SUSY in dimensions other than 4

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May 17, 2010

Abstract

In this paper we will consider how to construct supersymmetric models in dimension higher than four. For this it is necessary to construct spinors in higher dimensions, since supersymmetry is by definition a symmetry which relates tensorial and spinorial representations. Then we are going to construct a massless multiplet state in a similar way as for 4 dimensions but for other number of dimensions and at the end we will show how 4 dimensional lagrangian can be obtained by dimensional reduction of a higher dimensional one.

Contents

1 Spinor in higher dimension

The finite dimensional representation of the Lorentz algebra fall into two classes: some representations are contained in multiple direct product of the fundamental vector representation of the group and the other are not. The former are the tensor representations, the latter are the spinor representations. The spinor representations of the Lorentz algebra are characterized by the fact that generators have half-integer eigenvalues, while tensorial representations have integer eigenvalues. An easy handle on the spinor representation is provided by the Dirac matrices and their properties. We will use the notation Γ_a for the Dirac matrices for dimensions different from four and γ_μ for four dimensions, except in chapter 2. These are irreducible representations of the Clifford algebra. The Clifford algebra in d dimensions is given by:

$$\{\Gamma_a, \Gamma_b\} = \eta_{ab} \quad \text{with} \quad \eta_{ab} = \text{diag}(1, -1, ..., -1) \quad a, b = 0, 1, ..., d-1. \tag{1}$$

We find that

$$\Sigma_{ab} \equiv \frac{i}{4}[\Gamma_a, \Gamma_b] \tag{2}$$

form a representation of the Lorentz algebra in d dimensions because

$$[\Sigma_{ab}, \Sigma_{cd}] = i(\eta_{bc}\Sigma_{ad} - \eta_{bd}\Sigma_{ac} - \eta_{ac}\Sigma_{bd} + \eta_{ad}\Sigma_{bc}). \tag{3}$$

It will comes out that this representation is not necessarily irreducible. To find out if the representation is irreducible, we have to consider separately the case where d is an odd or even dimension. For even dimension we make use of a theorem from representation theory.

Theorem 1.1. For even dimension d and a given metric η_{ab}; $a, b = 0, 1, ..., d-1$, all irreducible representations of the Clifford algebra are equivalent and are $n \times n$ matrices ($\mathbb{C}$-matrices) with

$$n = 2^{d/2}$$

.

This means that given any set of matrices $\{\Gamma_a\}$ and $\{\Gamma_a^{'}\}$ both satisfying the Clifford algebra, then it exists a non-singular matrix S such that

$$\Gamma_a = S\Gamma_a S^{-1} \quad \forall \quad a = 0, 1, ..., d-1.$$

For odd dimension we again make use of a theorem.

Theorem 1.2. For a given odd dimension d and a given metric η_{ab}; $a, b = 0, 1, ..., d-1$, there are two equivalence classes of irreducible representations in terms of $n \times n$ matrices with

$$n = 2^{(d-1)/2}.$$

Example: If Γ_a is in an equivalence class, then $-\Gamma_a$ is in the other.

We now return to the even dimension. Knowing that for d even all irreducible representation are equivalent, take Γ_a to be such an irreducible representation of the Clifford algebra. Then

$$\Gamma_a, -\Gamma_a, \Gamma_a^\dagger, -\Gamma_a^\dagger, \Gamma_a^T, -\Gamma_a^T, \Gamma_a^*, -\Gamma_a^* \tag{4}$$

also satisfy the Clifford algebra.

Now we can introduce interwiners A, C, D such that:

$$A\Gamma_a A^{-1} = \Gamma_a^\dagger \tag{5}$$

$$C^{-1}\Gamma_a C = -\Gamma_a^T \tag{6}$$

$$(CA^T)^{-1}\Gamma_a(\underbrace{CA^T}_{\equiv D}) = -\Gamma_a^* \tag{7}$$

together with Γ_{d+1}, which satisfies:

$$\Gamma_{d+1}\Gamma_a\Gamma_{d+1}^{-1} = -\Gamma_a \tag{8}$$

where

$$\Gamma_{d+1} \equiv \Gamma_0 \ldots \Gamma_{d-1}. \tag{9}$$

From those equations we find that the interwiners satisfy

$$A = \alpha A^\dagger; \quad C = \eta C^T; \quad D = \delta(D^{-1})^*; \quad \Gamma_{d+1} = \beta\Gamma_{d+1}^{-1} \tag{10}$$

by taking then hermitian adjoint of (5), the negative of (8), the transposed of (6) and the complex conjugate of (7) with the additional condition

$$\alpha\alpha^* = \eta^2 = 1 \quad \text{and} \quad \delta = \delta^*.$$

We have here the freedom to scale α and δ and choose $\alpha = |\delta| = 1$. The remaining quantities β, η and δ are defined by the metric. For the matrix

$$\tilde{D} \equiv \Gamma_{d+1}^{-1}D \tag{11}$$

we have

$$\tilde{D}^{-1}\Gamma_a\tilde{D} = \Gamma_a^* \tag{12}$$

and

$$\tilde{D} = \tilde{\delta}\tilde{D}^{-1*} \quad \text{with} \quad \tilde{\delta} = \beta\delta. \tag{13}$$

For the odd dimensions we have to find out which of the

$$-\Gamma_a, \Gamma_a^\dagger, -\Gamma_a^\dagger, \Gamma_a^T, -\Gamma_a^T, \Gamma_a^*, -\Gamma_a^* \tag{14}$$

fall into the same equivalence class as Γ_a. This is determined by the behavior of Γ_{d+1} under the transformations

$$A\Gamma_a A^{-1} = \pm\Gamma_a^\dagger; \quad C^{-1}\Gamma_a C = \pm\Gamma_a^T \tag{15}$$

which always satisfy

$$A\Gamma_{d+1}A^{-1} = C^{-1}\Gamma_{d+1}C = \Gamma_{d+1} \propto \mathbb{1}. \tag{16}$$

The signs of β, η, δ and $\tilde{\delta}$ are listed in table 1,2 and 3:

Table 1: The value of $\beta = \Gamma_{d+1}^2$

	d=0,1 mod 4	d=2,3 mod 4
d_- even	+1	-1
d_- odd	-1	+1

The equivalence classes are given in table 4.

