Supersymmetric Lagrangians with gauge symmetry

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Abstract

In this report we concentrate on the ($N = 1$) supersymmetry algebra, its representations and how to build supersymmetric Lagrangians starting from superfield formalism.
This will be followed by a discussion of gauge symmetries in supersymmetric models leading us to a master Lagrangian for renormalizable supersymmetric gauge theories. Finally we have a look at super-QED as a simple example of a supersymmetric Abelian gauge theory.

Contents

1 Introduction

Today the most common method for describing particle propagations and interactions is through the use of quantum field theories: We consider a field content together with a symmetry group G that contains the Poincare group as a subgroup. Next we try to build an action that is invariant under G or, equivalently, a Lagrangian density that changes at most by a total derivative. We can then make use of the usual machinery of the action principle which gives rise to the equations of motion and hence the dynamical behaviour of the particles that are associated to the quantum fields.
Various attempts in extending the usual Poincare group by a further internal symmetry group that does not just come as a direct product $G = P \times H$ failed. This product-form has the property that the generators of the Lie algebra of each factor do not interfere:

Let P_μ be the translation generators and $M_{\mu\nu}$ the generators of Lorentz transformations. They form the Lie algebra of P and satisfy the usual Poincare commutation relations. Let then H_i be the generators of the Lie algebra of our internal symmetry group that satisfy the relation

$$[H_i, H_j] = i f_{ijk} H_k \tag{1}$$

where f_{ijk} are the structure constants of the Lie algebra. Then the product form manifests itself in the vanishing of the commutators

$$[P_\mu, H_i] = [M_{\mu\nu}, H_i] = 0 \tag{2}$$

so that internal symmetry generators are translational invariant Lortentz scalars. It also follows that they commute with the mass-squared P^2 and generalized spin operator W^2, where $W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\sigma} P_\nu M_\sigma$

$$\left[P^2, H_i\right] = \left[W^2, H_i\right] = 0 \tag{3}$$

and all components of an irreducible multiplet of the internal symmetry group have the same mass and spin.

The most general theorem in which was shown that it is impossible to enlarge the symmetry group without the product form was given by Coleman and Mandula ([CM]). Altough they made reasonable assumptions for a physical theory there was found a way out by considering a graded Lie algebra: The use of the usual commutation relations is extended by anticommutation relations of the form

$$t_A t_B - (-1)^{\eta_A \eta_B} t_B t_A = i f_{ABC} t_C \tag{4}$$

where $\eta_A \in \{1, 0\}$ is called the grading of the generator t_A[1]. This extended Lie algebra gives rise to the so called 'supersymmetry' generators Q. They are now fermionic, i.e. they change the spin by a half-odd amount and change fermions into bosons and vice versa:

$$Q|B\rangle = |F\rangle, \quad Q|F\rangle = |B\rangle \tag{5}$$

This means that supersymmetry models involve symmetry transformations involving particles with diffent spin and hence different statistics which leads to a unified description of fermions and bosons.

2 Supersymmetry algebra

For the first part we will use the two component notation. The Lorentz group has a variety of representations, corresponding to particles with integer or half-integer spins. The Haag-Lopuszanski-Sohnius theorem[2] states then that the supersymmetry generators can only belong to the $\left(0, \frac{1}{2}\right)$ or $\left(\frac{1}{2}, 0\right)$ representations of the algebra of the Lorentz group and as two-spinors we label them as $Q_a,\ a \in \left\{\frac{1}{2}, -\frac{1}{2}\right\}$. The Hermitian adjoint of a $\left(0, \frac{1}{2}\right)$ or $\left(\frac{1}{2}, 0\right)$ operator is a linear combination of $\left(\frac{1}{2}, 0\right)$ or $\left(0, \frac{1}{2}\right)$ respectively. Using this notation the fundamental anticommutation relations (for $N = 1$ supersymmetry) are:

$$\{Q_a, Q_b^*\} = 2\sigma^{\mu}_{ab} P_{\mu}, \tag{6}$$
$$\{Q_a, Q_b\} = 0. \tag{7}$$

We can see that Eq.(6) is a sensible expression when we recognize that P_μ is in the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation.
Later we will mainly use the dirac 4-component notation in which the fundamental (anti-)commutation relations read

$$\left\{Q, \bar{Q}\right\} = -2iP_\mu \gamma^\mu, \tag{8}$$
$$[P_\mu, Q] = \left[P_\mu, \bar{Q}\right] = 0, \tag{9}$$
$$[Q, M_{\mu\nu}] = \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right] Q. \tag{10}$$

[1] η_A is zero or even for bosonic and odd for fermionic generators
[2] R. Haag, J. T. Lopuszanski, and M. Sohnius, Nucl. Phys.B88, 257 (1975).

Here Q is the 4-component generator of the supersymmetry algebra

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} \tag{11}$$

$$Q_1 = Q^*_{-\frac{1}{2}}, \quad Q_2 = -Q^*_{\frac{1}{2}}, \quad Q_3 = Q_{\frac{1}{2}}, \quad Q_4 = Q_{-\frac{1}{2}}. \tag{12}$$

Equations (8)-(10) build together with the standard Poincare algebra the algebra for supersymmetry. From (9) we see that also the mass-squared operator P^2 will commute with Q and it follows that all particles in our multiplet will have the same mass.

2.1 Representations of N=1 SUSY algebra

Now we can start to construct a 'supermultiplet' which contains the minimal field content for our theory that is used to build a Lagrangian density. Its component fields should give rise to an irreducible representation of the SUSY algebra. As an example for such a representation we have a look at the construction of field supermultiplets.

We start with a complex spin-0 (scalar) field $\phi(x)$ as a 'ground state' of our representation. The SUSY generators Q_a annihilate it[3] so we expect the scalar field to commute with the Q_a but not with the complex conjugates Q_a^*

$$\begin{aligned} [Q_a, \phi(x)] &= 0, & (13) \\ [Q_a^*, \phi(x)] &= \sigma_2^{ab}\zeta_b(x) \neq 0 & (14) \end{aligned}$$

where $\zeta_b(x)$ is a two-component spinor field in the $(\frac{1}{2}, 0)$ representation. From this and (6) follows that

$$\{Q_a, \zeta_b(x)\} = -2i(\sigma^\mu\sigma^2)_{ab}\partial_\mu\phi(x) \tag{15}$$

By the use of (7) it can be shown further that

$$\{Q_a^*, \zeta_b(x)\} = -2i\delta_{ab}\mathcal{F}(x) \tag{16}$$

where $\mathcal{F}(x)$ is a scalar field. Using the previous results we find that

$$\begin{aligned} [Q_a^*, \mathcal{F}(x)] &= 0, & (17) \\ [Q_a, \mathcal{F}(x)] &= -\sigma^\mu_{ab}\partial_\mu\zeta_b(x). & (18) \end{aligned}$$

[3] [WB] 25.5.

