Supersymmetric Lagrangians

Julián Cancino
Supervisor : Dr. Giuliano Panico

Proseminar FS10

Abstract

We review the procedure for constructing supersymmetric Lagrangians. First, the Lagrangian is constructed from a given basic field content, the supermultiplet, and the Wess-Zumino model is presented in this framework. Then the superfield formalism is introduced, which simplifies the algebraic manipulations of the fields. Supersymmetric Lagrangian for chiral superfields are constructed from a general sum of field components which transform as a 4-divergence under supersymmetry. Finally the general renormalizable Lagrangian for chiral superfields is introduced.

Contents

1 Introduction **1**

2 The Coleman-Mandula theorem **2**

3 Supermultiplet method **4**
3.1 Tools : "Fermion = boson" rule and graded Jacobi identity 4
3.2 Supermultiplets 5
3.3 Constructing invariants 7
3.4 The Wess-Zumino model with the supermultiplet method 9
3.5 On-shell Lagrangians and auxiliary fields 9

4 Superfield method **10**
4.1 Tools : Grassmann variables 10
4.2 Superfields 10
4.3 Constructing invariants 12

5 General Lagrangian for chiral superfields **12**
5.1 Renormalizable Lagrangian for chiral superfields 13

A Properties of spinor bilinears **14**

B On-shell algebra of the Wess-Zumino model : an explicit calculation **14**

References **15**

1 Introduction

The standard model of high-energy particle physics (with the addition of neutrino masses and mixing) gives a description in very good agreement with the experimental data up to the TeV scale. However, we know it must be incomplete at least from the Planck scale on,

$$M_{Planck} = \frac{1}{\sqrt{8\pi G}} \approx 10^{18}\,\text{GeV},$$

since gravitation must be treated quantum mechanically. Let's consider the cut-off of the model Λ and recall that it can be identified with the scale at which new physics appears. Assuming that the electroweak theory gives a complete description of the physics in the 16 orders of magnitudes of energy between the electroweak scale,

$$M_{EW} \approx 100\,\mathrm{GeV},$$

and the Planck scale, we would expect $\Lambda \approx M_{Planck}$. The squared-mass correction induced by a Yukawa coupling of the form $-\lambda_f H\bar{f}f$ becomes,

$$(\Delta m_H^2)_f = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda^2 + \cdots ,$$

whereas for a Higgs-scalar coupling of the form $-\lambda_s|H|^2|s|^2$, it has the form,

$$(\Delta m_H^2)_s = +\frac{\lambda_s}{16\pi^2}\Lambda^2 + \cdots .$$

This suggests that $m_H \approx M_{Planck}$, since the corrections to m_H are of this order of magnitude. The experimentally preferred value for m_H is however of the order of the electroweak scale, which would need an extremely good fine tuning of the parameters of the standard model.

In (unbroken) supersymmetry each fermion is associated 2 (complex) scalars with $\lambda_s = |\lambda_f|^2$, impling that the divergences would cancel exactly. Even if supersymmetry is softly broken (since we do not observe superpartnets of the same mass it must be the case) the quadratic divergence still cancel, and there is only a much milder logarithmic divergence. Supersymmetry also ensures that this cancellation property persists at higher orders [6].

The LHC at CERN will investigate the TeV scale and could observe some of the supersymmetric partners (e.g. the MSSM [1] ones) if they exist. The mass of the supersymmetric partners should be of the order of the scale of the soft supersymmetry breaking which cannot be much larger than the TeV scale in order for the above cancellation to occur on indirect constraint grounds.

In this report we will concentrate on a procedure to construct Lagrangian exhibiting supersymmetry, i.e. mixing fermionic and bosonic degrees of freedom. We will skip many of the details of the supersymmetry algebra and of its representation.

Recipe to construct a quantum field theory The modern method for constructing a quantum field theory is the following [1, 2]: First, one chooses a field content for the theory, with given transformations under a symmetry group G, typically containing the Poincaré group $\mathcal{P}$ as a subgroup. States of the system correspond to irreducible representations of G. A (linear) representation of a group G is a group homomorphism $U : G \to GL(V)$, where V is a vector space – the representation space – to which the fields belong, and $GL(V)$ is the group of all linear invertible transformations of V to itself, satisfying,

$$U(g_1)U(g_2) = U(g_1g_2) \qquad\qquad \forall g_1, g_2 \in G.$$

Then, one builds a Lagrangian (density) $\mathcal{L}[\varphi]$ out of the fields such that the action $S = \int d^4x\mathcal{L}[\varphi]$ is invariant under the action of G. This is the case if,

$$\mathcal{L}[U(g)\varphi] = \mathcal{L}[\varphi] + \partial_\mu f^\mu \qquad\qquad \forall g \in G,$$

with $f : \mathbb{R}^{1,3} \to \mathbb{R}$, a function over space-time. In a slightly misleading terminology, one speaks then of a Lagrangian invariant under G, although it is the action which is truly invariant. One then gets the Euler-Lagrange equations of motion for the classical field from of the principle of least action,

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = 0.$$

Quantization is then achieved by imposing commutation relations or through the path-integral formalism.

2 The Coleman-Mandula theorem

Up to now, we have not made any comment about the symmetry group G. Typically, it takes the form of a direct product,

$$G \cong \mathcal{P} \times H,$$

[1] Minimal Supersymmetric Standard Model.

where H is a Lie group called the internal symmetry group. The Lie algebra $\mathfrak{g}$ of G is (isomorphic to) the tangent space of G at the identity $T_{id}G$. It is a vector space spanned by the generators of the group, which is closed under the Lie bracket $[\cdot,\cdot]$ [5]. Since the group is a direct product, we can write,

$$\mathfrak{g} \cong T_{id}G = T_{id}\mathcal{P} \oplus T_{id}H \cong \mathfrak{p} \oplus \mathfrak{h}.$$

The generators [2] of G are thus the union of the usual generators of translation P_μ ($\mu = 0, 1, 2, 3$) and Lorentz boost $M_{\mu\nu} = -M_{\nu\mu}$ from $\mathfrak{p}$,

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [P_\mu, M_{\rho\sigma}] &= i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho}), \end{aligned} \tag{1}$$

and the generators of the internal symmetry group from $\mathfrak{h}$,

$$[T_i, T_j] = f_{ijk}T_k. \tag{2}$$

The direct product structure means that each element of G cann be written as (p, h) with $p \in \mathcal{P}$ and $h \in H$, and that the multipication rule is simply,

$$(p_1, h_1)(p_2, h_2) = (p_1p_2, h_1h_2) \qquad\qquad \forall p_1, p_2 \in \mathcal{P}, h_1, h_2 \in H,$$

i.e. $\mathcal{P}$ and H do not interfere with each other. Or, using the generators, with a slight abuse of notation,

$$[P_\mu, T_i] = [M_{\mu\nu}, T_i] = 0, \tag{3}$$

which we can summarize schematically as,

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p} \qquad [\mathfrak{h}, \mathfrak{h}] = \mathfrak{h} \qquad [\mathfrak{p}, \mathfrak{h}] = 0.$$

Typical examples are a single Dirac field with $H = U(1)_{charge}$, an isospin doublet with $H = SU(2)_{isospin}$ and the quark color triplet with $H = SU(3)_{color}$.

