z^2\right\rangle . \tag{80c}$$

Physically, the ket $\left|p_a, m^2, z^2\right\rangle$ describes a particle of mass m with energy-momentum p_a. To find an interpretation of z, we have to distinguish massive and massless particles. In the end of this paragraph, we consider the action of the operators on the particular state $\left|p_a, m^2, z^2\right\rangle$ of a massive particle at rest,

$$p_a = (-m, 0, 0, 0) . \tag{81}$$

On such a state, the Casimir operator C takes the form

$$C = m^2 Z_0^2 + m^2 Z^2 = m^4 S^2 , \tag{82}$$

where

$$S_0 = 0 , \qquad S_i = \frac{1}{m} Z_i . \tag{83}$$

The equation (78) becomes

$$[S_i, S_j] = i\varepsilon_{ijk} S_k , \tag{84}$$

and so on S_i has the interpretation of an angular momentum. Consequently, the second Casimir operator is on a massive particle state

$$C = m^4 y (y+1) , \tag{85}$$

where $y \in \mathbb{N}/2$ is the superspin of the representation.

4.2 Positivity of the energy[12]

By using (52d), we obtain

$$P_a = -\frac{1}{2}\left(\bar{\sigma}_a\right)^{\dot{\alpha}\alpha}\left\{Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}\right\} \ , \tag{86}$$

and this proves that the expectation value of the energy P^0 is positive,

$$\begin{aligned} 2\langle\psi| P^0 |\psi\rangle &= \langle\psi| \left\{Q^I_1, \bar{Q}^I_{\dot{1}}\right\} |\psi\rangle + \langle\psi| \left\{Q^I_2, \bar{Q}^I_{\dot{2}}\right\} |\psi\rangle \\ &= \left\| Q^I_1 |\psi\rangle \right\|^2 + \left\| \bar{Q}^I_{\dot{1}} |\psi\rangle \right\|^2 + \left\| Q^I_2 |\psi\rangle \right\|^2 + \left\| \bar{Q}^I_{\dot{2}} |\psi\rangle \right\|^2 \geq 0 \end{aligned} \tag{87}$$

The positivity of the energy is a consequence of the supersymmetric extension of the Poincaré algebra; the Poincaré algebra admits representation with negative energy. Moreover, we have

$$P^0 |\psi\rangle = 0 \ , \tag{88}$$

if and only if

$$Q^I_\alpha |\psi\rangle = 0 \ , \qquad \bar{Q}^I_{\dot{\alpha}} |\psi\rangle = 0 \ . \tag{89}$$

So a state is supersymmetrically invariant if and only if it has zero energy. This property is important for the spontaneously breaking of the supersymmetry which is characterized by a vacuum state with non-zero energy.

4.3 Bosons and fermions[13]

The Hilbert space $\mathcal{H}$ can be divided into two parts: bosons $\mathcal{B}$ for which the spin is integer and fermions $\mathcal{F}$ for which the spin is half-integer

$$\mathcal{H} = \mathcal{B} \sqcup \mathcal{F} \ . \tag{90}$$

The bosons are transforming under a representation $(m/2, n/2)$ for the Lorentz group for which $m+n$ is even, and the fermions in a representation $(m/2, n/2)$ with $m+n$ odd. Since the Pauli-Lubanski vector W_a commutes with all the even generators M_{ab}, P_a and T_l, the spin s defined by (175) can not be changed by the even generators,

$$P\mathcal{B} \subset \mathcal{B} \ , \qquad P\mathcal{F} \subset \mathcal{F} \ , \tag{91}$$

where P denotes any even generator. Since the odd generators are in the $(1/2, 0)$ or $(0, 1/2)$ representation, for any odd generator Q we have

$$Q\mathcal{B} \subset \mathcal{F} \ , \qquad Q\mathcal{F} \subset \mathcal{B} \ . \tag{92}$$

For many representations which are physically interesting, the operators P_a are one-to-one and consequently, by using (86) we obtain

$$\mathcal{F} = P_a\mathcal{F} = \left\{Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}\right\}\mathcal{F} \subset \left(Q^I_\alpha + \bar{Q}^I_{\dot{\alpha}}\right)\mathcal{B} \subset \mathcal{F} \ . \tag{93}$$

This prove that the number of bosons is equal to the number of fermions.

Then we consider two one-particle states $|b\rangle$ and $|f\rangle$ of respective mass m_b and m_f which are related by a supersymmetric generator,

$$Q^I_\alpha |b\rangle = |f\rangle \ . \tag{94}$$

Since P^2 is a Casimir operator,

$$-m_f^2 |f\rangle = P^2 |f\rangle = P^2 Q^I_\alpha |b\rangle = Q^I_\alpha P^2 |b\rangle = -m_b^2 Q^I_\alpha |b\rangle = -m_b^2 |f\rangle \ , \tag{95}$$

hence $m_f = m_b$. However, we do not observe in nature a boson field with the same mass as the electron, so we conclude that if the Hamiltonian describing these interaction is supersymmetric,

$$\left[b^\dagger, Q^I_\alpha\right] = f^\dagger \ , \tag{96}$$

the supersymmetry should be spontaneously broken.