Equations (6) - (18) show that the fields ϕ, ζ_a and $\mathcal{F}$ form a representation for the SUSY algebra and they give us the field content to derive an invariant action. The multiplet

$$\Phi_{chiral} = (\phi, \zeta_a, \mathcal{F}) \tag{19}$$

is known as the chiral multiplet for reasons that become clear later. It is however not unique, we only need a set of fields that mix entirely among themselves under a supersymmetry transformation.

Another important representation is the vector supermultiplet which contains the field content to describe interactions of vector bosons. Accordingly we are looking for a mutliplet that contains a (four-component) Majorana spinor λ and a massless vector field A_μ. These are enough if we are on-shell, i.e. when λ and A_μ satisfy their (free field) equations of motion:

$$\partial\!\!\!/ \lambda = 0, \quad \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0. \tag{20}$$

Yet if we want the SUSY algebra to close off-shell, we need to introduce an auxiliary pseudoscalar field[4] D. The complete vector multiplet is then

$$\Phi_{vector} = (A_\mu, \lambda, D) \tag{21}$$

Finally there is a further multiplet that appears to be useful for our later considerations, namely the curl supermultiplet

$$\Phi_{curl} = (F_{\mu\nu}, \lambda, D) \tag{22}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

All the mentioned multiplets furnish a complete irreducible representation of the SUSY algebra.

3 Superfield formalism

We can construct our supermultiplets always directly as explained above. However it turns out that the easiest way to calculate products of supermultiplets, which should form other supermultiplets, is through the use of so called superfields defined on a superspace. A superfield combines the field content of our supermultiplet in one single field.
In $N = 1$ superspace we have the usual spacetime coordinates x^μ together

[4] [WE] Chapter 6.

with four anticommuting parameters θ known as Grassmann numbers. Grassmann numbers anticommute with themselves and other fermionic quantities like spinors:

$$\{\theta_a, \theta_b\} = \{\theta_a, \psi_b\} = 0 \tag{23}$$

and commute with all bosonic quantities (c-numbers, scalars etc.). These relations also imply that $\theta_a \theta_a = 0$ (no summation), so that any power series in θ will terminate after a finite number of terms.

In the same way that we have defined the four-component momentum operators P_μ as the generators of translations of the ordinary spacetime coordinates x^μ

$$\phi(x) \to \phi'(x) = \phi(x+a) = e^{ia^\mu P_\mu} \phi e^{-ia^\mu P_\mu} \cong \phi + a^\mu \frac{\partial \phi}{\partial x^\mu} = \phi + \delta\phi \tag{24}$$

$$\Leftrightarrow [P_\mu, \phi] = -i\partial_\mu \phi \tag{25}$$

we can see the supersymmetry generators Q as the translation generators of the fermionic superspace coordinates. The action on a superfield is now described not just by the translation operator $\frac{\partial}{\partial \theta}$ but by a superspace differential operator $\mathcal{D}$:

$$\mathcal{D} = -\frac{\partial}{\partial \bar{\theta}} + \gamma^\mu \theta \frac{\partial}{\partial x^\mu}. \tag{26}$$

This is because the symmetry generators have now non-vanishing anticommutators (see Eq.(8)).
For an infinitessimal transformation we get

$$[\bar{\alpha} Q, S] = i\bar{\alpha}\mathcal{D}S = i\delta S \tag{27}$$

where α is a Majorana spinor parameter.

3.1 General Superfields

In our case θ is a four component Majorana spinor and because each of the components anticommutes we get a power series that terminates after the quartic term. We can therefore express the most general function of x^μ and anticommuting grassmann parameter θ as

$$\begin{aligned} S(x,\theta) =& C(x) - i(\bar{\theta}\gamma_5 \omega(x)) - \frac{i}{2}(\bar{\theta}\gamma_5\theta)M(x) - \frac{1}{2}(\bar{\theta}\theta)N(x) \\ &+ \frac{i}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)V^\mu(x) - i(\bar{\theta}\gamma_5\theta)(\bar{\theta}[\lambda(x) + \frac{1}{2}\not{\partial}\omega(x)]) \\ &- \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2(D(x) + \frac{1}{2}\Box C(x)), \end{aligned}$$

which we call a 'general superfield'. If $S(x,\theta)$ is a scalar then C(x), M(x), N(x) and D(x) are (pseudo-)scalars, $\omega(x)$ and $\lambda(x)$ are 4-component spinor fields and V^μ is a vector field. The superfield $S(x,\theta)$ transforms under a supersymmetry transformation as $i\delta S = [\bar{\alpha}Q, S]$. In components this reads:

$$
\begin{aligned}
\delta C &= i(\overline{\alpha}\gamma_5\omega) &(28)\\
\delta\omega &= (-i\gamma_5\partial\!\!\!/ C - M + i\gamma_5 N + V\!\!\!/)\alpha &(29)\\
\delta M &= -\left[\overline{\alpha}(\lambda + \partial\!\!\!/\omega)\right] &(30)\\
\delta N &= i\left[\overline{\alpha}\gamma_5(\lambda + \partial\!\!\!/\omega)\right] &(31)\\
\delta V_\mu &= (\overline{\alpha}\gamma_\mu\lambda) + (\overline{\alpha}\partial_\mu\omega) &(32)\\
\delta\lambda &= \left(\frac{1}{2}\left[\partial_\mu V\!\!\!/, \gamma^\mu\right] + i\gamma_5 D\right)\alpha &(33)\\
\delta D &= i(\overline{\alpha}\gamma_5\partial\!\!\!/\lambda). &(34)
\end{aligned}
$$

As expected for a SUSY transformation, the variation of a bosonic/fermionic field is proportional to a fermionic/bosonic field. Given two superfields S_1 and S_2 their product $S = S_1S_2$ is also a superfield:

$$
\begin{aligned}
\delta S &= [\bar{\alpha}Q, S_1S_2] = S_1[\bar{\alpha}Q, S_2] + [\bar{\alpha}Q, S_1]S_2\\
&= (\bar{\alpha}\mathcal{D}S_1)S_2 + S_1(\bar{\alpha}\mathcal{D}S_2) = \bar{\alpha}\mathcal{D}S
\end{aligned}
$$

3.2 Chiral Superfields

If we now compare the field content of our general superfield $S(x,\theta)$ with the one in the supermultiplet we constructed in chapter (2.1), we find that it contains more fields than we need. In other words it is a reducible representation of the supersymmetry algebra. We should try then to find a smaller set of component fields which mix only among themselves under SUSY transformations.