Now, one might ask oneself, if it would not be possible to combine the Poincaré and internal symmetry groups in a non-trivial way. The answer under very general assumptions is no, at least not with Lie groups. This is the content of the [3]

Theorem 1 (Coleman-Mandula theorem [3], 1967) *Let G be the symmetry group of an $\mathcal{S}$-matrix, a connected Lie group with a subgroup isomorphic to the Poincaré group $\mathcal{P}$, such that,*

1. *for any M there is only a finite number of particle types with mass less than M,*
2. *any two particle state undergoes some reaction at all energies except perhaps an isolated set,*
3. *the amplitudes for elastic two-body scattering are analytic functions of the scattering angle at almost all energies and angles.*

Then G is locally isomorphic to $\mathcal{P} \times H$ for a Lie group H.

Hence it looks like there is no way to knit the two parts together in a non trivial way.

From the commutation relation (3), one can see that the Casimir operators of the Poincaré group – the mass-square operator, $P^2 = P_\mu P^\mu$, and the generalized spin operator, $W^2 = W_\mu W^\mu$ with $W_\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$ – take constant values within a irreducible multiplet of the internal symmetry, since,

$$[T_i, P^2] = [T_i, W^2] = 0,$$

i.e. all particle in the multiplet have the same mass (O'Raifeartaigh's theorem) and spin. In order to combine fermions and bosons inside a same multiplet, we thus need to focus on another type of generators.

[2] We use $\eta = diag(-1, 1, 1, 1)$, $\mu, \nu = 0, 1, 2, 3$, $\alpha, \dot{\alpha} = 1, 2$; Summation over repeated indices is implied. $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$, with $\gamma_5^2 = \mathbb{1}$.

[3] A proof can be found in Ref. [9], pp. 12-22.

The way out The first appearance of supersymmetry was pointed out by Gol'fand and Likhtman in an attempt to explain why so few of the Lagrangians authorised by imposing only Poincaré invariance are realized in Nature [4]. They introduced new generators $Q_\alpha, \bar{Q}^{\dot{\alpha}}$ satisfying the commutation relations,

$$[P_\mu, Q_\alpha] = [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0, \qquad [M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, \qquad [M_{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = i(\sigma_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}}, \tag{4}$$

and the *anti*commutation relation (in order to circumvene the Coleman-Mandula theorem),

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, \qquad \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0. \tag{5}$$

These new generators change the spin by 1/2 and thus transform a fermionic state into a bosonic state and vice-versa. This can be shown remembering $M_{12} \equiv J_3$, $\bar{Q}^{\dot{1}} = \varepsilon^{\dot{1}\dot{\beta}} \bar{Q}_{\dot{\beta}} = -(Q_2)^\dagger$ and $\bar{Q}^{\dot{2}} = \varepsilon^{\dot{2}\dot{\beta}} \bar{Q}_{\dot{\beta}} = (Q_1)^\dagger$,

$$[J_3, Q_1] = +\frac{1}{2} Q_1, \qquad [J_3, Q_2] = -\frac{1}{2} Q_2, \qquad [J_3, (Q_1)^\dagger] = -\frac{1}{2}(Q_1)^\dagger, \qquad [J_3, (Q_2)^\dagger] = +\frac{1}{2}(Q_2)^\dagger,$$

which implies, if $J_3|\psi\rangle = s|\psi\rangle$, for example,

$$J_3 Q_1 |\psi\rangle = (Q_1 J_3 + [J_3, Q_1])|\psi\rangle = \left(s + \frac{1}{2}\right) Q_1|\psi\rangle.$$

So, instead of having, say, a fermion multiplet and a boson multiplet not talking to each other, the new generators knit them into a single *supermultiplet*.

The simplest model exhibiting $N = 1$ supersymmetry is the so-called Wess-Zumino model, describing a chiral field in self interaction [10, 11]. As the φ^4-theory (Klein-Gordon field in quartic self-interaction), it is the basic toy model of supersymmetry. We shall derive it in two different ways.

3 Supermultiplet method

In this section we investigate the consequences of the supersymmetry algebra that we can use in order to construct supersymmetric Lagrangians.

3.1 Tools : "Fermion = boson" rule and graded Jacobi identity

Our representation includes particles satisfying different statistics which are mixed together by the action of supersymmetry transformations. The supersymmetry generators can be grouped in two categories *bosonic* (satisfying commutation relations, like P_μ and $M_{\mu\nu}$) and *fermionic* (satisfying anticommutation relations, like Q_α and $\bar{Q}_{\dot{\alpha}}$). Bosonic operators do not affect the spin of the particle, whereas fermionic do, as can be seen from the commutation relations (4),

$$[Q_\alpha, P^2] = [\bar{Q}_{\dot{\alpha}}, P^2] = 0, \qquad [Q_\alpha, W^2] \neq 0, \quad [\bar{Q}_{\dot{\alpha}}, W^2] \neq 0.$$

For unbroken supersymmetry, the generators of supersymmetry transformations commute with P^2 so there is a common set of eigenstates, and states of the same supermultiplet have the same mass.

We are now ready to show the

Theorem 2 ("Fermion = boson" rule) *Let the fields be in a linear representation of supersymmetry, in which the momentum generator P_μ is a one-to-one map of the representation space onto itself. Then a supermultiplet always contains an equal number of bosonic and fermionic degrees of freedom.*

Proof [12] Let's take w.l.o.g. a arbitrary field in the bosonic sector B. As already stated, a fermionic generator, say Q_α, will map it to a field of the fermion sector, hence $Q_\alpha(B) \subset F$. Applying now another fermionic operator, say $\bar{Q}_{\dot{\beta}}$, will map it back to a field the boson sector, and thus $\bar{Q}_{\dot{\beta}} Q_\alpha(B) \subset B$. Based on the anticommutation relation (5), we can thus conclude that the map $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}$ is one-to one as P_μ. Therefore, the two composing maps must be one-to-one themselves, which implies that there are as many bosonic degrees of freedom as fermionic ones. □

This theorem applies as well 'on-shell' (when the fields satisfy their equations of motion) as well as 'off-shell'. As we will see, some field loose degrees of freedom when they go on-shell, and one must ensure that for each lost bosonic degree of freedom, a fermionic also dissapear.

Another useful tool is

Theorem 3 (Graded Jacobi identity) *Let B_i, B_j, B_k and $F_\alpha, F_\beta, F_\gamma$ be bosonic resp. fermionic generators. Then,*

$$[[B_i, B_j], B_k] + [[B_k, B_i], B_j] + [[B_j, B_k], B_i] = 0 \tag{6}$$

$$[[F_\alpha, B_i], B_j] + [[B_j, F_\alpha], B_i] + [[B_i, B_j, F_\alpha] = 0 \tag{7}$$

$$[\{F_\alpha, F_\beta\}, B_i] + \{[B_i, F_\alpha], F_\beta\} - \{[F_\beta, B_i], F_\alpha\} = 0 \tag{8}$$

$$[\{F_\alpha, F_\beta\}, F_\gamma] + [\{F_\gamma, F_\alpha\}, F_\beta] + [\{F_\beta, F_\gamma\}, F_\alpha] = 0. \tag{9}$$

3.2 Supermultiplets

Now, that we have a few restricting tools at hand, we can start by building the basic building blocks for expressing a Lagrangian, the construction of the supermultiplet. A supermultiplet is an arrangement of the component fields (basic field that we "observe", like a Dirac field or a complex scalar field), and can be pictured as a vector, with each field as components. Because of theorem 2, we are not free to give any field content we wish to the supermultiplet, but must instead pay attention to the number of fermionic and bosonic degrees of freedom. Furthermore, the fields themselves must yield a representation of the supersymmetric algebra.