Table 2: The sign of η

d=	1	2	3	4	5	6	7	8	9	10	11	12
	+	-	-	-	-	+	+	+	+	-	-	-

Table 3: The signs of δ and $\tilde{\delta}$

d=	1	2	3	4	5	6	7	8	9	10	11	12
	+	++	+	+-	-	−	-	-+	+	++	+	+-

Going back to the Theorems, we first see that the number of complex spinor components n increase exponentially with d. Moreover in many dimension the spinor representation are not irreducible representation of the Lorentz group. We can impose chirality (Weyl) condition, reality (Majorana) condition or even both simultaneously.
The Chirality condition comes from the fact that we can generalize the γ_5 of the four dimensions as

$$\Gamma_{d+1}. \tag{17}$$

Note that for odd dimensions $\Gamma_{d+1} \propto \mathbb{1}$, so that chirality condition is not possible. But in even dimensions we find that

$$\{\Gamma_{d+1}, \Gamma_a\} = 0 \tag{18}$$

and from this we can evaluate that

$$[\Gamma_{d+1}, \Sigma_{ab}] = 0. \tag{19}$$

So that the generators Σ_{ab} cannot provide an irreducible representation of the Lorentz algebra. We may define a pair of Weyl irreducible representations $\Sigma^{\pm}_{ab}$ by projecting out the subspaces with $\Gamma_{d+1} = \pm\sqrt{\beta}$:

$$\Sigma^{\pm}_{ab} = \frac{1}{2}(\mathbb{1} \pm \sqrt{\beta}\Gamma_{d+1})\Sigma_{ab}. \tag{20}$$

In this way, we have defined a irreducible representation of the Lorentz algebra whose dimension is a half of the initial one.
The reality (Majorana) condition for a spinor have the general form

$$\psi = X\psi^* \tag{21}$$

with X some non-singular $n \times n$ matrix. Since an infinitesimal Lorentz transformation acts on a spinor as

$$\delta\psi = \frac{i}{2}\lambda^{ab}\Sigma_{ab}\psi = -\frac{1}{2}\lambda^{ab}\frac{1}{4}[\Gamma_a, \Gamma_b]\psi \tag{22}$$

and on the complex conjugate as

$$\delta\psi^* = -\frac{1}{2}\lambda^{ab}\frac{1}{4}([\Gamma_a, \Gamma_b])^*\psi^*. \tag{23}$$

λ^{ab} is a infinitesinal parameter of the transformation. The matrix X must have the property

$$([\Gamma_a, \Gamma_b])^* = X^{-1}[\Gamma_a, \Gamma_b]X \tag{24}$$

Table 4: The equivalence classes which contain Γ_a for arbitrary odd dimensions

	d≡1 mod 4	d≡3 mod 4
d_- even	$\Gamma_a, \Gamma_a^\dagger, \Gamma_a^T, \Gamma_a^*$	$\Gamma_a, \Gamma_a^\dagger, -\Gamma_a^T, -\Gamma_a^*$
d_- odd	$\Gamma_a, -\Gamma_a^\dagger, \Gamma_a^T, -\Gamma_a^*$	$\Gamma_a, -\Gamma_a^\dagger, -\Gamma_a^T, \Gamma_a^*$

such that the Majorana condition satisfy $\delta\psi = X\delta\psi^*$. We see that $X = D$ or $X = \Gamma_{d+1}^{-1} D \equiv \tilde{D}$ as defined earlier satisfy this condition.
Not in all cases, will this be consistent. Only if at least one of the conditions

$$DD^* = \delta = \mathbb{1} \quad \text{or} \quad \tilde{D}\tilde{D}^* = \tilde{\delta} = \mathbb{1} \tag{25}$$

is satisfied, we can have a Majorana condition. The reason is

$$(\psi)^* = (D\psi^*)^* \Leftrightarrow \psi^* = D^*\psi \overset{Majorana}{=} D^{-1}\psi \tag{26}$$

$$\Rightarrow DD^*\psi = \psi \tag{27}$$

and the same for $\tilde{D}$. A standard way of writing Majorana condition is

$$\psi = \psi^c \equiv C\overline{\psi}^T \tag{28}$$

with

$$\overline{\psi} \equiv \psi^\dagger A \tag{29}$$

and C defined earlier is the charge conjugation operator. Finally, we would like to impose both condition simultaneously. For this we must have

$$(\mathbb{1} \pm \sqrt{\beta}\Gamma_{d+1})\psi = D(\mathbb{1} \pm \sqrt{\beta}^*\Gamma_{d+1}^*)\psi^* \tag{30}$$

or the corresponding equation with $\tilde{D}$. Evaluating this gives

$$= D\psi^* \pm \sqrt{\beta}^* D\Gamma_{d+1}^* D^{-1} D\psi^* = (\mathbb{1} \pm \sqrt{\beta}^*\Gamma_{d+1})D\psi^* \tag{31}$$

Therefore we must have $\sqrt{\beta}$ real, i.e. $\beta = +1$ and $\delta = 1$. This implies that $\delta = \tilde{\delta} = 1$. Starting with $\tilde{D}$ would have led to the same results. To summarize our results see table 5.