Consider a superfield with $\lambda = D = 0$. Although D remains zero under these transformations if $\lambda = 0$, the condition that λ vanishes is invariant only if we impose $V_\mu = \partial_\mu Z$. A superfield satisfying these conditions is called a 'chiral superfield' $X(x,\theta)$ and may be decomposed further as

$$X(x,\theta) = \frac{1}{\sqrt{2}}\left[\Phi(x,\theta) + \tilde{\Phi}(x,\theta)\right] \qquad (35)$$

with

$$\begin{aligned}\Phi(x,\theta) =&\phi(x) - \sqrt{2}(\overline{\theta}\psi_L(x)) + \mathcal{F}(x)\left(\overline{\theta}\left(\frac{1+\gamma_5}{2}\right)\theta\right) + \frac{1}{2}(\overline{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\phi(x) \\ &- \frac{1}{\sqrt{2}}(\overline{\theta}\gamma_5\theta)(\overline{\theta}\partial\!\!\!/\psi_L(x)) - \frac{1}{8}(\overline{\theta}\gamma_5\theta)^2\Box\phi(x)\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}(x,\theta) =&\tilde{\phi}(x) - \sqrt{2}(\overline{\theta}\psi_R(x)) + \tilde{\mathcal{F}}(x)\left(\overline{\theta}\left(\frac{1-\gamma_5}{2}\right)\theta\right) - \frac{1}{2}(\overline{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\tilde{\phi}(x) \\ &+ \frac{1}{\sqrt{2}}(\overline{\theta}\gamma_5\theta)(\overline{\theta}\partial\!\!\!/\psi_R(x)) - \frac{1}{8}(\overline{\theta}\gamma_5\theta)^2\Box\tilde{\phi}(x)\end{aligned}$$

and component fields

$$\phi = \frac{A+iB}{\sqrt{2}}, \quad \psi_L = \left(\frac{1+\gamma_5}{2}\right)\psi, \quad \mathcal{F} = \frac{F-iG}{\sqrt{2}}, \tag{36}$$

$$\tilde{\phi} = \frac{A-iB}{\sqrt{2}}, \quad \psi_R = \left(\frac{1-\gamma_5}{2}\right)\psi, \quad \mathcal{F} = \frac{F+iG}{\sqrt{2}}. \tag{37}$$

To avoid confusion with the names of the component fields of a general superfield we made the identifications

$$C = A, \quad \omega = -i\gamma_5\psi, \quad M = G, \quad N = -F, \quad Z = B. \tag{38}$$

Notice that the field content of these superfields corresponds to the chiral multiplet constructed in chapter (2.1) when we identify the two-component Majorana spinor ζ_a as

$$\psi_L = \frac{1}{2}\begin{pmatrix}\zeta_a \\ -(i\sigma_2)_{ab}\zeta_b^*\end{pmatrix}. \tag{39}$$

These superfields are known as left- and right-chiral superfields respectively because they satisfy the conditions

$$\mathcal{D}_R\Phi = \mathcal{D}_L\tilde{\Phi} = 0, \tag{40}$$

where $\mathcal{D}_{R,L}$ are the right- and left-handed parts of the superderivative (26):

$$\mathcal{D}_{R,L} = \left(\frac{1\mp\gamma_5}{2}\right)\mathcal{D}. \tag{41}$$

The component fields of either Φ or $\tilde{\Phi}$ give rise to a complete and irreducible representation of the SUSY algebra. By a similar argument as for the general superfield, the product of two left-chiral superfields (or of right-chiral superfields) is again a left-(or right-)chiral superfield. This also means that any function of only left-chiral superfields is again a left-chiral superfield. The transformation laws for the component fields of the left chiral superfield under a SUSY transformation read:

$$\begin{aligned}
\delta\phi &= \sqrt{2}\,(\overline{\alpha_R}\psi_L) && (42)\\
\delta\psi_L &= \sqrt{2}\partial_\mu\phi\gamma^\mu\alpha_R + \sqrt{2}\mathcal{F}\alpha_L && (43)\\
\delta\mathcal{F} &= \sqrt{2}\left(\overline{\alpha_L}\partial\!\!\!/\psi_L\right) && (44)
\end{aligned}$$

4 Construction of SUSY invariant Lagrangians

From equations (34) and (44) we see that the D-term of a general and the F-term of a chiral superfield transform as total derivatives under SUSY transformations. Therefore they are good candidates for building a SUSY invariant action. The left-chiral superfield[5] $f(\Phi)$ is called the superpotential. Since the superderivative of a chiral superfield is not chiral we cannot include them into $f(\Phi)$. By power counting arguments it can be shown that for a renormalizable theory each term in $f(\Phi)$ can contain at most three factors of Φ. The general[6] real superfield $K(\Phi, \Phi^*)$ is known as the Kahler potential. Again by power counting it can be shown that for a renormalizable theory $K(\Phi, \Phi^*)$ must be of the form

$$K(\Phi, \Phi^*) = \sum_{n,m} g_{nm}\Phi_n^*\Phi_m \qquad (45)$$

and without loss of generality we can choose a basis such that $g_{nm} = \delta_{nm}$. The D-term of the Kahlerpotential is then:

$$\begin{aligned}
\frac{1}{2}\left[K(\Phi, \Phi^*)\right]_D =& -\partial_\mu\phi_n^*\partial^\mu\phi_n + \mathcal{F}_n^*\mathcal{F}_n\\
&-\frac{1}{2}(\overline{\psi_{nL}}\gamma^\mu(\partial_\mu\psi_L)_n) + \frac{1}{2}(\overline{(\partial_\mu\psi_L)_n}\gamma^\mu\psi_{nL})\\
=&:\mathcal{L}_D.
\end{aligned}$$

and the F-term of $f(\Phi)$ is

[5]Equivalently we could always use right-chiral superfields.