An explicit construction : the $N = 1$ chiral supermultiplet This is the most important case since the matter fields of the standard model are chiral fermions. The other important case being the construction of the vector supermultiplet for the interaction fields which are vector bosons. We review the computation here, in order to get a feeling of how the general case goes [4].

We start with the field with the lowest spin, in this case a complex scalar field A. We impose that A satisfy the constraint $[A, \bar{Q}_{\dot\alpha}] = 0$. Using this, (8), (5) and the fact that $[\varphi, P_\mu] = i\partial_\mu\varphi$, we get,

$$\{[A, Q_\alpha], \bar{Q}_{\dot\beta}\} = 2i(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu A. \tag{10}$$

Then we "rise the spin by 1/2" using the fermionic generator, to get a complex femionic field through,

$$[A, Q_\alpha] \equiv 2i\psi_\alpha. \tag{11}$$

From this field we construct 2 bosonic fields through,

$$\{\psi_\alpha, Q_\beta\} \equiv -iF_{\alpha\beta}, \qquad \{\psi_\alpha, \bar{Q}_{\dot\beta}\} \equiv X_{\alpha\dot\beta}. \tag{12}$$

At this stage we use the supersymmetry algebra – consisting now only in the insertion of (11) into (10) – to reexpress $X_{\alpha\dot\beta} = (\sigma^\mu)_{\alpha\dot\beta}\partial_\mu A$. Inserting, (11) in (12) and using theorem 3, we get,

$$2F_{\alpha\beta} = \{[A, Q_\alpha], Q_\beta\} = \{[Q_\beta, A], Q_\alpha\} - [\underbrace{\{Q_\alpha, Q_\beta\}}_{=0}, A] = -2i\{\psi_\beta, Q_\alpha\} = -2F_{\beta\alpha},$$

which implies that $F_{\alpha\beta} = \varepsilon_{\alpha\beta}F$ for a complex scalar field F, with $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$. Having "eliminated" $X_{\alpha\dot\beta}$, we start the game again for F, by defining,

$$[F, Q_\alpha] \equiv \lambda_\alpha, \qquad [F, \bar{Q}_{\dot\alpha}] \equiv \bar\chi_{\dot\alpha},$$

and again, using (9) we find that $\bar\chi_{\dot\alpha} = 2\partial_\mu\psi^\beta(\sigma^\mu)_{\beta\dot\alpha}$ and $\lambda_\alpha = 0$. Finally, we need to check that all the independent fields A, ψ, F satisfy theorem 3 (notice that some of the identities have already been used in the construction of the component fields). We have thus found the $N = 1$ chiral supermultiplet $\phi = (A; \psi; F)$.

We now want to see how the fields transform under supersymmetry for this will be relevant when we will study the variation of the Lagrangian constructed from such fields. Since we are dealing with fermionic operators (unlike for other transformations which would come from an internal Lie symmetry group), we need to introduce infinitesimal fermionic (or Grassmannian) variation parameters ζ^α and $\bar\zeta^{\dot\alpha} \equiv (\zeta^\alpha)^\dagger$ which make it possible to define the variation under a supersymmetry transformation $\varphi \to \varphi' = \varphi + \delta\varphi$ through,

$$\delta\varphi \equiv -i[\varphi, \zeta Q + \bar{Q}\bar\zeta] \qquad\qquad \zeta Q \equiv \zeta^\alpha Q_\alpha, \quad \bar{Q}\bar\zeta \equiv \bar{Q}_{\dot\alpha}\bar\zeta^{\dot\alpha} = -\bar{Q}^{\dot\alpha}\bar\zeta_{\dot\alpha}, \tag{13}$$

yielding,

$$\delta A = 2\zeta\psi, \qquad \delta\psi_\alpha = -\zeta_\alpha F - i\partial_\mu A(\sigma^\mu)_{\alpha\dot\beta}\bar\zeta^{\dot\beta}, \qquad \delta F = -2i\partial_\mu\psi^\alpha(\sigma^\mu)_{\alpha\dot\beta}\bar\zeta^{\dot\beta}.$$

[4] It can be found in Sect. 3.6 of Ref. [8] (or in Sect. 26.1 of Ref. [9]).

Using the supermultiplet notation, we see that the the commutator of two supersymmetry transformations is simply,

$$[\delta_1, \delta_2]\phi \equiv \delta_1(\delta_2\phi) - \delta_2(\delta_1\phi) = 2i(\zeta_1\sigma^\mu\bar{\zeta}_2 - \zeta_2\sigma^\mu\bar{\zeta}_1)\partial_\mu\phi \tag{14}$$
$$= 2i\left((\zeta_1)^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}(\bar{\zeta}_2)^{\dot{\beta}} - (\zeta_2)^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}(\bar{\zeta}_1)^{\dot{\beta}}\right)\partial_\mu\phi,$$

where we last line is just the spelling out of the first with all indices, in order to avoid confusion.

We can now check the validity of theorem 2 for the case of the chiral supermultiplet. We have 2 complex scalar fields and thus 4 bosonic degrees of freedom. The complex 2-spinor yields in turn 4 fermionic degrees of freedom. This supermultplet is irreducible in 4-dimensional space-time.

The anti-chiral supermultiplet is constructed by imposing $[A, Q_\alpha] = 0$ instead of $[A, \bar{Q}_{\dot{\alpha}}] = 0$ and it can be shown by direct computation, that $\bar{\phi} = (A^\dagger; \bar{\psi}; F^\dagger)$.

4-component notation Up to now, we have used 2-component or chiral notation for the operators and fields. For the rest of the discussion about supermultiplet, we are going to switch to 4-component notation. It is easier than it seems, if we remember that fermionic operators and fields can simply be expressed as Majorana operators and fields. The 4-component notation consists in just putting the Q and $\bar{Q}$ operators together,

$$Q \equiv \begin{pmatrix} Q_1 \\ Q_2 \\ \bar{Q}^{\dot{1}} \\ \bar{Q}^{\dot{2}} \end{pmatrix}, \qquad \bar{Q} \equiv \begin{pmatrix} Q^1 \\ Q^2 \\ \bar{Q}_{\dot{1}} \\ \bar{Q}_{\dot{2}} \end{pmatrix}^T, \tag{15}$$

yielding the supersymmetry algebra (4) in 4-component notation ($\not{a} \equiv \gamma^\mu a_\mu$),

$$\{Q, \bar{Q}\} = 2\not{P}, \qquad [Q, P_\mu] = 0, \qquad [Q, M_{\mu\nu}] = \frac{1}{2}\sigma_{\mu\nu}Q, \qquad \ldots \tag{16}$$

Defining a 4-spinor ζ, we can then rewrite (13) and (14) in,

$$\delta\phi = -i[\phi, \bar{\zeta}Q] \tag{17}$$
$$[\delta_1, \delta_2]\phi = 2i\bar{\zeta}_1\gamma^\mu\zeta_2\partial_\mu\phi. \tag{18}$$