[12] This subsection comes form the paragraph 3.2 of [11].

[13] The ideas of this subsection come from subsection 3.1 of [7] and chapter 4 of [8].

A Notations

A.1 Indices

In general, the Lorentz indices are denoted by a, b, c and d and run form 0 to 3. They are lowered with the metric η_{ab} and raised with the inverse metric η^{ab},

$$\eta = \left(\eta_{ab}\right) , \qquad \eta^{-1} = \left(\eta^{ab}\right) , \tag{97}$$

where

$$\eta = \mathrm{diag}\left(-1, 1, 1, 1\right) . \tag{98}$$

The spinor indices are α, β, γ and δ and run form 1 to 2. The spinor indices are raised and lowered with the totally antisymmetric tensor[14],

$$\varepsilon = \left(\varepsilon^{\alpha\beta}\right) , \qquad \varepsilon^{-1} = \left(\varepsilon_{\alpha\beta}\right) , \tag{99}$$

where

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{100}$$

A.2 Pauli matrices and generators of $\mathsf{SL}\left(2, \mathbb{C}\right)$

The Pauli matrices are

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{101}$$

and we define

$$\bar{\sigma}_a = \varepsilon \sigma_a^t \varepsilon^{-1} . \tag{102}$$

The aim defining the $\bar{\sigma}$ is to have the following useful property,

$$\mathrm{tr}\left(\bar{\sigma}_a \sigma_b\right) = -2\eta_{ab} . \tag{103}$$

In components, we write

$$\sigma_a = \left(\left(\sigma_a\right)_{\alpha\dot{\alpha}}\right) , \qquad \bar{\sigma}_a = \left(\left(\bar{\sigma}_a\right)^{\dot{\alpha}\alpha}\right) , \tag{104}$$

and therefore

$$\left(\bar{\sigma}_a\right)^{\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \left(\sigma_a\right)_{\beta\dot{\beta}} . \tag{105}$$

The generators of $\mathsf{SL}\left(2, \mathbb{C}\right)$ are defined as

$$\sigma_{ab} = -\frac{1}{4}\left(\sigma_a \bar{\sigma}_b - \sigma_b \bar{\sigma}_a\right) , \qquad \bar{\sigma}_{ab} = -\frac{1}{4}\left(\bar{\sigma}_a \sigma_b - \bar{\sigma}_b \sigma_a\right) . \tag{106}$$

B Space-time symmetry groups

In this section we will discuss standard results concerning the Lorentz and Poincaré group and their representations. While studying the representations we will introduce the two-component notation for spinors, which is useful for supersymmetry.

[14] The reason of this convention is explained in subsection B.4.

B.1 Lorentz group[15]

The Lorentz group is the space-time symmetry which leaves the Lorentzian metric invariant,

$$\mathsf{O}\,(3,1)=\left\{\Lambda\in\mathsf{GL}\,(4,\mathbb{R}):\ \|Ax\|^2=\|x\|^2\right\}\ , \tag{107}$$

where the norm is defined by

$$\|x\|^2=x^t\eta x\ . \tag{108}$$

This group has four connected components which are characterized by

$$\det\Lambda=\pm 1\ ,\qquad\qquad \operatorname{sign}\Lambda^0{}_0=\pm 1\ . \tag{109}$$

The subset of $\mathsf{O}\,(3,1)$ for which $\det\Lambda=1$ is $\mathsf{SO}\,(3,1)$ and the component connected to the identity $\mathsf{SO}_0\,(3,1)$,

$$\mathsf{SO}_0\,(3,1)=\left\{\Lambda\in\mathsf{SO}\,(3,1):\ \det\Lambda=\operatorname{sign}\Lambda^0{}_0=1\right\}\ . \tag{110}$$

By definition, the group $\mathsf{SO}_0\,(3,1)$ is connected but it can be shown that it is not simply connected[16]. Now the aim is to find the universal covering[17] group $\mathsf{Spin}\,(3,1)$ of $\mathsf{SO}_0\,(3,1)$. We are interested in the universal covering because the representations of a simply connected Lie group are in one-to-one correspondence to representations of the Lie algebra.