[6]Note that conjugate of a left-chiral superfield is a right-chiral superfield.

$$
\begin{aligned}
[f(\Phi)]_F =& -\frac{1}{2}\left(\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\right)(\overline{\psi_{nL}}\psi_{mL}) + \mathcal{F}_n\frac{\partial f(\phi)}{\partial\phi_n} \\
=& : \mathcal{L}_F.
\end{aligned}
$$

Note here that also the conjugate of $[f(\Phi)]_F$ is SUSY invariant so we include it into $\mathcal{L}_F$.
The complete Lagrangian density is the sum of these contributions

$$
\mathcal{L} = \mathcal{L}_D + \mathcal{L}_F \tag{46}
$$

Notice also that no derivatives of $\mathcal{F}$ and $\mathcal{F}^*$ appear. Therefore, when they satisfy their classical equations of motion, we can express them by other fields in our multiplet. For this reason $\mathcal{F}$ and $\mathcal{F}^*$ are called auxiliary fields that we had to introduce in order for the algebra to close offshell (see [MA], p. 18).

4.1 The Wess-Zumino Model

The most general model involving only a single chiral multiplet in renormalizable self-interaction was found first by Wess and Zumino ([WZ]). The off-shell Lagrangian for this model reads:

$$
\begin{aligned}
\mathcal{L}_{WZ} =& \mathcal{L}_0 + \mathcal{L}_g \\
=& \frac{1}{2}\left((\partial_\mu A)^2 + (\partial_\mu B)^2 + i\bar{\psi}\partial\!\!\!/\psi + F^2 + G^2\right) - m\left(AF + BG + \frac{1}{2}\bar{\psi}\psi\right) \\
& - g[(A^2 - B^2)F + 2ABG + \bar{\psi}(A - \gamma_5 B)\psi]
\end{aligned}
$$

We will meet the free Wess-Zumino Lagrangian density $\mathcal{L}_0$ again in chapter (6) when we construct a model involving Abelian gauge interactions.

5 Supersymmetric Gauge Theories

The most interesting models are based on gauge symmetries following from different underlying symmetrygroups. We would like to include these symmetries in our model in order to find a connection with the standard model. Consequently we have to construct actions that are simultaneously invariant under supersymmetry and gauge transformations.
We consider a set of Abelian or non-Abelian gauge transformations that leave the supersymmetry generator Q invariant. Because supersymmetry

and gauge transformations must commute, each component field in our supermultiplet has to transform in the same way under such gauge transformations. Without loss of generality we may restrict ourselves in the following to left-chiral superfields:

$$\phi_n(x) \rightarrow [exp(it_A\Lambda^A(x))]_{nm}\phi_m(x) \tag{47}$$

$$\psi_{nL}(x) \rightarrow [exp(it_A\Lambda^A(x))]_{nm}\psi_{mL}(x) \tag{48}$$

$$\mathcal{F}_n(x) \rightarrow [exp(it_A\Lambda^A(x))]_{nm}\mathcal{F}_m(x) \tag{49}$$

where the t_A are Hermitian matrices representing the generators of the gauge group and $\Lambda^A(x)$ are real functions that parametrize a finite gauge transformation.

Although our general left-chiral superfield involves derivatives which complicates the transformations we may avoid that complication by a redefinition of variables:

$$\hat{x}^\mu := x^\mu + \frac{1}{2}(\bar{\theta}\gamma_5\gamma^\mu\theta) \tag{50}$$

in terms of this new variable the superfield can be written without derivatives

$$\Phi(x,\theta) = \phi(\hat{x}) - \sqrt{2}\,\bar{\theta}\psi_L(\hat{x}) + i\bar{\theta}\mathcal{F}(\hat{x}) \tag{51}$$

and transforms as

$$\Phi_n(x,\theta) \rightarrow (e^{it_A\Lambda^A(\hat{x})})_{nm}\Phi_m(x,\theta) \tag{52}$$

Since the superpotential $f(\Phi_n)$ depends only on left-chiral superfields it will be invariant under local gauge transformations if it is invariant under global transformations. Nevertheless we have a problem with the Kahler potential

$$K(\Phi_n,\Phi_n^*) = \sum_n \Phi_n^*\Phi_n \tag{53}$$

which is no longer gauge invariant: The Hermitian adjoint of (52) is

$$\Phi_n^\dagger(x,\theta) \rightarrow \Phi_m^\dagger(x,\theta)(e^{-it_A\Lambda^A(\hat{x})^*})_{mn}. \tag{54}$$

Observe that the matrices t_A are Hermitian and the $\Lambda^A(\hat{x})$ are in general complex ($\Lambda^A(\hat{x}) \neq \Lambda^A(\hat{x})^*$) because $\hat{x}$ contains a complex part. To overcome this problem let us introduce a 'gauge connection matrix' $\Gamma(x,\theta)$ which has the transformation property

$$\Gamma(x,\theta) \rightarrow e^{it_A\Lambda^{A*}(\hat{x})}\Gamma(x,\theta)e^{-it_A\Lambda^A(\hat{x})} \tag{55}$$

This matrix is not unique and we can take it to be Hermitian $\Gamma^\dagger = \Gamma$. One special form of Γ can be taken to be

$$\Gamma(x,\theta) = e^{-2t_A V^A(x,\theta)} \tag{56}$$

where the V^A are a set of *real* 'gauge superfields'. As a consequence all bosonic component fields of V^A are real and all fermionic components are Majorana. This ensures that the vector potential and the gauge field strength are real. Redefining the Kahler potential

$$K \to K' = \sum_n \Phi^*_{nL} e^{-2t_A V^A(x,\theta)} \Phi_{nL} \tag{57}$$

will render it gauge invariant:

$$\begin{aligned} &\Phi^*_n(x,\theta)\left[e^{-it_P\Lambda^P(\hat{x})} e^{-2t_A V^A(\hat{x})} e^{it_Q\Lambda^Q(\hat{x})}\right]\Phi_n(x,\theta) \\ &= \Phi^*_n(x,\theta)\left[e^{-2t_A V^A(\hat{x})}\right]\Phi_n(x,\theta). \end{aligned}$$