Since we are going to enconter it again in what follows, we reexpress the chiral supermultiplet in terms of real fields $\tilde{A} \equiv \operatorname{Re} A$, $B \equiv \operatorname{Im} A$, $\tilde{F} \equiv \operatorname{Re} F$ and $G \equiv \operatorname{Im} F$ as well as a real Dirac 4-spinor ψ (dropping immediately the $\sim$),

$$\mathbb{X} \equiv (A, B; \psi; F, G),$$

with transformations,

$$\delta A = \bar{\zeta}\psi, \quad \delta B = -i\bar{\zeta}\gamma_5\psi, \quad \delta\psi = \left(F - i\gamma_5 G + i\not{\partial}(A - i\gamma_5 B)\right)\zeta, \quad \delta F = \bar{\zeta}\not{\partial}\psi, \quad \delta G = -i\bar{\zeta}\gamma_5\not{\partial}\psi. \tag{19}$$

The $N = 1$ general supermultiplet In the previous derivation of an $N = 1$ supermultiplet, we made explicit use of the chirality condition. We did thus get the most general $N = 1$ supermultiplet. There are two reasons to have looked at this special case first. The first is purely "pedagogical" : not imposing the chirality condition would have lead us to lengthier calculations. The second is that the chiral supermultiplet is irreducible – one can thus interpret its field content as particles –, while the general supermultiplet is not.

The general supermultiplet is [5],

$$\mathbb{G} \equiv (C; \chi; M, N, A_\mu; \lambda; D).$$

We can impose a reality/Majorana condition $\mathbb{G}^\dagger = \mathbb{G}$, thus getting the $N = 1$ real general multiplet. This means that the scalar field M, pseudoscalar fields C, N and D, vector field A_μ are real, and that the Dirac spinors χ and λ satisfy the Majorana condition. The aim of introducing this reality condition comes from the fact that we will become clear in a moment since one of the component

[5] Remember that although the Dirac 4-spinors χ, λ do no "wear" indices like the vector A_μ, each of them has a 4 real degrees of freedom. So $\mathbb{G}$ has 8 bosonic and 8 fermionic degrees of freedom and we fulfill theorem 2.

fields will be used to construct a Lagrangian which should be real. The component fields of this supermultiplet transform as,

$$
\begin{aligned}
&\text{Spin } 0: \ \delta C = i\bar{\zeta}\gamma_5\chi, \quad \delta M = -\bar{\zeta}(\lambda + \not{\partial}\chi), \quad \delta N = i\bar{\zeta}\gamma_5(\lambda + \not{\partial}\chi), \quad \delta D = \bar{\zeta}\gamma_5\not{\partial}\lambda, \\
&\text{Spin } \frac{1}{2}: \ \delta\chi = (-i\gamma_5\not{\partial}C - M + i\gamma_5 N + \not{A})\zeta, \qquad \delta\lambda = (i\sigma^{\mu\nu}\partial_\mu A_\nu + i\gamma_5 D)\,\zeta, \\
&\text{Spin } 1: \ \delta A_\mu = \bar{\zeta}(\gamma_\mu\lambda + \partial_\mu\chi).
\end{aligned} \tag{20}
$$

We chose to write the transformation in this form to make a few remarks, some of which will be of great importance in what follows. First, we remark that each modification of the field depends only on the fields of the line above and/or below. This is a wonderful occurence of the fact that the supersymmetry mixes particle states with different spins. Second, of all the component fields, only D transforms as a total spacial derivative, while the other either do not contain derivatives or mix them with other fields.

The fields D, λ and A_μ mix almost only among themselves. By defining $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ to get rid of χ, we get a true submultiplet of the general supermultiplet, called the curl submultiplet,

$$\mathbb{G}_{curl} \equiv (\lambda; F_{\mu\nu}; D).$$

Another more involved submultiplet is the chiral submultiplet,

$$\mathbb{G}_{chiral} \equiv (M, N; \lambda - i\not{\partial}\chi; \not{A}; D + \Box C).$$

By imposing $\mathbb{G}_{curl} = 0$, we get (rearranging the fields) a chiral supermultiplet, that we have already computed earlier,

$$\mathbb{X} = (A, C; \chi; -M, -N),$$

with A being the solution of $A_\mu = \partial_\mu A$.

As the notation suggests, the tranformation of each component field corresponds to the transformation for the corresponding "slot" of the chiral supermultiplet given above; e.g. in this case $-M$ is the "F-component" of $\mathbb{X}$, denoted $(\mathbb{X})_F$, and thus transforms as $\delta(-M) = \delta(\mathbb{X})_F = \bar{\zeta}\not{\partial}(\mathbb{X})_\psi = \bar{\zeta}\not{\partial}\chi$. The superfield formalism presented in the next section makes it possible to avoid such esoteric and perhaps confusing nomenclature.

Analogously, setting $\mathbb{G}_{chiral} = 0$, we obtain the (irreducible) linear supermultiplet,

$$\mathbb{L} \equiv (C; \chi; A_\mu),$$

with the transformations given by the corresponding transformation in the general supermultiplet, using the constraint $\mathbb{G}_{chiral} = 0$.

3.3 Constructing invariants

Combining supermultiplets Now that we have taken a look to the building blocks, we want to see how one can construct new supermultiplets from the basic ones.

The explicit mixing of the component fields of the multiplet products are rather algebraically involved and can be found in sections 4.4 of Ref. [8], we shall give only the content of the first component. This complexity is one of the reason of introducing the superfields to "take care of the algebra".

There are three possibilities to construct a product of two chiral supermultiplets. The first results in a chiral supermultiplet,

$$A_3 = A_1A_2 - B_1B_2, \ldots \Rightarrow \mathbb{X}_1 \cdot \mathbb{X}_2 = \mathbb{X}_3.$$

This product is associative, and thus we can combine as many chiral supermultiplet as we want. The second and the third are symmetric resp. antisymmetric in the multiplicants and define a general supermultiplet,

$$
\begin{aligned}
C_3 = A_1A_2 + B_1B_2, \ldots \Rightarrow \mathbb{X}_1 \times \mathbb{X}_2 = \mathbb{G}_3, \\
C_3 = A_1B_2 - B_1A_2, \ldots \Rightarrow \mathbb{X}_1 \wedge \mathbb{X}_2 = \mathbb{G}_3,
\end{aligned}
$$

We can also combine general supermultiplets together to form another general supermultiplet through,

$$C_3 = C_1C_2, \ldots \Rightarrow \mathbb{G}_1 \cdot \mathbb{G}_2 = \mathbb{G}_3.$$

The kinetic supermultiplet In the chiral supermultiplet ϕ, we have two complex scalars. Earlier, we started from the A to construct the supermultiplet. What would happen if we repeated the construction of the supermultiplet from the $F^\dagger$? The obtained supermultiplet is again chiral, and called the kinetic multiplet,

$$\mathsf{T}\mathbb{X} = (F, G; \not{\partial}\psi; \Box A, \Box B).$$

The name comes from the fact that T can be interpreted as generalisation of the Dirac operator $i\not{\partial}$. Indeed,

$$\mathsf{T}\mathsf{T}\mathbb{X} = \mathsf{T}(F, G; \not{\partial}\psi; \Box A, \Box B) = (\Box A, \Box B; \underbrace{\not{\partial}\not{\partial}}_{=\mathbb{1}\Box}\psi; \Box F, \Box G) = \Box\mathbb{X}.$$

"Dimensional" analysis As we did in quantum field theory, we want to construct the Lagrangians out of pieces that we sum up. In order for this to make sense, we need to have a guideline which of these terms are allowed and which ones are not. Doing a "dimensional" analysis allows us to restrict the possible terms (and will be later used for renomalizability using a power counting argument).