Table 5: Chirality and reality of spinors in Minkowski space-time with $d \leq 12$

d	1	2	3	4	5	6	7	8	9	10	11	12
number of spinor dimension	1	2	2	4	4	8	8	16	16	32	32	64
Weyl spinors	-	x	-	x	-	x	-	x	-	x	-	x
Majorana spinors	x	x	x	x	-	-	-	x	x	x	x	x
Majorana Weyl spinors	-	x	-	-	-	-	-	-	-	x	-	-
Minimal spinor dimension	1	1	2	4	8	8	16	16	16	16	32	64

2 Construction of gamma matrices

We saw how to handle with the gamma matrices and what we can construct with it. We are now interested in how construct the matrices themselves. Here a possible recursive construction for even dimension. In d=2 we take

$$\Gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{32}$$

Then in d=2k+2, k=1,2,.. we define

$$\Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mu = 0, ..., d-3 \tag{33}$$

$$\Gamma^{d-2} = \mathbb{1}_{2^k \times 2^k} \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{34}$$

$$\Gamma_{d-1} = \mathbb{1}_{2^k \times 2^k} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{35}$$

Here γ^μ denote the $2^k \times 2^k$ gamma matrices in $d-2$ dimensions.
For odd $d = 2k+3$ we can take the gamma matrices of the even dimensions below $d = 2k+2$ and add the generalization of γ^5 times a i-factor to get the set

$$\Gamma^0, ..., \Gamma^{d-1}, i^{d/2}\Gamma^{d+1} \tag{36}$$

of gamma matrices, where Γ^d is the γ^5 in the even d dimensions.

3 General supersymmetry algebras

The Coleman-Mandula theorem implies that also for dimensions different than four the bosonic symmetries extending the Poincare' group mecessarily commute with the Poincare' group itself and therefore the correspondingg generators are scalars. So in S-matrix theory of particles, there are only the momentum d-vector P^μ, a Lorentz generator $J^{\mu\nu}$ ($\mu, \nu = 0, 1, ..., d-1$), and various Lorentz scalar charges. In this chapter I make the choice for simplicity to set the Lorentz scalar charges equal 0. The anticommutators of the fermionic symmetry generators with each other are bosonic symmetry generators, and therefore must be a linear combination of P^μ and $J^{\mu\nu}$. This puts severe limits on the Lorentz transformation properties of the fermionic generators. I will now explain why the fermionic generator must transform according to the fundamental spinor representation of the Lorentz group. Assuming that there are at most a finite number of fermionic symmetry generators, they must transform according to a finite-dimensional representation of the homogeneous Lorentz group O(1,d-1). For d even or odd, we can find $d/2$ or $(d-1)/2$ Lorentz generators $J^{01}, J^{23}, J^{45}, ...$ which commute which each other. It can be show (see Weinberg III chap. 32 for the explicit argumentation) that we can find a basis of Q's that are simultaneous eingenvector of these Lorentz generators.

$$[J^{01}, Q] = -i\omega Q \tag{37}$$

and

$$[J^{23}, Q] = -\sigma^{23} Q, \quad [J^{45}, Q] = -\sigma^{45}, ...Q \tag{38}$$

where $\omega, \sigma^{23}, \sigma^{45}, ...$ are real numbers. We call the eigenvalue ω the weight of the fermionic symmetry operator. Since J^{01} must be represented on Hilbert space by a Hermitian operator, we find that $Q^\dagger$ has the same weight as Q.

Now consider $\{Q, Q^\dagger\}$ which must be a linear combination of P^μ and $J^{\mu\nu}$. To calculate the weights of the components P^μ we recall the commutation relation

$$[P^\mu, J^{\rho\sigma}] = i(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho) \tag{39}$$

which shows that $P^0 \pm P^1$ has weight $\omega = \pm 1$, while the other components $P^2, P^3, ..., P^{d-1}$ has weight zero. In the same way, we find that $J^{0i} \pm J^{1i}$ with $i = 2, 3, ..., d-1$ have weight $\omega = \pm 1$, the J^{ij} with both i and j between 2 and $d-1$ have weight 0 and J^{01} have weight 0. We conclude that all bosonic symmetry generators have weight ± 1 or 0. Remember here that $\{Q, Q^\dagger\}$ must be a linear combination of operators with such a weight. Since Q and $Q^\dagger$ have the same weight, given that Q has weight ω we find that $\{Q, Q^\dagger\}$ has weight 2ω. We find that ω must be equal to $\pm\frac{1}{2}$, since weight 0 is excluded for non-zero fermionic Q. By an argumentation (see Weinberg III chap. 32) that the 01-plane has nothing more special then the other, it can be show that all the σ's have values $\pm\frac{1}{2}$. This is characteristic for the spinor representations and so Q must belong to some direct sum of these representations. With a similar approach it can be show that

$$[Q, P^\mu] = 0. \tag{40}$$

Equation (40) implies that J^{ab} cannot appear on the right hand side of the anticommutator $\{Q, Q^\dagger\}$. The general anticommutation relation (in the case where the central charges are 0) is then of the form

$$\{Q_n, Q_m\} \propto \Gamma^\mu_{nm} P^\mu \tag{41}$$

where n, m runs over the number of fermionic degrees of freedom given by the dimension d.
To find the anticommutator for a specific dimension d, we must take the Weyl- and Majorana condition into account. For example in the sixth dimension we can impose the Weyl condition. The fermionic generators can be arranged in a single complex Weyl spinor

$$Q_a, \quad a = 1, ...8 \tag{42}$$

$$\{Q_a, \overline{Q}^b\} = \frac{1}{2}(\mathbb{1} + \Gamma_7)_a^c (\Gamma^a)_c^b P_a. \tag{43}$$

and this is true only for minimal supersymmetry in six dimensions.

4 Massless Multiplets

We now want to construct supermultiplets of massless particle states in dimensions bigger or equal than 4 ($d \geq 4$). We found that $\Sigma_{\mu\nu}$,defined earlier, are symmetry generators of the Lorentz group.
Consider now

$$[J^{\mu\nu}, Q_i] = -\frac{1}{2}(\Sigma^{\mu\nu})_{ij} Q_j \tag{44}$$

where i runs over the number of spinor degrees of freedom. We saw this structure in four dimensions and it is the same in all other dimensions.