Furthermore if some function of Φ and $\Phi^\dagger\Gamma$ is invariant under global gauge transformations it is also invariant under an extension of the usual gauge where we take instead of $\Lambda^A(\hat{x})$ an arbitrary left-chiral superfield $\Omega^A(x,\theta)$:

$$\Phi_{nL}(x,\theta) \to (e^{it_A\Omega^A(x,\theta)})_{nm}\Phi_{mL}(x,\theta). \tag{58}$$

5.1 Abelian Gauge Transformations

Let us first consider the case of Abelian gauge transformations where we have only one gauge potential superfield and one coupling parameter[7]. First we want to construct a kinetic part of the Lagrangian for V. It transforms as

$$\begin{aligned} e^{-2t_A V^A} &\to e^{+it_P\Omega^P(\hat{x})} e^{-2t_A V^A(\hat{x})} e^{-it_Q\Omega^Q(\hat{x})} &(59)\\ \Leftrightarrow V &\to V + \frac{i}{2}[\Omega - \Omega^*]. &(60) \end{aligned}$$

Since the second term is real we see that V remains real under such transformations. Ω then is a left-chiral superfield and can hence be expanded as before

[7]We absorb it into the parameter superfield for simplicity.

$$\Omega(x,\theta) = W(x) - \sqrt{2}(\bar{\theta}\frac{1+\gamma_5}{2}w(x)) + \mathcal{W}(x)(\bar{\theta}\frac{1+\gamma_5}{2}\theta) + \frac{1}{2}(\bar{\theta}\gamma_5\theta)\partial^\mu W(x)$$
$$- \frac{1}{\sqrt{2}}(\bar{\theta}\gamma_5\theta)(\bar{\theta}\partial\!\!\!/\frac{1+\gamma_5}{2}w(x)) - \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box W(x).$$

We also expand V in its component fields

$$V(x,\theta) = C(x) - i(\bar{\theta}\gamma_5\omega(x)) - \frac{i}{2}(\bar{\theta}\gamma_5\theta)M(x) - \frac{1}{2}(\bar{\theta}\theta)N(x) + \frac{i}{2}(\bar{\theta}\gamma_5\gamma^\mu\theta)V_\mu(x)$$
$$- i(\bar{\theta}\gamma_5\theta)(\bar{\theta}[\lambda(x) + \frac{1}{2}\partial\!\!\!/\omega(x)]) - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2(D(x) + \frac{1}{2}\Box C(x))$$

and find using the expansions of Ω in (60) that the component fields undergo the transformations

$$\begin{aligned}
C(x) &\rightarrow C(x) - Im(W(x)) \\
\omega(x) &\rightarrow \omega(x) + \frac{1}{\sqrt{2}}w(x) \\
M(x) &\rightarrow M(x) - Re(\mathcal{W}(x)) \\
N(x) &\rightarrow N(x) + Im(\mathcal{W}(x)) \\
V_\mu(x) &\rightarrow V_\mu(x) + \partial_\mu Re(W(x)) \\
\lambda(x) &\rightarrow \lambda(x) \\
D(x) &\rightarrow D(x).
\end{aligned}$$

We can now use a gauge transformation to put the gauge field in a more convenient form by setting some of the component fields to zero:

$$C(x) = \omega(x) = M(x) = N(x) = 0. \tag{61}$$

This particular gauge is called the Wess-Zumino gauge and we recognize the usual Yang-Mills gauge transformation for an Abelian theory: The gauge field changes by a gradient and the transformation parameter is the real part of the scalar component of the chiral superfield whereas λ and D remain gauge invariant.

We are still left to find a gauge invariant supersymmetric kinetic term for the gauge fields.
In QED the field-strength tensor

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \tag{62}$$

appears and we expect it to enter the Lagrangian density in the combination $F_{\mu\nu}F^{\mu\nu}$ since this is the only parity conserving gauge invariant function of the gauge field V_μ. We may normalize this by a factor of $-\frac{1}{4}$ for conventional reasons. In the same manner we may guess the form of the Lagrangian up to prefactors of the remaining terms. These can be determined from the supersymmetry condition on $\mathcal{L}_{GK}$ and we get

$$\mathcal{L}_{GK} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\bar{\lambda}\partial\!\!\!/\lambda) + \frac{1}{2}D^2. \tag{63}$$

5.1.1 Curl Superfields

As a more systematic approach for constructing the gauge kinetic part we should ask what kind of superfield W has $F_{\mu\nu}$, λ and D as component fields. We know that such a superfield exists because we saw in section (2.1) that an adequate multiplet exists, namely the curl multiplet. It turns out that W is a spinorfield so we may add a spinor index. It takes the form:

$$\begin{aligned} W_\alpha(x,\theta) = \Big[& \lambda(x) + \frac{1}{2}\gamma^\mu\gamma^\nu\theta F_{\mu\nu}(x) - i\gamma_5\theta D(x) - \frac{1}{2}(\theta^T\epsilon\theta)\partial\!\!\!/\gamma_5\lambda(x) \\ & + \frac{1}{2}(\theta^T\epsilon\gamma_5\theta)\partial\!\!\!/\lambda(x) + \frac{1}{2}(\theta^T\epsilon\gamma^\mu\theta)\gamma_5\partial_\mu\lambda(x) \\ & - \frac{1}{4}(\theta^T\epsilon\theta)\gamma_5\gamma^\mu\gamma^\nu\gamma^\sigma\theta\partial_\sigma F_{\mu\nu}(x) \\ & + \frac{1}{2}i(\theta^T\epsilon\theta)\gamma^\sigma\theta\partial_\sigma D(x) - \frac{1}{8}(\theta^T\epsilon\theta)^2\Box\lambda(x) \Big]_\alpha \end{aligned}$$

which is the usual expansion for a general superfield with the identifications:

$$\begin{aligned} C_{(\alpha)} &= \lambda_\alpha \\ \omega_{(\alpha)\beta}(x) &= \frac{1}{2}i(\gamma^\mu\gamma^\nu\epsilon)_{\alpha\beta}F_{\mu\nu}(x) + (\gamma_5\epsilon)_{\alpha\beta}D(x) \\ V_{(\alpha)\mu}(x) &= -i\partial_\mu\left(\gamma_5\lambda(x)\right)_\alpha \\ M_{(\alpha)}(x) &= -i\left(\partial\!\!\!/\gamma_5\lambda(x)\right)_\alpha \\ N_{(\alpha)}(x) &= -\left(\partial\!\!\!/\lambda(x)\right)_\alpha \\ \lambda_{(\alpha)\beta}(x) &= D_{(\alpha)}(x) = 0. \end{aligned}$$