Theorem. *We have the identification*

$$\mathsf{Spin}\,(3,1)\simeq\mathsf{SL}\,(2,\mathbb{C})\ , \tag{111}$$

and moreover $\mathsf{SL}\,(2,\mathbb{C})$ *is a double covering of* $\mathsf{SO}_0\,(3,1)$*,*

$$\mathsf{SO}_0\,(3,1)\simeq\mathsf{SL}\,(2,\mathbb{C})\,/\mathbb{Z}_2\ . \tag{112}$$

Proof. Let H denote the space of Hermitian matrices in two dimensions. A basis for H is given by the four Pauli matrices (101). Consequently, we obtain an isomorphism between $\mathbb{R}^{(3,1)}$ and H by

$$\begin{aligned}\phi:\mathbb{R}^{(3,1)}&\to H\\ x&\mapsto x^a\sigma_a\end{aligned}\ ,\qquad\qquad \begin{aligned}\phi^{-1}:H&\to\mathbb{R}^{(3,1)}\\ x&\mapsto -\tfrac{1}{2}\operatorname{tr}\left(\bar\sigma^a x\right)\end{aligned}\ , \tag{113}$$

where we have used (103). For $N\in\mathsf{SL}\,(2,\mathbb{C})$ we consider the map ρ defined as

$$\begin{aligned}\rho\,(N):H&\to H\\ x&\mapsto NxN^\dagger\ .\end{aligned} \tag{114}$$

Since the map ϕ satisfies

$$\|x\|^2=\det\left(\phi\,(x)\right)\ , \tag{115}$$

the map π defined as

$$\pi\,(N)=\phi^{-1}\circ\rho\,(N)\circ\phi\ , \tag{116}$$

has the following property

$$\|\pi\,(N)\,x\|^2=\det\left(\rho\,(N)\circ\phi\,(x)\right)=\det\left(N\phi\,(x)\,N^\dagger\right)=\det\left(\phi\,(x)\right)=\|x\|^2\ . \tag{117}$$

Consequently, $\pi\,(N)\in\mathsf{O}\,(3,1)$ and since $\mathsf{SL}\,(2,\mathbb{C})$ is connected, we have constructed a map

$$\pi:\mathsf{SL}\,(2,\mathbb{C})\to\mathsf{SO}_0\,(1,3)\ , \tag{118}$$

which is an homomorphism. For the latter we can show that

$$\ker\pi=\{\pm 1\}\ , \tag{119}$$

and hence this shows the identification

$$\mathsf{SO}_0\,(3,1)\simeq\mathsf{SL}\,(2,\mathbb{C})\,/\mathbb{Z}_2\ . \tag{120}$$

Since $\mathsf{SL}\,(2,\mathbb{C})$ is simply connected it is the universal covering of $\mathsf{SO}_0\,(3,1)$. □

[15] This part is highly inspired by the subsection 1.1 of [3].

[16] A connected space is said simply connected if any closed path can be shrunk to a point.

[17] The universal covering of a space X is a simply connected space $\tilde X$ together with a local one-to-one map $\pi:\tilde X\to X$. The universal covering is unique up to an isomorphism. For example, the universal covering of the circle S^1 is $\mathbb{R}$ with the map $\theta\mapsto e^{i\theta}$.

B.2 Lorentz algebra

The Lorentz algebra of $\mathsf{SO}_0(3,1)$ is

$$\mathfrak{so}(3,1) = \left\{\Lambda \in \mathfrak{gl}(4,\mathbb{R}) :\ \Lambda^t\eta = -\eta\Lambda\right\} , \tag{121}$$

and a basis is given by the six generators J_{ab} defined as

$$(J_{ab})_{ij} = \delta_{ai}\eta_{bj} - \delta_{bi}\eta_{aj} . \tag{122}$$

The commutation relations between the generators are

$$[J_{ab}, J_{cd}] = \eta_{ad}J_{bc} - \eta_{ac}J_{bd} + \eta_{bc}J_{ad} - \eta_{bd}J_{ac} . \tag{123}$$

B.3 Lie algebra of $\mathsf{SL}(2,\mathbb{C})$

The Lie algebra of $\mathsf{SL}(2,\mathbb{C})$ is

$$\mathfrak{sl}(2,\mathbb{C}) = \{N \in \mathfrak{gl}(2,\mathbb{C}) :\ \operatorname{tr} N = 0\} . \tag{124}$$

The algebra $\mathfrak{sl}(2,\mathbb{C})$ has a complex-dimension of three. Thus a basis of $\mathfrak{sl}(2,\mathbb{C})$ viewed as a complex algebra is given by the three Pauli matrices $\{\sigma_i\}_{i=1}^3$. As a real algebra, a basis is given by the six generators $\{\sigma_{ab}\}_{a<b}$ defined through (106). Consequently, nearly[18] all elements $N \in \mathsf{SL}(2,\mathbb{C})$ can be expressed as

$$N = \exp\left\{a^i\sigma_i\right\} = \exp\left\{\frac{1}{2}w^{ab}\sigma_{ab}\right\} , \tag{125}$$

where $a^i \in \mathbb{C}$ and $w^{ab} \in \mathbb{R}$, with $w^{(ab)} = 0$. Form this, the structure constants of $\mathfrak{sl}(2,\mathbb{C})$ and therefore also of $\mathfrak{so}(3,1)$ are

$$[\sigma_{ab}, \sigma_{cd}] = \eta_{ad}\sigma_{bc} - \eta_{ac}\sigma_{bd} + \eta_{bc}\sigma_{ad} - \eta_{bd}\sigma_{ac} . \tag{126}$$