Since Σ_{01} is anti-Hermitian it has only imaginary eigenvalues. This is true for any bosonic and fermionic operator. We can now use the weight (w) of any fermionic operator (Q)

$$[J_{01}, O] = -iwO. \tag{45}$$

For the other generators $J^{23}, J^{45}, ...$ we found out that

$$[J^{ij}, O] = -\sigma^{ij} O \quad \text{with } \sigma^{ij} \in \mathbb{R} \tag{46}$$

We now return to the construction of the multiplets. A massless particle state can be rotated into a standard frame where its momentum is given by:

$$p^\mu = (p^0, p^1, 0, ..., 0) \quad \text{with} \quad p^0 = p^1. \tag{47}$$

In 4 dimension, the state was characterized by the helicity. In higher dimensions we can define a spin as: the maximum absolute value of the eigenvalue of any Lorentz generator J^{ij} in the representation. Since the fermionic supersymmetry generators have $\omega = \pm\frac{1}{2}$, we find that the anticommutator of any these generator with its hermitian adjoint have weight $\omega = \pm 1$. If we compare this to the commutator of the momentum and Lorentz generator which is given by

$$[P^\mu, J^{\rho\sigma}] = i(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho) \tag{48}$$

which in our special case give

$$[J^{01}, P^0 + P^1] = -i(P^0 + P^1) \tag{49}$$
$$[J^{01}, P^0 - P^1] = i(P^0 - P^1). \tag{50}$$

This means that $P^0 + P^1$ has weight 1 and $P^0 - P^1$ has weight -1 and therefore that $\{Q_i, Q_i^\dagger\}$ is proportional to $P^0 \pm P^1$. But we are working in a frame were $P^0 - P^1$ vanishes, so all the fermionic supersymmetric generator with $\omega = -\frac{1}{2}$ are zero, because of the positive definite metric on the space of physical states. In this way we already halved the number of fermionic generators.
At the beginning we had $2^(d/2)$ or $2^(d-1)/2$ (even or odd number of dimensions d and without imposing Weyl or Majorana condition) fermionic generators. We now still have $2^{d/2-1}$ or $2^{(d-1)/2-1}$ fermionic generators.
We can further divide the remaining supersymmetry generators into two classes, those who have $\sigma^{23} = +\frac{1}{2}$ and those who have $\sigma^{23} = -\frac{1}{2}$, where the sigma is given by:

$$[J^{23}, Q] = -\sigma^{23} Q. \tag{51}$$

We denote the two classes by $Q^\pm$.
Since the operator $P^0 + P^1$ has $\sigma^{23} = 0$, the fermionic supersymmetric generators of each class anticommute with each other (so $\{Q_{\pm i}, Q_{\pm j}\} = 0$).
Now consider a representation with spin j, and consider any state $|\lambda\rangle$ that is an eigenstate of J^{23} with eigenvalue $\lambda > 0$ (so $J^{23}|\lambda\rangle = \lambda|\lambda\rangle$) and is annihilated by all supersymmetry generators with $\sigma^{23} = -\frac{1}{2}$ (so $Q_-|\lambda\rangle = 0$). We can now create state with $J^{23} = \lambda - \frac{k}{2}$ by acting on $|\lambda\rangle$ with k different fermionic generators with $\sigma^{23} = +\frac{1}{2}$. For example:

$$J^{23} Q_+ |\lambda\rangle = (Q_+ J^{23} + [J^{23}, Q_+])|\lambda\rangle = (\lambda - \frac{1}{2}) Q_+ |\lambda\rangle \tag{52}$$

and with a similar calculation we find

$$J^{23}Q_+^\dagger Q_+|\lambda\rangle = \lambda Q_+^\dagger Q_+|\lambda\rangle \tag{53}$$

$$\longmapsto \quad Q_+^\dagger Q_+|\lambda\rangle = |\lambda\rangle. \tag{54}$$

To count the number of state we see that if there are a total of $\mathcal{N}$ ($\mathcal{N}$ is the number of fermionic degrees of freedom after have used the Weyl or/and condition) fermionic supersymmetry generators , then there are $\mathcal{N}/4$ of them with $\omega = +\frac{1}{2}$ and $\sigma^{23} = +\frac{1}{2}$, and since these operators all anticommute the number of states form in this way with $J^{23} = \lambda - \frac{k}{2}$ will be given by the binomial coefficient

$$\binom{\mathcal{N}/4}{k} \tag{55}$$

and the total number of state by

$$\sum_{k=0}^{\mathcal{N}/4} \binom{\mathcal{N}/4}{k} = 2^{\mathcal{N}/4}. \tag{56}$$

The minimum eigenvalue obtained in this way is $\lambda - \mathcal{N}/8$ We can now ask, what is the maximal dimension such that the absolute value of the spin does not exceed two. By setting $\lambda = j$, we find that 11 dimension and not extended supersymmetry (N=1) is the maximal dimensionality.

5 Supersymmetric Yang-Mills theory in d=6

The goal of this chapter is to compare a six dimensional Lagrangian to a four dimensional one. Consider the following Lagrangian in a six dimensional Minkowski space for a gauge field A_a (a=0,1,...,3,5,6) and his superpartner a chiral spinor λ in the adjoint representation of the gauge group:

$$L = tr(-\frac{1}{4}F_{ab}F^{ab} + i\overline{\lambda}\Gamma^a\nabla_a\lambda) \tag{57}$$

$$\lambda = \frac{1}{2}(\mathbb{1} - \Gamma_7)\lambda \tag{58}$$

$$\nabla_a\lambda = \partial_a\lambda + i[A_a, \lambda]. \tag{59}$$

This Lagrangian is a density under the supersymmetric transformations

$$\delta A_a = i\overline{\zeta}\Gamma_a\lambda - i\overline{\lambda}\Gamma_A\zeta \tag{60}$$

$$\delta\lambda = -\frac{1}{2}i\Sigma^{ab}\zeta F_{ab} \tag{61}$$

To work out this Lagrangian in detail we need a particular representation of the Dirac matrices in six dimensions:

$$\Gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix} \quad \text{for} \quad \mu = 0, .., 3 \tag{62}$$

$$\Gamma_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix} \tag{63}$$

$$\Gamma_6 = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{64}$$

$$\Gamma_7 = \Gamma_0 ... \Gamma_6 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{65}$$

$$A = \Gamma_0. \tag{66}$$

Since λ is a chiral spinor, it can be written as

$$\lambda = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \tag{67}$$

with χ an unconstrained, complex 4-spinor. Now rewrite

$$i\overline{\lambda}\Gamma^a \nabla_a \lambda = i \begin{pmatrix} 0 & \overline{\chi} \end{pmatrix} \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix} \nabla_\mu \begin{pmatrix} \chi \\ 0 \end{pmatrix} \tag{68}$$

$$+ i \begin{pmatrix} 0 & \overline{\chi} \end{pmatrix} \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix} \nabla_5 \begin{pmatrix} \chi \\ 0 \end{pmatrix} \tag{69}$$

$$+ i \begin{pmatrix} 0 & \overline{\chi} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \nabla_6 \begin{pmatrix} \chi \\ 0 \end{pmatrix} \tag{70}$$

$$= i\overline{\chi}\gamma^\mu \nabla_\mu \chi - i\overline{\chi}\gamma_5 \nabla_5 \chi - i\overline{\chi}\nabla_6 \chi. \tag{71}$$