Let us define the (mass) dimension of a supermultiplet as the mass dimension of its "generating" component, i.e. the A-component in the case of a chiral supermultiplet and the C-component for a general supermultiplet,

$$d(\mathbb{X}) \equiv d(A), \qquad d(\mathbb{G}) \equiv d(C).$$

From (18), using that $d(\partial_\mu) = d([L^{-1}]) = d([M]) \equiv 1$, we must have,

$$d(\zeta) = d(\bar{\zeta}) = -\frac{1}{2},$$

in order to have the same dimensionality on both sides of the equation.

Using the same type of argument with the transformations of the chiral and general supermultiplets, we get,

$$\begin{aligned}
\mathbb{X}: \quad & d(A) = d(B) = d(\mathbb{X}), \quad d(\psi) = d(\mathbb{X}) + \frac{1}{2}, \quad d(F) = d(G) = d(\mathbb{X}) + 1,\\
\mathbb{G}: \quad & d(C) = d(\mathbb{G}), \quad d(\chi) = d(\mathbb{G}) + \frac{1}{2}, \quad d(M) = d(N) = d(A_\mu) = d(\mathbb{G}) + 1,\\
& d(\lambda) = d(\mathbb{G}) + \frac{3}{2}, \qquad d(D) = d(\mathbb{G}) + 2.
\end{aligned}$$

We now turn on to the dimensionality of product and kinetic supermultiplets,

$$\begin{aligned}
d(\mathbb{X}_1 \cdot \mathbb{X}_2) &= d(\mathbb{X}_1 \times \mathbb{X}_2) = d(\mathbb{X}_1 \wedge \mathbb{X}_2) = d(\mathbb{X}_1) + d(\mathbb{X}_2),\\
d(\mathbb{G}_1 \cdot \mathbb{G}_2) &= d(\mathbb{G}_1) + d(\mathbb{G}_2),\\
d(\mathsf{T}\mathbb{X}) &= d(\mathbb{X})_F = d(\mathbb{X}) + 1.
\end{aligned}$$

"Invariant" components and supersymmetric Lagrangians We can finalize our reflection as follows. Let Φ be the component field of a supermultiplet with the highest mass dimension. The transformation $\delta\Phi$ being linear in $\bar{\zeta}$ (we look at infinitesimal transformations), Φ itself cannot appear in the transformation because $\bar{\zeta}$ changes the statistics.

The transformation must hence be proportional to the field with next-to-highest mass dimension Ψ of the opposite statistics, with,

$$d(\Psi) = d(\Phi) - \frac{1}{2}.$$

The only operator we can include in order to get a dimensional match is a derivative, since $d(\partial_\mu) = 1$. Hence the transformation of Φ must take the form of a total divergence $\delta\Phi = \partial_\mu K^\mu$.

By looking at the transformation properties of the chiral and general multiplets, we see that this is the case and we can conclude that,

- $(\mathbb{X})_F$ transforms as a total divergence with $K^\mu = \bar{\zeta}\gamma^\mu(\mathbb{X})_\psi$,
- $(\mathbb{G})_D$ transforms as a total divergence with $K^\mu = \bar{\zeta}\gamma_5\gamma^\mu(\mathbb{G})_\lambda$.

If we now define a Lagrangian through $\mathcal{L}_{\mathbb{X}} = (\mathbb{X})_F$ or $\mathcal{L}_{\mathbb{G}} = (\mathbb{G})_D$, the corresponding action would be invariant under supersymmetry transformations. In this case we speak of a supersymmetric Lagrangian.

Since the action is a dimensionless quantity, $d(S) = 0$, the Lagrangian must have $d(\mathcal{L}) = 4$.

We will now focus on (combinations of) the chiral supermultiplet since we want primarily to describe matter particles. Other supermultiplets are not irreducible or – as $\mathbb{L}$ – contain vector fields, which becomes useful when we want to implement supersymmetric gauge theories. This needs the implementation of a supersymmetric version of gauge invariance, which goes beyond the scope of this report.

3.4 The Wess-Zumino model with the supermultiplet method

We have now all the tools we need to formulate our first supersymmetric model. This model is based on the $N = 1$ chiral supermultiplet and is the most general renormalizable Lagrangian involving a single supermultiplet in self-interaction. It was originally found by Wess and Zumino [10, 11] and is therefore called the Wess-Zumino model.

We need a kinetic term (with derivatives), and we want it to "survive" when we apply the Euler-Lagrange equations. The term $\frac{1}{2}(\mathbb{X} \cdot \mathsf{T}\mathbb{X})_F$ is the simplest such term and it implies that $d(\mathbb{X}) = 1$. In order to have a renormalizable theory, all couplings must have a non-negative dimension. This allows to write the Wess-Zumino Lagrangian,

$$\mathcal{L}_{WZ} = \left(\frac{1}{2}\mathbb{X} \cdot \mathsf{T}\mathbb{X} + \frac{m}{2}\mathbb{X} \cdot \mathbb{X} + \frac{g}{3}\mathbb{X} \cdot \mathbb{X} \cdot \mathbb{X} \right)_F , \tag{21}$$

where $d(m) = 1$ and $d(g) = 0$.

Carrying out the supermultiplet multiplication and expressing everything as a function of the component fields of $\mathbb{X}$, we find the Wess-Zumino Lagrangian [6],

$$\begin{aligned}\mathcal{L}_{WZ} = &-\frac{1}{2}\left[(\partial_\mu A)(\partial^\mu A) + (\partial_\mu B)(\partial^\mu B) + i\bar{\psi}\partial\!\!\!/\psi\right] + \frac{1}{2}(F^2 + G^2) \\ &- m\left(AF + BG - \frac{1}{2}\bar{\psi}\psi\right) - g\left[(A^2 + B^2)F + 2ABG - \bar{\psi}(A - i\gamma_5 B)\psi\right],\end{aligned} \tag{22}$$

where we removed a total divergence to bring the kinetic part in its usual form. The equations of motion become then, using the supermultiplet notation,

$$\mathsf{T}\mathbb{X} = m\mathbb{X} + g\mathbb{X} \cdot \mathbb{X}. \tag{23}$$

One remarks that with this notation, the equality of the masses and couplings of all the fields is explicit. If we had taken (22) with different masses and couplings for each term, we would have been forced to set them all equal to m resp. g in order to have a supersymmetric Lagrangian.

3.5 On-shell Lagrangians and auxiliary fields

By looking at the Lagrangian (22), we notice that the equations of motion of F and G are purely algebraic (they do not involve derivatives). As a consequence, *when on-shell* – i.e. satisfying their equations of motion – F and G can be expressed in terms of the other fields (in this case A and B), and "forgotten" from the point of view of the physics involved. For this reason, F and G are called auxiliary fields.