This result can be deduced directly from (123), because the Lorentz group $\mathsf{SO}_0(3,1)$ is locally isomorphic to its universal covering $\mathsf{SL}(2,\mathbb{C})$ the corresponding Lie algebra are isomorphic

$$\mathfrak{so}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C}) . \tag{127}$$

B.4 Representations of the Lorentz group[19]

Any representation T of $\mathsf{SO}_0(3,1)$ induces a representation $\tilde{T}$ of $\mathsf{SL}(2,\mathbb{C})$ by

$$\tilde{T}(N) = T(\pi(N)) . \tag{128}$$

where π is the covering map. The representation $\tilde{T}$ constructed in this way satisfies

$$\tilde{T}(N) = \tilde{T}(-N) , \tag{129}$$

but there exist representations of $\mathsf{SL}(2,\mathbb{C})$ for which

$$\tilde{T}(N) = -\tilde{T}(-N) . \tag{130}$$

Now we will construct the irreducible representations of $\mathsf{SL}(2,\mathbb{C})$. The simplest representations of $\mathsf{SL}(2,\mathbb{C})$ are of dimension two,

$$\rho_\circ(N) = N , \qquad \rho^\circ(N) = \left(N^\mathsf{T}\right)^{-1} , \qquad \rho_\bullet(N) = \bar{N} , \qquad \rho^\bullet(N) = \left(\bar{N}^\mathsf{T}\right)^{-1} . \tag{131}$$

[18] More precisely, the exponential map covers $\mathsf{SL}(2,\mathbb{C})$ apart a two-dimensional surface

$$\left\{N\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}N^{-1},\ N \in \mathsf{SL}(2,\mathbb{C})\right\} .$$

Particularly, for each $N \in \mathsf{SL}(2,\mathbb{C})$ at least N or $-N$ admits such an exponential form.

[19] This section is inspired by the subsection 4.1 of [12], and by the section 1.2 of [3].

The representations $\rho_\circ$ and ρ° respectively $\rho_\bullet$ and $\rho^\bullet$ are equivalent because

$$\rho^\circ(N) = \varepsilon\rho_\circ(N)\varepsilon^{-1}\,, \qquad \rho^\bullet(N) = \varepsilon\rho_\bullet(N)\varepsilon^{-1}\,, \tag{132}$$

where ε is given by (100). We denote by $\mathcal{V}_\circ$, $\mathcal{V}^\circ$, $\mathcal{V}_\bullet$ and $\mathcal{V}^\bullet$ the vectors spaces isomorphic to $\mathbb{C}^2$ on which the corresponding representations act. The elements of these spaces are the two-component spinors and we adopt the following component notation

$$(\xi_\alpha) \in \mathcal{V}_\circ\,, \qquad (\xi^\alpha) \in \mathcal{V}_\circ\,, \qquad (\bar{\xi}_{\dot\alpha}) \in \mathcal{V}_\bullet\,, \qquad (\bar{\xi}^{\dot\alpha}) \in \mathcal{V}^\bullet\,. \tag{133}$$

In components, a matrix $N \in \mathsf{SL}(2,\mathbb{C})$ is denoted

$$N = \left(N_\alpha{}^\beta\right)\,, \qquad \bar{N} = \left(\bar{N}_{\dot\alpha}{}^{\dot\beta}\right)\,. \tag{134}$$

By deciding to raise and lower indices with the totally antisymmetric tensor (99), the representations are explicitly given in term of spinors by

$$\begin{aligned}\rho_\circ(N) : \mathcal{V}_\circ &\to \mathcal{V}_\circ \\ \xi_\alpha &\mapsto N_\alpha{}^\beta\xi_\beta\end{aligned}\,, \qquad \begin{aligned}\rho^\circ(N) : \mathcal{V}^\circ &\mapsto \mathcal{V}^\circ \\ \xi^\alpha &\mapsto N^\alpha{}_\beta\xi^\beta\end{aligned}\,, \tag{135}$$

$$\begin{aligned}\rho_\bullet(N) : \mathcal{V}_\bullet &\to \mathcal{V}_\bullet \\ \bar{\xi}_{\dot\alpha} &\mapsto \bar{N}_{\dot\alpha}{}^{\dot\beta}\bar{\xi}_{\dot\beta}\end{aligned}\,, \qquad \begin{aligned}\rho^\bullet(N) : \mathcal{V}^\bullet &\mapsto \mathcal{V}^\bullet \\ \bar{\xi}^{\dot\alpha} &\mapsto \bar{N}^{\dot\alpha}{}_{\dot\beta}\bar{\xi}^{\dot\beta}\end{aligned}\,. \tag{136}$$