It can also be expressed with the help of superspace differential operators (see appendix Eq.(96)) as

$$W_\alpha(x,\theta) = \frac{i}{4}(\mathcal{D}^T\epsilon\mathcal{D})\mathcal{D}_\alpha V(x,\theta) \tag{64}$$

The desired term for the Lagrangian can be obtained by splitting it up into left- and right-chiral superfields

$$W = W_L + W_R = \frac{1+\gamma_5}{2}W + \frac{1-\gamma_5}{2}W. \tag{65}$$

The left-chiral part takes the form

$$W_L = \lambda_L(\hat{x}) + \frac{1}{2}\gamma^\mu\gamma^\nu\theta_L F_{\mu\nu}(\hat{x}) + (\theta_L^T\epsilon\theta_L)\partial\!\!\!/\lambda_R(\hat{x}) - i\theta_L D(\hat{x}). \tag{66}$$

Next we take the F-term of W_L^2:

$$-\frac{1}{2}Re\left[\epsilon^{\alpha\beta}W_{L\alpha}W_{L\beta}\right]_F = -\frac{1}{2}(\overline{\lambda}\partial\!\!\!/\lambda) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^2 \tag{67}$$

and recognize the same term that we derived before in Eq.(63). The generalization of the field strength tensor to a so called 'curl superfield' W_α is a useful tool for constructing gauge kinetic parts. The extension to non-Abelian theories is straight forward.

5.1.2 Fayet-Iliopoulos term

As shown above the D-term transforms as a total derivative (Eq.(34)) and hence yields a supersymmetric term in the action. For Abelian theories there is a further term coming from the D-term of the gauge superfield

$$\mathcal{L}_{FI} = \xi_P D_p \tag{68}$$

where p runs over each U(1) factor of the gauge group and the coupling constants ξ_P have mass dimension 2. It is called the Fayet-Iliopoulos term and plays an important role in the spontaneous breaking of supersymmetry ([WB], 27.5).

5.2 Non-Abelian Gauge Transformations

Again the gauge potential superfield transforms as

$$\begin{aligned} e^{-2t_A V^A} &\rightarrow e^{+it_P\Omega^P(\hat{x})}e^{-2t_A V^A(\hat{x})}e^{-it_Q\Omega^Q(\hat{x})} && (69)\\ \Leftrightarrow V^A &\rightarrow V^A + \frac{i}{2}[\Omega^A - \Omega^{A*}] + ... && (70) \end{aligned}$$

but this time we also find further terms which arise from the Campbell-Baker-Hausdorff formula (see appendix Eq.(97)) indicated by ellipsis. Consequently there will also be more terms in each of the component fields so getting to the Wess-Zumino gauge is more complicated than in the Abelian case: We have to correct the needed gauge transformations which can be done by an iterative procedure to all orders.
We express the left-chiral superfields $\Omega^A(\hat{x})$ as[8]

$$\Omega^A(\hat{x}) = \Lambda^A(x) + \frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\Lambda^A(x) - \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box\Lambda^A(x). \tag{71}$$

Inserting (71) into (70) we find for an infinitesimal gauge that $V^A(x,\theta)$ transforms as

$$V^A(x,\theta) \to V^A(x,\theta) + C^A_{BC}V^B(x,\theta)\Lambda^C(x) + \frac{i}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\Lambda^A(x). \tag{72}$$

In terms of the component fields this reads

$$\begin{aligned} V^A_\mu(x) &\to V^A_\mu(x) + \partial_\mu\Lambda^A(x) + C^A_{BC}V^B_\mu\Lambda^C(x) && (73)\\ \lambda^A(x) &\to \lambda^A(x) + C^A_{BC}\lambda^B(x)\Lambda^C(x) && (74)\\ D^A(x) &\to D^A(x) + C^A_{BC}D^B(x)\Lambda^C(x) && (75)\end{aligned}$$

with structure constants $[t_B, t_C] = iC^A_{BC}t_A$ and implicit summation over all double indices. As expected we find the Yang-Mills gauge transformation for a non-Abelian theory and also that λ^A and D^A transform covariantly.
By virtue of physical intuition we write down the generalization of (63) for a non-Abelian theory immediately:

$$\mathcal{L}_{GK} = -\frac{1}{4}F_{A\mu\nu}F^{\mu\nu}_A - \frac{1}{2}\overline{\lambda_A}\not{D}_{AC}\lambda_C + \frac{1}{2}D_AD_A \tag{76}$$

where $F^{\mu\nu}_A$ is the usual gauge covariant field-strength tensor

$$F_{A\mu\nu} = \partial_\mu V_{A\nu} - \partial_\nu V_{A\mu} + C_{ABC}V_{B\mu}V_{C\nu} \tag{77}$$

and $(D_\mu\lambda)_A = \partial_\mu\lambda_A + C_{ABC}V_{B,\mu}\lambda_C$ is the gauge covariant derivative of the gaugino field in the adjoint representation. The spin zero fields D_A enter without derivatives and turn out to be auxiliary fields with purely algebraic equations of motion.
We could have derived $\mathcal{L}_{GK}$ from the extended curl superfield W^A_α in the same manner as we did for the Abelian case in section (5.1.1).

[8]This is the usual expansion of a left-chiral superfield but with $\mathcal{F}$- and ψ_L-components set to zero. This is because after WZ-gauge the only left gauge-freedom corresponds to the scalar field.