5.1 trivial dimensional reduction

A possible way of obtaining the 4 dimensional model from the 6 dimensional one is by taking account of only 4 of the 6 dimensions. This is called the trivial dimensional reduction. First we assume that nothing depends on x^5 x^6. So we can set $\partial_5 = \partial_6 = 0$. We can rewrite the covariant derivative $\nabla_a \lambda$ for $a = 5, 6$ and it leaves us with:

$$\nabla_{5,6}\chi = i[A_{5,6}, \chi]. \tag{72}$$

For the field strength tensor $F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$ we get:

$$F_{\mu 5} = \nabla_\mu A_5, \qquad F_{\mu 6} = \nabla_\mu A_6 \tag{73}$$

$$F_{56} = i[A_5, A_6]. \tag{74}$$

We now have a lagrangian that depends only on four coordinates x^μ and reads

$$L = tr(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F_{56}F^{56} - \frac{1}{2}F_{\mu 5}F^{\mu 5} - \frac{1}{2}F_{\mu 6}F^{\mu 6} \tag{75}$$

$$+ i\overline{\chi}\gamma^\mu\nabla_\mu\chi - i\overline{\chi}\gamma_5\nabla_5\chi - i\overline{\chi}\nabla_6\chi) \tag{76}$$

$$= tr(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\nabla_\mu A_5\nabla^\mu A^5 - \frac{1}{2}\nabla_\mu A_6\nabla^\mu A^6 + \frac{1}{2}[A_5, A_6]^2 \tag{77}$$

$$+ i\overline{\chi}\gamma^\mu\nabla_\mu\chi + \overline{\chi}\gamma_5[A_5, \chi] + \overline{\chi}[A_6, \chi]) \tag{78}$$

$$= tr(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\nabla_\mu A_5\nabla^\mu A_5 + \frac{1}{2}\nabla_\mu A_6\nabla^\mu A_6 \tag{79}$$

$$+ i\overline{\chi}\gamma^\mu\nabla_\mu\chi - \overline{\chi}[\chi, A_6] - \overline{\chi}\gamma_5[\chi, A_5] + \frac{1}{2}[A_5, A_6]^2). \tag{80}$$

If we now use the identifications

$$\chi = \frac{1}{\sqrt{2}}(\lambda_1 - i\lambda_2) \tag{81}$$

$$A_5 = N, \quad A_6 = M \tag{82}$$

we get

$$L = tr(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\overline{\lambda}_i\gamma^\mu\nabla_\mu\lambda_i + \frac{1}{2}\nabla_\mu M\nabla^\mu M + \frac{1}{2}\nabla_\mu N\nabla^\mu N \tag{83}$$

$$-i\overline{\lambda}_2[\lambda_1, M] - i\overline{\lambda}_2\gamma_5[\lambda_1, N] + \frac{1}{2}[M, N]^2). \tag{84}$$

In the calculation, a number of terms vanish due to the symmetry properties of bi-spinors. Here an example (set $\lambda_1 \equiv \lambda$)

$$tr(\overline{\lambda}[\lambda, M]) = \overline{\lambda}^a\lambda^b M^c \underbrace{tr(T^a[T^b, T^c])}_{\propto f^{abc}} \propto f^{abc}\overline{\lambda}^a\lambda^b M^c \tag{85}$$

$$= \frac{1}{2}f^{abc}\overline{\lambda}^a\lambda^b M^c + \frac{1}{2}f^{abc}\overline{\lambda}^a\lambda^b M^c \tag{86}$$

$$= \frac{1}{2}f^{abc}\overline{\lambda}^a\lambda^b M^c + \frac{1}{2}f^{abc}\overline{\lambda}^b\lambda^a M^c \tag{87}$$

$$= \frac{1}{2}f^{abc}\overline{\lambda}^a\lambda^b M^c + \frac{1}{2}f^{bac}\overline{\lambda}^a\lambda^b M^c \tag{88}$$

$$= \frac{1}{2}f^{abc}\overline{\lambda}^a\lambda^b M^c - \frac{1}{2}f^{abc}\overline{\lambda}^a\lambda^b M^c = 0. \tag{89}$$

In the third line we use that $\overline{\lambda^a}\lambda^b = \overline{\lambda^b}\lambda^a$ for two real spinors whose components anticommute. For the term

$$tr(\overline{\lambda}\gamma_5[\lambda, N]) = ... = 0 \tag{90}$$

we use that $\overline{\lambda^a}\gamma_5\lambda^b = \overline{\lambda^b}\gamma_5\lambda^a$. The Lagrangian (76) is exactly the N=2 (4 dimensional) Lagrangian for a gauge multiplet and a chiral multiplet (here we use N=1 definition of multiplets). This Lagrangian is a density under the supersymmetric transformation

$$\delta A_\mu = i\overline{\zeta}_i \gamma_\mu \lambda_i \tag{91}$$
$$\delta M = \varepsilon_{ij}\overline{\zeta}_i \lambda_j \tag{92}$$
$$\delta N = \varepsilon_{ij}\overline{\zeta}_i \gamma_5 \lambda_j \tag{93}$$
$$\delta\lambda_i = -\frac{i}{2}\sigma^{\mu\nu}\zeta_i F_{\mu\nu} + i\varepsilon_{ij}\gamma^\mu \nabla_\mu (M + \gamma_5 N)\zeta_j - i\gamma_5 \zeta_i [M, N]. \tag{94}$$