Now we can construct other representations of $\mathsf{SL}(2,\mathbb{C})$ by using the tensor product,

$$\rho_{\circ^m\bullet^n} = \bigotimes_{i=1}^m \rho_\circ \otimes \bigotimes_{j=1}^n \rho_\bullet\,, \qquad \rho^{\bullet^n\circ^m} = \bigotimes_{j=1}^n \rho^\circ \otimes \bigotimes_{i=1}^m \rho^\bullet\,, \tag{137}$$

and there are equivalent by construction due to (132). A spinor with m undotted indices and n dotted indices is an element of tensor products,

$$\left(\xi_{\alpha_1\cdots\alpha_m\dot\alpha_1\cdots\dot\alpha_n}\right) \in \bigotimes_{i=1}^m \mathcal{V}_\circ \otimes \bigotimes_{j=1}^n \mathcal{V}_\bullet\,, \qquad \left(\bar{\xi}^{\dot\alpha_1\cdots\dot\alpha_n\alpha_1\cdots\alpha_m}\right) \in \bigotimes_{i=1}^n \mathcal{V}^\bullet \otimes \bigotimes_{j=1}^m \mathcal{V}^\circ\,, \tag{138}$$

and transforms as

$$\rho_{\circ^m\bullet^n}(N) : \xi_{\alpha_1\cdots\alpha_m\dot\alpha_1\cdots\dot\alpha_n} \mapsto N_{\alpha_1}{}^{\beta_1}\cdots N_{\alpha_m}{}^{\beta_m}\bar{N}_{\dot\alpha_1}{}^{\dot\beta_1}\cdots\bar{N}_{\dot\alpha_n}{}^{\dot\beta_n}\xi_{\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n}\,, \tag{139a}$$

$$\rho^{\bullet^m\circ^n}(N) : \bar{\xi}^{\dot\alpha_1\cdots\dot\alpha_n\alpha_1\cdots\alpha_m} \mapsto \bar{N}^{\dot\alpha_1}{}_{\dot\beta_1}\cdots\bar{N}^{\dot\alpha_n}{}_{\dot\beta_n}N^{\alpha_1}{}_{\beta_1}\cdots N^{\alpha_m}{}_{\beta_m}\bar{\xi}^{\dot\beta_1\cdots\dot\beta_n\beta_1\cdots\beta_m}\,. \tag{139b}$$

The first equation proves that

$$\rho_{\circ^m\bullet^n}(N) = (-1)^{m+n}\rho_{\circ^m\bullet^n}(-N)\,. \tag{140}$$

The discussion of the previous section tells us that the representation $\rho_{\circ^m\bullet^n}$ of $\mathsf{SL}(2,\mathbb{C})$ is associated to a representation of $\mathsf{SO}_0(3,1)$ if and only if $m+n$ is even. In the case where $m=n$, we define a real tensor by the condition

$$\xi_{\alpha_1\cdots\alpha_m\dot\alpha_1\cdots\dot\alpha_n} = \bar{\xi}_{\dot\alpha_1\cdots\dot\alpha_n\alpha_1\cdots\alpha_m}\,. \tag{141}$$

By defining the components of the Pauli matrices (101) as

$$\sigma_a = \left((\sigma_a)_{\alpha\dot\alpha}\right)\,, \qquad \bar{\sigma}_a = \left((\bar{\sigma}_a)^{\dot\alpha\alpha}\right)\,, \tag{142}$$

any real vector in the representation $(1/2,1/2)$ is equivalent to a standard four-vector ξ_a with

$$\xi_{\alpha\dot\alpha} = (\sigma^a)_{\alpha\dot\alpha}\xi_a\,, \qquad \xi_a = -\frac{1}{2}(\bar{\sigma}_a)^{\dot\alpha\alpha}\xi_{\alpha\dot\alpha}\,, \tag{143}$$

which is consistent with the completeness relation

$$\left(\sigma^a\right)_{\alpha\dot{\alpha}}\left(\bar{\sigma}_a\right)^{\dot{\beta}\beta} = -2\delta_\alpha^\beta\delta_{\dot{\alpha}}^{\dot{\beta}} \,. \tag{144}$$