The "on-shell Lagrangian" is then free of these auxiliary fields,

$$\tilde{\mathcal{L}}_{WZ} = -\frac{1}{2}\left[(\partial_\mu A)(\partial^\mu A) + m^2A^2\right] - \frac{1}{2}\left[(\partial_\mu B)(\partial^\mu B) + m^2B^2\right] - \frac{1}{2}\bar{\psi}(i\partial\!\!\!/ + m)\psi \tag{24}$$

$$- mgA(A^2 + B^2) - g\bar{\psi}(A + i\gamma_5 B)\psi - \frac{g^2}{2}(A^2 + B^2)^2, \tag{25}$$

where the fields satisfy the "on-shell transformations" of supersymmetry,

$$\tilde{\delta}A = \bar{\zeta}\psi \qquad \tilde{\delta}B = -i\bar{\zeta}\gamma_5\psi \qquad \tilde{\delta}\psi = -\left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right](A - i\gamma_5 B)\zeta.$$

In appendix B, we show that this algebra only closes if the fields satisfy their equations of motion.

In cases where the complete field content is not yet known, one starts by constructing an on-shell Lagrangian $\tilde{\mathcal{L}}$ with fields satisfying some on-shell transformation rules. The next (highly non trivial) step consists in trying to find auxiliary fields to close the algebra off-shell. This is in particular the case in extended $N \neq 1$ supersymmetries.

[6] A linear term $c(\mathbb{X})_F$ can be absorbed in a redefinition of the fields. It would generate a vacuum expectation value which plays a role in spontaneous breaking of supersymmetry.

4 Superfield method

After having fought with some esoteric transformations properties and representations of the supersymmetry algebra, one might ask oneself if there is a way to "geometrize" supersymmetry or if the correspondence principle can be extended the supersymmetry generators.

This approach was implemented very shortly after the supermultiplet approach of Wess and Zumino by Salam and Strathdee [7]. The formalism shows many advantages over the supermultiplet method, as now the algebraic structure of supersymmetry is "coded" in the theory and not in some weird multiplications and transformation rules.

4.1 Tools : Grassmann variables

As seen in quantum field theory [1], the action of generators of the Poincaré group on a quantum field can be expressed by differential operators operating on them. Since supersymmetry is characterized by anticommuting generators, one must find a way to encapsulate this feature in a convenient way.

To this end one defines the $N = 1$ superspace as an 8-dimensional space, where the first 4 coordinates are the usual space-time coordinates, denoted by x, and the remaining four coordinates are described by a 4-tuple of Grassmann numbers, denoted by θ.

Grassmann numbers and power expansion Grassmann numbers are characterized by,

$$\{\alpha, \beta\} = 0 \Leftrightarrow \alpha\beta = -\beta\alpha \qquad\qquad \forall \alpha, \beta \in \mathfrak{G}. \tag{26}$$

This has a very important implication for the power series expansion of a function depending on Grassmann parameters. Let us first handle the case when there is only one Grassmann parameter involved, and let $f(\alpha)$ be a function of the Grassmann parameter α. Then, since $\alpha\alpha = -\alpha\alpha = 0$, the power series stops after the first order, and,

$$f(\alpha) = f(0) + \alpha \frac{\partial f}{\partial \alpha}\bigg|_{\alpha=0} .$$

We emphazise here that the Taylor expansion in α is terminated and that no approximation is made.

The case of a functions of multiple Grassmann parameters $\{\alpha_i\}_{i=1,\dots,n}$ must be handled with a bit more care. First it is clear from the above example that monomials of Grassmann parameters can contain at most once each parameter. Hence a polynomial in n Grassmann parameters can have at most degree n, the monomial being $\alpha_1 \cdots \alpha_n$ (since Grassmann parameters do not commute, the order is important!). Then, with a certain amount of arbitrariness, one defines the partial (left) derivative with respect to α_k of a function of many Grassmann parameters "as usual" with the supplementary prescription of having to move the corresponding Grassmann parameter directly after the derivative, e.g.,

$$\frac{\partial}{\partial \alpha}(\alpha\beta) = \beta, \qquad \frac{\partial}{\partial \beta}(\alpha\beta) = -\frac{\partial}{\partial \beta}(\beta\alpha) = -\alpha.$$

4.2 Superfields

We can now define superfields as being functions on superspace. From the properties of Grassmann algebra, each such function must have an expansion of the form [7],

$$\begin{aligned}
\mathrm{G}(x,\theta) = {} & C(x) - i\bar{\theta}\gamma_5\chi(x) - \frac{i}{2}(\bar{\theta}\gamma_5\theta)M(x) - \frac{1}{2}(\bar{\theta}\theta)N(x) + \frac{i}{2}(\bar{\theta}\gamma_5\gamma^\mu\theta)A_\mu(x) \\
& - i(\bar{\theta}\gamma_5\theta)\left[\bar{\theta}\left(\lambda(x) + \frac{1}{2}\partial\!\!\!/\chi(x)\right)\right] - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\left(D(x) + \frac{1}{2}\Box C(x)\right),
\end{aligned} \tag{27}$$

with $C(x)$, $M(x)$, $N(x)$, $D(x)$ complex scalar fields, $A_\mu(x)$ complex vector field, and $\chi(x)$, $\lambda(x)$ complex 4-spinor fields.

Imposing that for any two Grassmann numbers $(\alpha\beta)^\dagger \equiv \beta^\dagger\alpha^\dagger$, we get from the reality condition $\mathrm{G}^\dagger = \mathrm{G}$ that, C, M, N and D are real, whereas χ and λ are Majorana spinors.

[7] We follow in this section the notations of Ref. [9]. $\bar{\theta} \equiv \theta^T\epsilon\gamma_5$, $\epsilon \equiv \mathbb{1} \otimes (i\sigma_2)$. The properties of Majorana spinors are described in the appendix of chapter 26, pp. 107-111.

Representation of the supersymmetry algebra on the superfields The action of a supersymmetry generator on G can be represented using the superspace differential operator,

$$\mathcal{Q} \equiv -\frac{\partial}{\partial\bar{\theta}} + \gamma^\mu\theta\frac{\partial}{\partial x^\mu}, \qquad \mathcal{Q}_\nu = (\gamma_5\epsilon)_\nu{}^\rho\frac{\partial}{\partial\theta^\rho} + (\gamma^\mu)_\nu{}^\sigma\theta_\sigma\frac{\partial}{\partial x^\mu}, \tag{28}$$

as,

$$\delta\mathrm{G} \equiv -i[\mathrm{G},\bar{\zeta}Q] = (\bar{\zeta}\mathcal{Q})\mathrm{G} = -\left(\bar{\zeta}\frac{\partial\mathrm{G}}{\partial\bar{\theta}}\right) + (\bar{\zeta}\gamma^\mu\theta)\frac{\partial\mathrm{G}}{\partial x^\mu}. \tag{29}$$

The transformation of the components arise then naturally by comparing the θ-coefficients of,

$$\begin{aligned}\delta\mathrm{G} = \delta C - i\bar{\theta}\gamma_5\delta\chi + -\frac{i}{2}(\bar{\theta}\gamma_5\theta)\delta M - \frac{1}{2}(\bar{\theta}\theta)\delta N + \frac{i}{2}(\bar{\theta}\gamma_5\gamma^\mu\theta)\delta A_\mu \\ - i(\bar{\theta}\gamma_5\theta)\left[\bar{\theta}\delta\left(\lambda + \frac{1}{2}\partial\!\!\!/\chi\right)\right] - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2\delta\left(D + \frac{1}{2}\Box C\right),\end{aligned}$$

with those of the left hand side of (29). Computationnally, we need to use the identities listed in appendix A. One then gets the same "weird" transformation rules (20) as in the supermultiplet case.