5.3 Master Lagrangian for SUSY gauge theories

In order to construct a gauge invariant Lagrangian for the scalar and spinor components of a left-chiral superfield we have to extract the D-term from the redefined Kahlerpotential (Eq.(57)). This reads ([WB] p.122)

$$\begin{aligned}
\frac{1}{2}\left[\Phi^\dagger\Gamma\Phi\right]_D = &- \left[(D_\mu\phi)^\dagger D_\mu\phi\right] - \frac{1}{2}\left[(\overline{\psi_L}\gamma^\mu D_\mu\psi_L)\right] + \frac{1}{2}\left[(\overline{D_\mu\psi_L}\gamma^\mu\psi_L)\right] \\
&+ \left[\mathcal{F}^\dagger\mathcal{F}\right] + i\sqrt{2}\left[(\overline{\psi_L}t_A\lambda^A)\phi + h.c.\right] \\
&- D_A\left[\phi^\dagger t_A\phi\right] \\
=: &\ \mathcal{L}_{gauge}
\end{aligned}$$

where we used the gauge invariant derivative

$$D_\mu\psi_L = \partial_\mu\psi_L - it_AV_\mu^A\psi_L, \quad D_\mu\phi = \partial_\mu\phi - it_AV_\mu^A\phi$$

to see that this term is indeed gauge invariant.

We can now collect the various contributions to the Lagrangain density that we have obtained so far

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{gauge} + \mathcal{L}_{GK} + \mathcal{L}_F + \mathcal{L}_{FI} \\
&= -(D_\mu\phi)_n^*(D_\mu\phi)_n - \frac{1}{2}(\overline{\psi_n}\gamma^\mu(D_\mu\psi)_n) + \mathcal{F}_n^*\mathcal{F}_n - \phi_n^*(t_A)_{nm}\phi_m D_A \\
&\quad - 2\sqrt{2}Im\left((t_A)_{nm}(\overline{\psi_{nL}}\lambda_A)\phi_m\right) + 2\sqrt{2}Im\left((t_A)_{mn}(\overline{\psi_{nR}}\lambda_A)\phi_m^*\right) \\
&\quad - \frac{1}{4}F_{A\mu\nu}F^{A\mu\nu} - \frac{1}{2}\overline{\lambda_A}(\not{D}\lambda)_A + \frac{1}{2}D_AD_A \\
&\quad - Re\left(\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}(\psi_{nL}^T\epsilon\psi_{mL})\right) + 2Re\left(\frac{\partial f(\phi)}{\partial\phi_n}\mathcal{F}_n\right) \\
&\quad - \xi_A D_A.
\end{aligned}$$

To obtain a Lagrangian density that contains only physical fields we eliminate the auxiliary fields $\mathcal{F}_n$ and D_A by using their respective equations of motion. We get:

$$\mathcal{F}_n = -\left(\frac{\partial f(\phi)}{\partial\phi_n}\right), \tag{78}$$

$$D_A = \xi_A + \phi_n^*(t_A)_{nm}\phi_m. \tag{79}$$

Substituting into $\mathcal{L}$ we arrive at the master formula for renormalizable supersymmetric gauge theories:

$$
\begin{aligned}
\mathcal{L} = &- (D_\mu\phi)^*_n(D_\mu\phi)_n - \frac{1}{2}(\overline{\psi_{nL}}\gamma^\mu(D_\mu\psi_L)_n) + \frac{1}{2}(\overline{(D_\mu\psi_L)_n}\gamma^\mu\psi_{nL}) - \frac{1}{4}F_{A\mu\nu}F_A^{\mu\nu} - \frac{1}{2}(\overline{\lambda_A}(\not{D}\lambda)_A) \\
&+ i\sqrt{2}(\overline{\psi_{nL}}(t_A)_{nm}\lambda_A)\phi_m - i\sqrt{2}\phi^*_n(\overline{\lambda_A}(t_A)_{nm}\psi_{mL}) \\
&- \left|\frac{\partial f(\phi)}{\partial\phi_n}\right|^2 - \frac{1}{2}(\xi_A + \phi^*_n(t_A)_{nm}\phi_m)^2 \\
&- \frac{1}{2}\left(\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\right)(\psi^T_{nL}\epsilon\psi_{mL}) - \frac{1}{2}\left(\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\right)^*(\psi^T_{nL}\epsilon\psi_{mL})^*.
\end{aligned}
$$

Let us discuss the relevant features of this Lagrangian density:

- The first line represents the usual gauge invariant kinetic parts for the components of the chiral and gauge superfields. It describes how all particles couple to gauge bosons through minimal coupling in the gauge covariant derivatives.
- The second line describes interactions of gauginos with scalar and fermion components of the chiral superfield. Gauginos couple matter fermions to their respective superpartners.
- The thrid line gives us the scalar potential which receives two contributions:
    1. an 'F-term contribution' from the superpotential
    2. a 'D-term contribution' that has its origin in the auxiliary fields D_A
- The last line describes non-gauge superpotential interactions of scalar and spinor fields. In the standard model this corresponds to Yukawa interactions of matter and Higgs fields.

6 Super-QED

To give an example of a supersymmetric Abelian gauge theory we consider the supersymmetric extension of QED. In usual QED the gauge invariance $\psi \to e^{ie\alpha(x)}\psi$ of the matter Lagrangian is achieved by a minimal substitution of the partial derivative by a covariant derivative that contains a gauge field A_μ:

$$\mathcal{L}_{matter} = i\overline{\psi}\not{D}\psi + m\overline{\psi}\psi, \quad D_\mu = \partial_\mu - ieA_\mu(x) \tag{80}$$

and requiring the gauge field to transform covariantly

$$A_\mu \to A_\mu + \partial_\mu \alpha. \tag{81}$$

Finally we have to add a gauge kinetic part to the Lagrangian to let the gauge field propagate given by

$$\mathcal{L}_{GK} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{82}$$

From what we learned in section (2.1) we expect for a supersymmetric version that we have a chiral multiplet interacting with a vector multiplet. Since the superpartner of the photon has spin-$\frac{1}{2}$ it has two physical degrees of freedom. Hence it must be described by either a chiral or a Majorana spinor, where the only possible gauge transformation has negative parity. However for QED we expect a field to transform with positive parity and we are therefore forced to introduce two chiral multiplets Φ_1 and Φ_2 such that the complex spinor $\psi = \psi_1 + i\psi_2$ transforms as

$$\begin{gathered}\psi \to \psi' = e^{-i\alpha(x)}(\psi_1 + i\psi_2) \approx (\psi_1 + \alpha(x)\psi_2) + i(\psi_2 - \alpha(x)\psi_1) \\ \Leftrightarrow \delta_g\psi_1 = \alpha(x)\psi_2, \quad \delta_g\psi_2 = -\alpha(x)\psi_1\end{gathered} \tag{83}$$