For $m > 1$ or $n > 1$ the representation $\rho_{\circ^m \bullet^n}$ on unconstrained spinors $\xi_{\alpha_1\cdots\alpha_m\dot{\alpha}_1\cdots\dot{\alpha}_n}$ is not reducible. For example, for $m = 2$ and $n = 0$, we can perform the following decomposition,

$$\xi_{\alpha\beta} = \xi_{(\alpha\beta)} + \xi_{[\alpha\beta]} = \xi_{(\alpha\beta)} + \varepsilon_{\alpha\beta}\xi_{[12]} \,. \tag{145}$$

Then we define the set of the spinors which are totally symmetric in their undotted indices and independently in their dotted indices,

$$(m/2, n/2) = \left\{ \xi_{\alpha_1\cdots\alpha_m\dot{\alpha}_1\cdots\dot{\alpha}_n} \in \bigotimes_{i=1}^{m} \mathcal{V}_\circ \otimes \bigotimes_{j=1}^{n} \mathcal{V}_\bullet : \ \xi_{\alpha_1\cdots\alpha_m\dot{\alpha}_1\cdots\dot{\alpha}_n} = \xi_{(\alpha_1\cdots\alpha_m)(\dot{\alpha}_1\cdots\dot{\alpha}_n)} \right\} . \tag{146}$$

In general, the restriction of the representation $\rho_{\circ^m \bullet^n}$ on $(m/2, n/2)$ is reducible.

For example, we will decompose a tensor of rank two ξ_{ab} in terms of irreducible representations. In terms of dotted and undotted indices, we defined

$$\xi_{\alpha\beta\dot{\alpha}\dot{\beta}} = \left(\sigma^a\right)_{\alpha\dot{\alpha}}\left(\sigma^b\right)_{\beta\dot{\beta}}\xi_{ab} \,. \tag{147}$$

Then we can write the tensor $\xi_{\alpha\beta\dot{\alpha}\dot{\beta}}$ as

$$\begin{aligned} \xi_{\alpha\beta\dot{\alpha}\dot{\beta}} &= \xi_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + \xi_{(\alpha\beta)[\dot{\alpha}\dot{\beta}]} + \xi_{[\alpha\beta](\dot{\alpha}\dot{\beta})} + \xi_{[\alpha\beta][\dot{\alpha}\dot{\beta}]} \\ &= \xi_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + \varepsilon_{\dot{\alpha}\dot{\beta}}\xi_{(\alpha\beta)[\dot{1}\dot{2}]} + \varepsilon_{\alpha\beta}\xi_{[12](\dot{\alpha}\dot{\beta})} + \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\xi_{[12][\dot{1}\dot{2}]} \,, \end{aligned} \tag{148}$$

and so on is an element of $(1,1) \oplus (1,0) \oplus (0,1) \oplus (0,0)$. In particular, if the tensor ξ_{ab} is symmetric we obtain

$$\xi_{\alpha\beta\dot{\alpha}\dot{\beta}} = \xi_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\xi_{[12][\dot{1}\dot{2}]} \,, \tag{149}$$

and if it is antisymmetric,

$$\xi_{\alpha\beta\dot{\alpha}\dot{\beta}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\xi_{(\alpha\beta)[\dot{1}\dot{2}]} + \varepsilon_{\alpha\beta}\xi_{[12](\dot{\alpha}\dot{\beta})} \,. \tag{150}$$

For the antisymmetric case, we define

$$\xi_{\alpha\beta} = \frac{1}{2}\left(\sigma^{ab}\right)_{\alpha\beta}\xi_{ab} \,, \qquad \bar{\xi}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}\left(\bar{\sigma}^{ab}\right)_{\dot{\alpha}\dot{\beta}}\xi_{ab} \,, \tag{151}$$

where σ^{ab} and $\bar{\sigma}^{ab}$ are the $\mathsf{SL}(2,\mathbb{C})$ generators (106). Thus we obtain the explicit forms

$$\xi_{\alpha\beta\dot{\alpha}\dot{\beta}} = 2\varepsilon_{\dot{\alpha}\dot{\beta}}\xi_{\alpha\beta} + 2\varepsilon_{\alpha\beta}\bar{\xi}_{\dot{\alpha}\dot{\beta}} \,, \qquad \xi_{ab} = \left(\sigma_{ab}\right)^{\alpha\beta}\xi_{\alpha\beta} - \left(\bar{\sigma}_{ab}\right)^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\alpha}\dot{\beta}} \,. \tag{152}$$

B.5 Poincaré group

The Poincaré group is the space-time group which leaves the distance invariant,

$$\mathsf{P}(3,1) = \left\{\Lambda : \mathbb{R}^4 \to \mathbb{R}^4 : \ \|\Lambda x - \Lambda y\|^2 = \|x-y\|^2\right\} . \tag{153}$$

The Poincaré group is the semi-direct product of the translation group $\mathsf{T}(4) \simeq \mathbb{R}^4$ with the Lorentz group,

$$\mathsf{P}(3,1) = \mathsf{O}(3,1) \ltimes \mathsf{T}(4) \ , \tag{154}$$

or more explicitly the set $\mathsf{O}(3,1) \times \mathsf{T}(4)$ with the following product,

$$\left(\Lambda_2, a_2\right)\left(\Lambda_1, a_1\right) = \left(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2\right) \ . \tag{155}$$

The Poincaré group can be realized explicitly as a matrix group,

$$\mathsf{P}(3,1) \simeq \left\{ \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}, \ \Lambda \in \mathsf{O}(3,1) \ , \ a \in \mathbb{R}^4 \right\} , \tag{156}$$

via which the semi-direct product (155) is realized.