Covariant superderivative It is useful to define a superderivative,

$$\mathcal{D} \equiv -\frac{\partial}{\partial\bar{\theta}} - \gamma^\mu\theta\frac{\partial}{\partial x^\mu}, \qquad \mathcal{D}_\nu = (\gamma_5\epsilon)_\nu{}^\rho\frac{\partial}{\partial\theta^\rho} - (\gamma^\mu)_\nu{}^\sigma\theta_\sigma\frac{\partial}{\partial x^\mu}, \tag{30}$$

which is "supersymmetrically covariant", i.e.

$$\delta(\mathcal{D}_\mu\mathrm{G}) = -i[\bar{\zeta}Q,\mathcal{D}_\nu\mathrm{G}] = -i\mathcal{D}_\nu[\bar{\zeta}Q,\mathrm{G}] = \mathcal{D}_\nu(\bar{\zeta}\mathcal{Q})\mathrm{G} = (\bar{\zeta}\mathcal{Q})\mathcal{D}_\nu\mathrm{G}.$$

This superderivative commutes with the supersymmetry generators, as can be shown by a direct computation:

$$\{\mathcal{Q}_\mu,\mathcal{D}_\nu\} = 0 \Rightarrow [\bar{\zeta}\mathcal{Q},\mathcal{D}_\nu] = 0.$$

Chiral superfield The chiral superfield is defined by imposing,

$$\lambda = 0, \qquad D = 0, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = 0,$$

where the last condition ensures that the first is invariant under supersymmetry transformations. Making the identifications,

$$C = A, \qquad \lambda = -i\gamma_5\psi, \qquad M = G, \qquad N = -F, \qquad A_\mu = \partial_\mu B,$$

we get from the (general) superfield, the chiral superfield expression,

$$\mathrm{X}(x,\theta) = A(x) - \bar{\theta}\psi(x) + \frac{1}{2}(\bar{\theta}\theta)F(x) - \frac{i}{2}(\bar{\theta}\gamma_5\theta)G(x) \tag{31}$$

$$+\frac{i}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu B(x) + \frac{1}{2}(\bar{\theta}\gamma_5\theta)\left[\bar{\theta}\gamma_5\partial\!\!\!/\psi(x)\right] - \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box A(x). \tag{32}$$

By defining,

$$\phi \equiv \frac{A+iB}{\sqrt{2}}, \qquad \tilde{\phi} \equiv \frac{A-iB}{\sqrt{2}}, \qquad F' \equiv \frac{F-iG}{\sqrt{2}}, \qquad \tilde{F}' \equiv \frac{F+iG}{\sqrt{2}}, \qquad x^\mu_\pm \equiv x^\mu \pm \frac{1}{2}(\theta_R^T\epsilon\gamma^\mu\theta_L)$$

(dropping immediately the $'$), where as usual for 4-spinors,

$$\psi_{L,R} = \frac{1}{2}(1\pm\gamma_5)\psi,$$

we can decompose,

$$\mathrm{X}(x,\theta) = \frac{1}{\sqrt{2}}\left[\Phi(x,\theta) + \tilde{\Phi}(x,\theta)\right],$$

with,

$$\Phi(x,\theta) = \phi(x_+) - \sqrt{2}\theta_L^T\epsilon\psi_L(x_+) + (\theta_L^T\epsilon\theta_L)F(x_+) \tag{33}$$

$$\tilde{\Phi}(x,\theta) = \tilde{\phi}(x_-) + \sqrt{2}\theta_R^T\epsilon\psi_R(x_-) - (\theta_R^T\epsilon\theta_R)\tilde{F}(x_-). \tag{34}$$

Those two components are called left- and right-chiral superfields. The name comes from,

$$\mathcal{D}_R\Phi = \mathcal{D}_L\tilde{\Phi} = 0, \qquad \mathcal{D}_{L,R} = \frac{1}{2}(1\pm\gamma_5)\mathcal{D},$$

and any chiral superfield is the sum of a left-chiral and of a right-chiral superfield. A chiral superfield X is called left-chiral if $\mathcal{D}_R\mathrm{X} = 0$ and analogously for a right-chiral superfield.

4.3 Constructing invariants

Combining supermultiplets The product of two superfields is again a superfield:

$$\delta(\mathrm{G}_1\mathrm{G}_2) = -i[\mathrm{G}_1\mathrm{G}_2, \bar{\zeta}Q] = -i[\mathrm{G}_1, \bar{\zeta}Q]\mathrm{G}_2 - i\mathrm{G}_1[\mathrm{G}_2, \bar{\zeta}Q] = \left[\bar{\zeta}\mathcal{Q}\mathrm{G}_1\right]\mathrm{G}_2 + \mathrm{G}_1\left[\bar{\zeta}\mathcal{Q}\mathrm{G}_2\right] = \bar{\zeta}\mathcal{Q}(\mathrm{G}_1\mathrm{G}_2).$$

By using the properties of Grassmann numbers and the ones listed in appendix A, we can bring them in the superfield "normal form". In doing that, we remark that the complicated multiplication structure is now automatically taken care of.

Furthermore, the covariant superderivative of a supefield is a superfield (see the definition above). We can thus conclude that an arbitrary polynomial of superfields and their covariant superderivatives is again a superfield.

We also have the stronger statement, that an arbitrary polynomial of a left-chiral superfields is a left-chiral superfield and analogously for right-chiral superfields.

Superspace integrals As in the case of supermultiplets, the components with the highest power of θ transforms as a total divergence under supersymmetry transformations. If we use this term to build a Lagrangian, the action constructed from it is invariant, and supersymmetry is a symmetry of the system.

In the case of a general superfield, it is the term proportional to $-\frac{1}{4}(\bar{\theta}\gamma_5\theta)^2$, known as D-term which can play this role. For a chiral superfield, the component with the highest θ power is proportional – in the case of a left-chiral superfield to $\theta_L^T\epsilon\theta_L$ and is called F-term.

An elegant way to get these components directly from the underlying superfield uses integration over the superfield coordinates θ. In doing this, the anticommutativity of Grassmann numbers need to be considered, in particular regarding integration order. We use the shortcut notation,

$$d^4\theta \equiv d\theta_4 d\theta_3 d\theta_2 d\theta_1,$$

and integrate always over the left most component first, having brought the corresponding coefficient in contact with it. We also note that the integration over $d^n\theta$ only keeps monomials of degree n.

To get a D-term of G, we decompose,

$$-\frac{1}{4}(\bar{\theta}\gamma_5\theta)^2 = -\frac{1}{4}(\theta^T\epsilon\theta)^2 = -\frac{1}{4}\left[\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\\\theta_4\end{pmatrix}^T\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\\\theta_4\end{pmatrix}\right]^2$$
$$= -\frac{1}{4}\left[\theta_1\theta_2 - \theta_2\theta_1 + \theta_3\theta_4 - \theta_4\theta_3\right]^2 = -\frac{1}{4}\cdot 2^2\cdot(\theta_1\theta_2\theta_3\theta_4 + \theta_3\theta_4\theta_1\theta_2) = -2\theta_1\theta_2\theta_3\theta_4,$$

and hence,

$$(\mathrm{G})_D = -\frac{1}{2}\int d^4\theta\,\mathrm{G}(x,\theta). \tag{35}$$

In the same fashion, in order to get the F-term of Φ, we do the identical decomposition game,

$$d^2\theta_L \equiv d(\theta_L)_2 d(\theta_L)_1, \qquad \theta_L^T\epsilon\theta_L = 2(\theta_L)_1(\theta_L)_2,$$

yielding,

$$(\Phi)_F = \frac{1}{2}\int d^2\theta_L\,\Phi(x,\theta). \tag{36}$$

5 General Lagrangian for chiral superfields

As before, we focus our attention on Lagrangians built out of chiral superfields only. It takes the form,

$$\mathcal{L} = (\mathrm{W})_F + (\mathrm{W})_F^* + \frac{1}{2}(\mathrm{K})_D = 2\mathrm{Re}\,(\mathrm{W})_F + \frac{1}{2}(\mathrm{K})_D = \mathrm{Re}\int d^2\theta_L\,\mathrm{W}(x,\theta) - \frac{1}{4}\int d^4\theta\,\mathrm{K}(x,\theta),$$

where the properties of W and K are discussed below.