The easiest choice for the matter part is the free Wess-Zumino Lagrangian for the two multiplets:

$$\mathcal{L}_{matter} = \frac{1}{2}\sum_{i=1}^{2}(\partial_\mu A_i)^2 + (\partial_\mu B_i)^2 + i\overline{\psi}_i\not{\partial}\psi_i + F_i^2 + G_i^2 - m(2A_iF_i + 2B_iG_i + \frac{1}{2}\overline{\psi}_i\psi_i). \tag{84}$$

Because supersymmetry and gauge transformations must commute, the multiplets transform according to the same transformation rules as the Majorana spinor

$$\delta_g\Phi_1 = \alpha(x)\Phi_2, \quad \delta_g\Phi_2 = -\alpha(x)\Phi_1. \tag{85}$$

This Lagrangian is already gauge invariant under global transformations. The next step is to introduce the extension of the gauge field A_μ. The gauge superfield $V(x,\theta)$ transforms as stated in (60) as $V \to V - \frac{i}{2}(\Omega - \Omega^*)$ with Ω (left-)chiral. The role of the parameter field $\alpha(x)$ is played by the W-component of Ω. We can now define a doublet from the chiral multiplets:

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi_1 + i\Phi_2 \\ \Phi_1 - i\Phi_2 \end{pmatrix} \tag{86}$$

such that gauge transformations become $\Phi \to e^{-i\Omega}\Phi$ with $\Omega = \begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix}$. Afterwards we build an invariant Kahler potential

$$K = \Phi^\dagger e^{-2V}\Phi, \qquad V = \begin{pmatrix} V & 0 \\ 0 & -V \end{pmatrix} \tag{87}$$

and in order to get rid of superfluous gauge freedoms we choose the Wess-Zumino gauge $C(x) = \omega(x) = M(x) = N(x) = 0$. The matter Lagrangian (84) becomes

$$\begin{aligned} \mathcal{L}_{matter} =& \frac{1}{2}\sum_{i=1}^{2} \left[(\nabla_\mu A_i)^2 + (\nabla_\mu B_i)^2 + i\overline{\psi}_i \gamma^\mu \nabla_\mu \psi_i + F_i^2 + G_i^2 - m(2A_i F + 2B_i G_i + \overline{\psi}_i \psi_i) \right] \\ & - \overline{\lambda}(A_1 + \gamma_5 B_1)\psi_2 + \overline{\lambda}(A_2 + \gamma_5 B_2)\psi_1 - (A_1 B_2 - B_1 A_2)D) \end{aligned}$$

with

$$\nabla_\mu X_1 = \partial_\mu X_1 - A_\mu X_2, \qquad \nabla_\mu X_2 = \partial_\mu X_2 - A_\mu X_1 \tag{88}$$

where $X = A, B, \psi$. The first line contains the analogues of the covariant derivative in QED. Because we introduced a doublet which mixes both multiplets, we now have also a mixture in the covariant derivatives. The contributions in the second line are non-minimal coupling terms and are features of our supersymmetric extension.

The gauge kinetic part of the Lagrangian can be constructed from the gauge invariant curl superfield (64) and reads

$$\mathcal{L}_{GK} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\overline{\lambda}\partial\!\!\!/\lambda + \frac{1}{2}D^2. \tag{89}$$

Our new physical spectrum now contains not only the photon but also its superpartner the photino represented by the field $\lambda(x)$. We recognize further that we have propagating charged scalar partner fields A_1, A_2, B_1 and B_2 for the electron field $\psi(x)$. No derivatives of D, F_i and G_i appear so they are auxiliary fields.

The full super-QED Lagrangian is then

$$\mathcal{L}_{sQED} = \mathcal{L}_{matter} + \mathcal{L}_{GK}. \tag{90}$$

7 Conclusion

To conclude our discussion we should summarize how to construct renormalizable supersymmetric gauge theories.
First of all we have to choose a gauge group and representations for the various supermultiplets. Matter fermions and Higgs bosons are parts of chiral scalar supermultiplets whereas gauge bosons are parts of the real gauge supermultiplet.
Next we have to choose a globally gauge invariant function of the left chiral superfields which forms our superpotential $f(\phi_n)$. The interactions of particles with gauge bosons are given by the usual 'minimal coupling' and the coupling of gauginos to matter by gauge interactions is described in the second line of our master Lagrangian.
Additional self interactions of scalar matter fields are given by the third line and non-gauge interactions of matter fields from the superpotential come from the last line.

8 Appendix

Throughout the text we used a few mathematical notation that is shown here in detail.

The pauli matrices σ_μ are:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{91}$$

Then the epsilon matrix is

$$\epsilon = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}. \tag{92}$$

We chose a representation for the Dirac algebra such that the γ_5-matrix reads

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{93}$$

For Majorana spinors the relation

$$\bar{\theta} = \theta^T \gamma_5 \epsilon \tag{94}$$

holds. We also introduced in (26) the superspace differential operator

$$\mathcal{D} = -\frac{\partial}{\partial\bar{\theta}} + \gamma^\mu \theta \frac{\partial}{\partial x^\mu} \tag{95}$$

in more detail this reads

$$\mathcal{D}_\alpha = (\gamma_5\epsilon)_{\alpha\gamma}\frac{\partial}{\partial\theta_\gamma} + \gamma^\mu_{\alpha\gamma}\theta_\gamma\frac{\partial}{\partial x^\mu}. \tag{96}$$

The Campbell-Baker-Hausdorff formula

$$e^A e^B = exp\left[\sum_n \frac{1}{n!}C_n(A,B)\right] \tag{97}$$

$$C_1 = A + B, \quad C_2 = [A,B]\,, \quad C_3 = \frac{1}{2}\left[[A,B]\,,B\right] + \frac{1}{2}\left[A,[A,B]\right], \quad ...$$

is the origin of the additional terms in section 5.2. In the Abelian case all commutators vanish and the sum terminates after the first term.

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