B.6 Poincaré algebra

The Poincaré algebra is denoted by $\mathfrak{p}(3,1)$ and with the explicit realization (156) we simply obtain

$$\mathfrak{p}(3,1) \simeq \left\{ \begin{pmatrix} \Lambda & a \\ 0 & 0 \end{pmatrix}, \ \Lambda \in \mathfrak{so}(3,1), \ a \in \mathbb{R}^4 \right\} . \tag{157}$$

A basis for the Poincaré algebra is given by six generators M_{ab} and four P_a defined by

$$M_{ab} = \begin{pmatrix} J_{ab} & 0 \\ 0 & 0 \end{pmatrix} , \qquad\qquad P_a = \begin{pmatrix} 0 & e_a \\ 0 & 0 \end{pmatrix} , \tag{158}$$

where J_{ab} are the generators (122) of $\mathfrak{so}(3,1)$ and $(e_a)_i = \delta_{ai}$. Each element of the Poincaré group viewed as (156) can be expressed as

$$\begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} = \exp\left\{ \frac{1}{2} w^{ab} M_{ab} + b^a P_a \right\} , \tag{159}$$

where $w^{ab} \in \mathbb{R}$ with $w^{(ab)} = 0$, and $b^a \in \mathbb{R}$. By using the relation (123), the commutation relations between the generators are

$$[M_{ab}, M_{cd}] = \eta_{ad} M_{bc} - \eta_{ac} M_{bd} + \eta_{bc} M_{ad} - \eta_{bd} M_{ac} , \tag{160a}$$
$$[P_a, P_b] = 0 , \tag{160b}$$
$$[M_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b . \tag{160c}$$

B.7 Unitary representations

In quantum field theory, we are interested in unitary representations which act on a complex Hilbert space $\mathcal{H}$. In a unitary representation, the generators are antihermitian,

$$M_{ab}^\dagger = -M_{ab} , \qquad\qquad P_a^\dagger = -P_a . \tag{161}$$

From a physical point of view we want to have Hermitian generators. To this end, we define new generators

$$M_{ab} \mapsto -i M_{ab} , \qquad\qquad P_a \mapsto i P_a , \tag{162}$$

and therefore equation (159) becomes

$$U(\Lambda, a) = \exp\left\{ \frac{i}{2} w^{ab} M_{ab} - i b^a P_a \right\} . \tag{163}$$

With these new generators, the commutation relations (160) become

$$[M_{ab}, M_{cd}] = i\eta_{ad} M_{bc} - i\eta_{ac} M_{bd} + i\eta_{bc} M_{ad} - i\eta_{bd} M_{ac} , \tag{164a}$$
$$[P_a, P_b] = 0 , \tag{164b}$$
$$[M_{ab}, P_c] = i\eta_{bc} P_a - i\eta_{ac} P_b . \tag{164c}$$

B.8 Casimir operators[20]

To study the representations of the Poincaré group it is useful to find the Casimir operators[21]. The first Casimir operator for the Poincaré group is the mass operator

$$P^2 = P_a P^a , \tag{165}$$

[20] This part is inspired by the section 1.3.5 of [13].

[21] In a simplistic way, a Casimir operator is an operator of the form $C_a C^a$ which commutes with all the element of the Lie algebra.

since

$$\left[P^2, P_a\right] = 0\,, \qquad\qquad \left[P^2, M_{ab}\right] = 0\,. \tag{166}$$

The second Casimir operator is more subtle, and for this we introduce the Pauli-Lubanski vector

$$W_a = \frac{1}{2}\varepsilon_{abcd}M^{bc}P^d\,. \tag{167}$$

By using the Poincaré algebra 164, we obtain

$$[W_a, P_b] = 0\,, \qquad [W_a, M_{bc}] = i\eta_{ab}W_c - i\eta_{ac}W_b\,, \qquad [W_a, W_b] = i\varepsilon_{abcd}P^cW^d\,, \tag{168}$$

and therefore

$$\left[W^2, P_a\right] = 0\,, \qquad\qquad \left[W^2, M_{ab}\right] = 0\,. \tag{169}$$

Consequently, W^2 is the second Casimir operator of the Poincaré algebra. The irreducible representations of the Poincaré group can be classified by P^2 and W^2.