We can now recast the 4 dimensional transformation in the six dimensional one. For this we need to define a spinor ζ from the two Majorana spinor parameters ζ_i of the four-dimensional transformations as

$$\zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_1 - i\zeta_2 \\ 0 \end{pmatrix} \tag{95}$$

and evaluate

$$i\overline{\zeta}\Gamma_\mu \lambda = \frac{i}{2}\overline{\zeta}_i \gamma_\mu \lambda_i + \frac{1}{2}\varepsilon_{ij}\overline{\zeta}_i \gamma_\mu \lambda_j \tag{96}$$
$$i\overline{\zeta}\Gamma_5 \lambda = \frac{i}{2}\overline{\zeta}_i \gamma_5 \lambda_i + \frac{1}{2}\varepsilon_{ij}\overline{\zeta}_i \gamma_5 \lambda_j \tag{97}$$
$$i\overline{\zeta}\Gamma_6 \lambda = \frac{i}{2}\overline{\zeta}_i \lambda_i + \frac{1}{2}\varepsilon_{ij}\overline{\zeta}_i \lambda_j \tag{98}$$

to find that

$$\delta A_a|_{a=\mu} = (i\overline{\zeta}\Gamma_a \lambda - i\overline{\lambda}\Gamma_a \zeta)|_{a=\mu} = i\overline{\zeta}_i \gamma_\mu \lambda_i \tag{99}$$
$$\delta A_5 = \varepsilon_{ij}\overline{\zeta_i}\gamma_5 \lambda_j = \delta N \tag{100}$$
$$\delta A_6 = \varepsilon_{ij}\overline{\zeta_i}\lambda_j = \delta M. \tag{101}$$

The transformation law for λ can be calculated from $\delta\lambda_i$ and is

$$\delta\lambda = -\frac{1}{2} i\Sigma^{\mu\nu}\zeta F_{\mu\nu} - i\Sigma^{\mu 5}\zeta \nabla_\mu A_5 - i\Sigma^{\mu 6}\zeta \nabla_\mu A_6 + \Sigma^{56}\zeta [A_5, A_6]. \tag{102}$$

We get the same result if we apply dimensional reduction to

$$\delta\lambda = -\frac{1}{2} i\Sigma^{ab}\zeta F_{ab}. \tag{103}$$

All this is consistent with the fact that the N=2 supersymmetry algebra with two central charges in four dimensions,

$$\{Q_i, \overline{Q_j}\} = 2\delta_{ij}\gamma^\mu P_\mu + 2i\varepsilon_{ij} Z + 2i\varepsilon_{ij}\gamma_5 Z' \tag{104}$$
$$[Q_i, P_\mu] = [Q_i, Z] = [Q_i, Z'] = 0 \tag{105}$$
$$[P_\mu, P_\nu] = [P_\mu, Z] = [P_\mu, Z'] = [Z, Z'] = 0, \tag{106}$$

can be recast into a six-dimensional form

$$\{Q, \overline{Q}\} = (\mathbb{1} + \Gamma_7)\Gamma^a P_a, \quad \{Q, Q\} = 0 \tag{107}$$
$$[Q, P_a] = 0, \quad [P_a, P_b] = 0 \tag{108}$$

with

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ Q_1 - iQ_2 \end{pmatrix}, \quad P_5 = -Z', \quad P_6 = -Z. \tag{109}$$

In this section we saw the close relationship between the N=2 super-Yang-Mills theory in d=4 and the N=1 super-Yang-Mills theory in d=6. This has been done by assuming that $\partial_5 = \partial_6 = 0$. To recast the superalgebra we than only considered four-dimensional Lorentz transformation in a six dimensional Minkowski space which led to the fact P_5, P_6 are and represent central charges. Then this condition breaks six-dimensional Lorentz invariance to four-dimensional Lorentz invariance.

6 The hypermultiplet in 6 dimension

There is another method of dimensional reduction. In the trivial reduction we always start in high dimension with a Lagrangian describing massless particles and end in 4 dimension with a Lagrangian still describing massless particles but with central charges. With this method we obtaine a four-dimensional Lagrangian with massive fields starting from a six-dimensional Lagrangian. Take the following Lagrangian for two complex scalars A and B and an anti-chiral spinor $\psi = \frac{1}{2}(\mathbb{1} + \Gamma_7)\psi$ in 6 dimension

$$L = \partial_a A^\dagger \partial^a A + \partial_a B^\dagger \partial^a B + \frac{i}{4} \overline{\psi} \Gamma^a \overleftrightarrow{\partial_a} \psi. \tag{110}$$

We assume that all the fields are periodic in x^5 and x^6 with periods $2\pi/m'$ and $2\pi/m$, respectively. So we can writte

$$A(x^\mu, x^5 + 2\pi n'/m', x^6 + 2\pi n/m) = A(x^\mu, x^5, x^6), \quad \text{for} \quad n, n' \in \mathbb{Z} \tag{111}$$

and the same for B and ψ. At each point in four- dimensional space-time, the remaining two dimensions then have the shape of a donut, and we can Fourier decompose the fields, for example

$$A(x^\mu, x^5, x^6) = \sum_{nn' \in \mathbb{Z}} exp(-in'm'x^5 - inmx^6) A_{nn'}(x^\mu). \tag{112}$$

Inserting the Fourier decomposed fields in the lagrangian we get a sum over $L_{nn'}$, where

$$\begin{aligned} L_{nn'} =& \partial_\mu A^\dagger_{nn'} \partial^\mu A_{nn'} + \partial_\mu B^\dagger_{nn'} \partial^\mu B_{nn'} + \frac{i}{4} \overline{\psi}_{nn'} \overleftrightarrow{\not\partial} \psi_{nn'} && (113) \\ & - (n'^2 m'^2 + n^2 m^2)(A^\dagger_{nn'} A_{nn'} + B^\dagger_{nn'} B_{nn'}) && (114) \\ & - \frac{n'm'}{2} \overline{\psi}_{nn'} \gamma_5 \psi_{nn'} + \frac{nm}{2} \overline{\psi}_{nn'} \psi_{nn'} && (115) \\ =& \partial_\mu A^\dagger_{nn'} \partial^\mu A_{nn'} + \partial_\mu B^\dagger_{nn'} \partial^\mu B_{nn'} + \frac{i}{4} \overline{\psi}_{nn'} \overleftrightarrow{\not\partial} \psi_{nn'} && (116) \\ & - (n'^2 m'^2 + n^2 m^2)(A^\dagger_{nn'} A_{nn'} + B^\dagger_{nn'} B_{nn'}) && (117) \\ & + \frac{1}{2} \overline{\psi}_{nn'} \begin{pmatrix} nm + in'm' & 0 \\ 0 & nm - in'm' \end{pmatrix} \psi_{nn'} && (118) \\ =& \partial_\mu A^\dagger_{nn'} \partial^\mu A_{nn'} + \partial_\mu B^\dagger_{nn'} \partial^\mu B_{nn'} + \frac{i}{4} \overline{\psi}_{nn'} \overleftrightarrow{\not\partial} \psi_{nn'} && (119) \\ & - (n'^2 m'^2 + n^2 m^2)(A^\dagger_{nn'} A_{nn'} + B^\dagger_{nn'} B_{nn'}) && (120) \\ & + \frac{1}{2} \sqrt{n^2 m^2 + n'^2 m'^2}\, \overline{\psi}_{nn'} \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \psi_{nn'}. && (121) \end{aligned}$$

(Notice that the last two matrices are 4×4-matrices and φ is the phase of the complex number $nm + in'm'$.) The fields $\psi_{nn'}$ are the Dirac spinors in four dimensions which form the bottom halves of the anti-chiral spinors in six dimension. Now we perform a γ_5-transformation on $\psi_{nn'}$ defined as

$$\psi_{nn'} \mapsto e^{\gamma_5 \varphi/2} \psi_{nn'} \tag{122}$$

which gives us the lagrangian

$$\begin{aligned} L_{nn'} =& \partial_\mu A^\dagger_{nn'} \partial^\mu A_{nn'} + \partial_\mu B^\dagger_{nn'} \partial^\mu B_{nn'} + \frac{i}{4} \overline{\psi}_{nn'} \overleftrightarrow{\not\partial} \psi_{nn'} && (123) \\ & - (n'^2 m'^2 + n^2 m^2)(A^\dagger_{nn'} A_{nn'} + B^\dagger_{nn'} B_{nn'}) && (124) \\ & + \frac{1}{2} \sqrt{n^2 m^2 + n'^2 m'^2}\, \overline{\psi}_{nn'} \psi_{nn'}. && (125) \end{aligned}$$

We see that the assumption of a torus in the fifth and sixth dimensions led to a spectrum with an infinite tower of massive multiplet with masses:

$$M_{nn'} = \sqrt{n^2 m^2 + n'^2 m'^2} \quad \text{with} \quad n, n' \text{integers.} \tag{126}$$

If we assume that $m' \leq m$, then the lowest mass becomes $M_{01} = m'$. If we choice $M_{01} = m'$ to be the massterm, we get the N=2 (d=4) Lagrangian for the hypermultiplet.

7 Supersymmetric Yang-Mills theory in d=10

In 10 dimensions we find that a spinor have 32 complex components, but that we can impose chirality and Majorana condition simultaneously. So we can work with a 16-component (real) spinor in ten dimension which must be chiral

$$\lambda = \frac{1}{2}(\mathbb{1} - \Gamma_{11})\lambda \tag{127}$$

and Majorana

$$\lambda = C\overline{\lambda}^T \equiv \lambda^c. \tag{128}$$

It can be found that

$$L = tr(-\frac{1}{4}F_{ab}F^{ab} + \frac{i}{2}\overline{\lambda}\Gamma^a\nabla_a\lambda) \tag{129}$$

is a density under the supersymmetric transformations

$$\delta A_a = i\overline{\zeta}\Gamma_a\lambda, \quad \delta\lambda = -\frac{i}{2}\Sigma^{ab}\zeta F_{ab}. \tag{130}$$

We can now perform a trivial dimensional reduction:

$$\partial_{4+m} = 0 \quad \text{for} \quad m = 1, ..., 6 \quad . \tag{131}$$

Comparing to the 6 dimensional example we did before we here expect the six components $A_5, ..., A_{10}$ to become scalars and the 16 real spinor components of λ to break down into four chiral spinors $\lambda_{\alpha i}$.
First we must give a particular representation of the 10 dimensional Dirac matrices.

$$\Gamma_\mu = \gamma_\mu \otimes 1\!\!1_{8\times 8} \quad \text{for } \mu = 0, ..., 3 \tag{132}$$

$$\Gamma_{4+m} = \gamma_5 \otimes \tilde{\Gamma}_m \quad \text{for } m = 1, 2, ..., 6 \tag{133}$$

with the γ_μ, γ_5 the standard 4×4 Dirac matrices in four dimensional Minkowski space and $\tilde{\Gamma}_m$ 8×8 matrices given by

$$\tilde{\Gamma}_m = \begin{pmatrix} 0 & \tilde{\sigma}_m \\ \tilde{\sigma}_m^{-1} & 0 \end{pmatrix} \tag{134}$$

with

$$\tilde{\sigma}_1 = i\gamma_1\gamma_5 C_{(4)}; \quad \tilde{\sigma}_2 = i\gamma_2\gamma_5 C_{(4)}; \quad \tilde{\sigma}_3 = i\gamma_3\gamma_5 C_{(4)} \tag{135}$$

$$\tilde{\sigma}_4 = i\gamma_0\gamma_5 C_{(4)}; \quad \tilde{\sigma}_5 = -iC_{(4)}; \quad \tilde{\sigma}_6 = -i\gamma_5 C_{(4)}. \tag{136}$$