Since the operators P_a, P^2 and W^2 commute, we can choose a basis of $\mathcal{H}$ of eigenvectors $\left\{\left|p_a, m^2, w^2\right\rangle\right\}$ with

$$P_a\left|p_a, m^2, w^2\right\rangle = p_a\left|p_a, m^2, w^2\right\rangle\,, \tag{170a}$$
$$P^2\left|p_a, m^2, w^2\right\rangle = -m^2\left|p_a, m^2, w^2\right\rangle\,, \tag{170b}$$
$$W^2\left|p_a, m^2, w^2\right\rangle = w^2\left|p_a, m^2, w^2\right\rangle\,. \tag{170c}$$

From a physical point of view, the ket $\left|p_a, m^2, w^2\right\rangle$ corresponds to a particle of mass m and momentum p_a. Some representations of the Poincaré group have a negative mass square m^2 and are therefore not physical.

To find an interpretation of w, we have to distinguish massive and massless particles. Here we will only consider a massive particle of mass m at rest,

$$p_a = (-m, 0, 0, 0)\,. \tag{171}$$

In this whole section we consider the action of the operators on the particular state $\left|p_a, m^2, w^2\right\rangle$. On the latter, the operator W_a takes the form

$$W_0 = 0\,, \qquad\qquad W_i = mS_i\,, \tag{172}$$

where S_i are three operators defined as

$$S_i = \frac{1}{2}\varepsilon_{ijk}M^{ij}\,. \tag{173}$$

Due to the equation (168), we obtain an $\mathfrak{su}(2)$ algebra

$$[S_i, S_j] = i\varepsilon_{ijk}S_k\,, \tag{174}$$

which proves that S_i is an angular momentum. Then, we have

$$W^2 = m^2S^2 = m^2s\,(s+1)\,, \tag{175}$$

where $s \in \mathbb{N}/2$, is the spin of the particle. Consequently, for a massive particle, the second Casimir operator is related to the spin,

$$w = m\sqrt{s\,(s+1)}\,. \tag{176}$$

For a massless particle, we can show that the number w is related to the helicity $\lambda \in \mathbb{Z}/2$,

$$w = m\sqrt{\lambda}\,. \tag{177}$$

Consequently, the two Casimir operators classify the irreducible representations in terms of the mass and the spin for massive particles or helicity for massless particles.

Nomenclature

$(m/2, n/2)$	Irreducible representation of the Lorentz group	(146)
$\left(a^l\right)^{IJ}$	Complex constants linking X^{IJ} to T_l	(57)
$[a, b\}$	Super-commutator in a superalgebra	(11)
ε	Completly antisymmetric tensor in dimension two	(100)
η	Lorentzian metric	(97)
$\mathfrak{g}_{0,1}$	Respectivly even and odd part of a $\mathbb{Z}_2$-graded algebra	(7)
Int	Internal symmetry group	(3)
$\mathfrak{int}$	Internal symmetry algebra	(4)
M_{ab}	Rotation generators of the Poincaré group	(162)
$M_{\alpha\beta}$, $\bar{M}_{\dot{\alpha}\dot{\beta}}$	Rotation generators M_{ab} in spinor notation	(21)
$\mathsf{P}\,(3,1)$	Poincaré group	(153)
$\mathfrak{p}\,(3,1)$	Poincaré algebra	(157)
P_a	Translation generators of the Poincaré algebra	(162)
$P_{\alpha\dot{\alpha}}$	Translation generators P_a in spinor notation	(20)
Q_α^I	Odd generators of the superalgebra	(19)
Q_r^I	Odd generators of the superalgebra in four-component notation	(53)
σ_a	Pauli matrices	(101)
$\bar{\sigma}_a$	Modified Pauli matrices	(102)
σ_{ab}, $\bar{\sigma}_{ab}$	Generators of $\mathsf{SL}\,(2,\mathbb{C})$	(106)
$\mathsf{SL}\,(2,\mathbb{C})$	Complex special linear group in two dimensions	(111)
$\mathfrak{sl}\,(2,\mathbb{C})$	Special algebra of traceless matrices in two dimensions	(124)
$(S_l)^I{}_J$	Complex constants between Q_α^I and T_l	(45)
$\mathsf{SO}_0\,(3,1)$	Proper orthochronous Lorentz group	(110)
$\mathfrak{so}\,(3,1)$	Lorentz algebra	(121)
T_l	Generators of the internal symmetry group	(5)
W_a	Pauli-Lubanski vector	(167)
X^{IJ}	Central charges	(42)
Z_a	Analogue to the Pauli-Lubanski vector for the superalgebra	(76)

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