Where $C_{(4)}$ is the four-dimensional charge conjugation operator given by

$$\begin{pmatrix} -\varepsilon_{\alpha\beta} & 0 \\ 0 & -\varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{137}$$

with $\varepsilon_{\alpha\beta}$ as the totally antisymmetric tensor normalize in the following way:

$$\varepsilon_{12} = \varepsilon^{12} = -\varepsilon_{\dot{1}\dot{2}} = - - \varepsilon^{\dot{1}\dot{2}} = +1. \tag{138}$$

Explicit calculations show that all the six $\tilde{\sigma}_m$ are antisymmetric and that

$$(\tilde{\sigma}_m^*)^{ij} = -(\tilde{\sigma}_m^{-1})^{ij} = \frac{1}{2}\varepsilon^{ijkl}(\tilde{\sigma}_m)_{kl}. \tag{139}$$

We also have the matrices $A, C_{(10)}$ and Γ_{11}:

$$A = \Gamma_0 \tag{140}$$

$$C_{(10)} = C_{(4)} \otimes \tilde{C}_{(6)} \tag{141}$$

$$\Gamma_{11} = \gamma_0 \ldots \gamma_3(\gamma_5)^6 \otimes \tilde{\Gamma}_1 \ldots \tilde{\Gamma}_6 = -\gamma_5 \otimes \tilde{\Gamma}_7. \tag{142}$$

The explicit form of $C_{(6)}$ and $\tilde{\Gamma}_7$ are

$$\tilde{C}_{(6)} = \begin{pmatrix} 0 & \mathbb{1}_{8\times 8} \\ \mathbb{1}_{8\times 8} & 0 \end{pmatrix} \quad \tilde{\Gamma}_7 = i \begin{pmatrix} \mathbb{1}_{8\times 8} & 0 \\ 0 & -\mathbb{1}_{8\times 8} \end{pmatrix}. \tag{143}$$

A general 32-component complex spinor take the form

$$\lambda = \begin{pmatrix} \lambda_{\alpha i} \\ \chi^i_\alpha \\ \overline{\omega}^{\dot{\alpha}}_i \\ \overline{\psi}^{\dot{\alpha} i} \end{pmatrix} \text{ with } \alpha, \dot{\alpha} = 1, 2 \textit{ and } i = 1, ..., 4 \tag{144}$$

which implies

$$\overline{\lambda} = \begin{pmatrix} \omega^{\alpha i}, & \psi^\alpha_i, & \overline{\lambda}^i_{\dot{\alpha}}, & \overline{\chi}_{\dot{\alpha} i} \end{pmatrix} \quad \text{and} \quad \lambda^c = \begin{pmatrix} \psi_{\alpha i} \\ \omega^i_\alpha \\ \overline{\chi}^{\dot{\alpha}}_i \\ \overline{\lambda}^{\dot{\alpha} i} \end{pmatrix}. \tag{145}$$

The chirality condition $\lambda = \frac{1}{2}(\mathbb{1} - \Gamma_{11})\lambda$ becomes

$$\chi^i_\alpha = \overline{\omega}^{\dot{\alpha}}_i = 0 \tag{146}$$

and the Majorana condition $\lambda = \lambda^c$ becomes

$$\overline{\psi}^{\dot{\alpha} i} = \overline{\lambda}^{\dot{\alpha} i} \equiv (\lambda_{\beta i})^\dagger \varepsilon^{\dot{\beta}\dot{\alpha}}; \quad \overline{\omega}^{\dot{\alpha}}_i = \overline{\chi}^{\dot{\alpha}}_i \equiv (\chi^i_\beta)^\dagger \varepsilon^{\dot{\beta}\dot{\alpha}}. \tag{147}$$

We see that 8 complex or 16 real component are left, those are the four chiral two-spinors $\lambda_{\alpha i}$. We proceed to decompose the terms in the 10 dimensional Lagrangian under the assumption of trivial dimensional reduction and the identification

$$M_m \equiv A_{4+m} \quad \text{for} \quad m = 1, ..., 6 \tag{148}$$

and get

$$-\frac{1}{4} F_{ab} F^{ab} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \nabla_\mu M_m \nabla^\mu M_m + \frac{1}{4} [M_m, M_n]^2 \tag{149}$$

$$\frac{i}{2} \overline{\lambda} \Gamma^a \nabla_a \lambda = \frac{i}{2} \overline{\lambda} \Gamma^\mu \nabla_\nu \lambda - \frac{1}{2} \overline{\lambda} \Gamma_{4+m} [\lambda, M_m] \tag{150}$$

$$= \frac{i}{2} \lambda_i \sigma^\mu \overleftrightarrow{\nabla}_\mu \overline{\lambda}^i + \frac{i}{2} \lambda_i [\lambda_j, (\tilde{\sigma}^{-1}_m)^{ij} M_m] - \frac{i}{2} \overline{\lambda}^i [\overline{\lambda}^j, (\tilde{\sigma}_m)_{ij} M_m]. \tag{151}$$

If we define

$$M_{ij} \equiv -\frac{1}{2} (\tilde{\sigma}_m)_{ij} M_m, \quad M^{ij} \equiv \frac{1}{2} (\tilde{\sigma}^{-1}_m)^{ij} M_m \tag{152}$$

and use the reality condition of M_m, we get the relationship

$$(M_{ij})^\dagger = \frac{1}{2} \varepsilon^{ijkl} M_{kl} \equiv M^{ij}. \tag{153}$$

We can also write $M_m M_m = M_{ij} M^{ij}$ because of $tr(\tilde{\sigma}_m \tilde{\sigma}^{-1}_n) = 4\delta_{mn}$. Inserting all this in our Lagrangian we get

$$L = tr(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\lambda_i \sigma^\mu \nabla_\mu \overline{\lambda}^i \tag{154}$$

$$+ \frac{1}{2} \nabla_\mu M_{ij} \nabla^\mu M^{ij} + i\lambda_i [\lambda_j, M^{ij}] \tag{155}$$

$$+ i\overline{\lambda}^i [\overline{\lambda}^j, M_{ij}] + \frac{1}{4} [M^{ij}, M^{kl}]^2) \tag{156}$$

which is the 4 dimensional N=4 super-Yang-Mills Lagrangian. The d=4, N=4 transformation law can be obtained in a similar way as in six dimensions.

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