Superpotential W is called the superpotential. It is a function of elementary left-chiral superfields $\Phi = \{\Phi_i\}_{i=1,\ldots,n}$ only (each corresponding to a multiplet of particles linked together through supersymmetry transformations) but not on their covariant superderivatives or spacial derivatives, and is hence itself left-chiral. We could also have picked the right-chiral part; it is only important to choose one chirality and stick to it.

Kahler potential K is a real general superfield called the Kahler potential. It is a function of the elementary left-chiral superfields $\Phi = \{\Phi_i\}_{i=1,\dots,n}$, their complex conjugates $\Phi^* = \{\Phi_i^*\}_{i=1,\dots,n}$, their covariant superderivatives as well as their spacial derivatives.

There is an equivalence relation on the class of Kahler potentials : given two Kahler potentials K, K', with $\mathrm{K} - \mathrm{K}' = \mathrm{X}$ for a chiral superfield X, their D-term will be identical since $(\mathrm{X})_D = 0$ and they will yield the same Lagrangian.

Kahler potentials are specially interesting when we want to consider effective theories whose Langrangians are not constrained to be renormalizable. A discussion of this general case can be found in section 26.8 of Ref. [9].

5.1 Renormalizable Lagrangian for chiral superfields

Using the same type of "dimensional" analysis as in section 3.3, we can restrain the form of the superpotential and of the Kahler potential to look specifically to renormalizable Lagrangians.

Elementary superfields and spacial derivatives have dimension 1, whereas covariant superderivative dimension 1/2. Since $d(\mathcal{L}) = d(\mathrm{W})_F = d(\mathrm{W}) + 1$, $d(\mathrm{W}) = 3$ and the most general renormalizable superpotential must be of the form,

$$\mathrm{W}(\Phi) = c_i \Phi_i + m_{ij}\Phi_i\Phi_j + \lambda_{ijk}\Phi_i\Phi_j\Phi_k. \tag{37}$$

Analogously $d(\mathcal{L}) = d(\mathrm{K})_D = d(\mathrm{K}) + 2$, and since it must be real, we have,

$$\mathrm{K}(\Phi, \Phi^*) = g_{ij}\Phi_i\Phi_j^* + \cdots, \tag{38}$$

with $g_{ij}^* = g_{ji}$, i.e. the matrix (g_{ij}) is Hermitian. The dots represent terms making no contribution to the D-term : they contain a pair of covariant superderivatives or a spacial derivative and are therefore linear in the fields, making them be left- or right-chiral.

Carrying out the multiplication and superspace integration using (33), (36) and (35) we get,

$$[\mathrm{W}(\Phi)]_F = \frac{1}{2}\int d\theta_L\, \mathrm{W}(\Phi) = -\frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\left(\bar{\psi}_{iL}\psi_{jL}\right) + F_i\frac{\partial W}{\partial\phi_i}$$

$$\frac{1}{2}[\mathrm{K}(\Phi,\Phi^*)]_D = -\frac{1}{4}\int d^4\theta\, \mathrm{K}(\Phi,\Phi^*) = g_{ij}\left[-\partial_\mu\phi_i^*\partial^\mu\phi_j + F_i^*F_j - \frac{1}{2}(\overline{\psi_{iL}}\partial\!\!\!/\psi_{jL}) + \frac{1}{2}(\partial_\mu(\overline{\psi_{iL}})\gamma^\mu\psi_{jL})\right],$$

where W is the same function as W with the scalar components $\{\phi_i\}_{i=1,\dots,n}$ as arguments instead of the superfields. By redefining the fields through $\Phi_i = N_{ij}\Phi_j'$, we get the same expression, with g_{ij} replaced by $g_{ij}' = (N^\dagger g N)_{ij}$. Since (g_{ij}) is Hermitian, it is possible to find an N such that $g_{ij}' = \delta_{ij}$. Dropping the primes, we get the most general renormalizable Lagrangian for n chiral superfields,

$$\begin{aligned}\mathcal{L} = &-\partial_\mu\phi_i^*\partial^\mu\phi_i + F_i^*F_i - \frac{1}{2}(\overline{\psi_{iL}}\partial\!\!\!/\psi_{iL}) + \frac{1}{2}(\partial_\mu(\overline{\psi_{iL}})\gamma^\mu\psi_{iL}),\\ &-\frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\left(\bar{\psi}_{iL}\psi_{jL}\right) - \frac{1}{2}\left(\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\right)^*\left(\bar{\psi}_{iL}\psi_{jL}\right)^* + F_i\frac{\partial W}{\partial\phi_i} + F_i^*\left(\frac{\partial W}{\partial\phi_i}\right)^*.\end{aligned} \tag{39}$$

The fields F_i, F_i^* are auxiliary fields and only appear algebraically in the Lagrangian and we can thus eliminate them using their equations of motion,

$$F_i = -\left(\frac{\partial W}{\partial\phi_i}\right)^* \qquad F_i^* = -\left(\frac{\partial W}{\partial\phi_i}\right),$$

yielding the on-shell Lagrangian,

$$\begin{aligned}\tilde{\mathcal{L}} = &-\partial_\mu\phi_i^*\partial^\mu\phi_i - \frac{1}{2}(\overline{\psi_{iL}}\partial\!\!\!/\psi_{iL}) + \frac{1}{2}(\partial_\mu(\overline{\psi_{iL}})\gamma^\mu\psi_{iL}),\\ &-\frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\left(\bar{\psi}_{iL}\psi_{jL}\right) - \frac{1}{2}\left(\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\right)^*\left(\bar{\psi}_{iL}\psi_{jL}\right)^* - \left(\frac{\partial W}{\partial\phi_i}\right)^*\frac{\partial W}{\partial\phi_i},\end{aligned}$$

with the scalar potential,

$$V(\phi) = \sum_i \left|\frac{\partial W}{\partial\phi_i}\right|^2.$$

Assuming that V has a maximum at $\phi_0 = (\phi_{1,0}, \ldots, \phi_{n,0})$ (this is the case if supersymmetry is not spontaneously broken), and expanding the scalar field by writing $\phi_i = \phi_{i,0} + \xi_i$, we get for the quadratic part of the Lagrangian,

$$\begin{aligned}\tilde{\mathcal{L}}_0 =& -\partial_\mu \xi_i^* \partial^\mu \xi_i - \frac{1}{2}(\overline{\psi_{iL}} \partial\!\!\!/ \psi_{iL}) + \frac{1}{2}(\partial_\mu(\overline{\psi_{iL}})\gamma^\mu \psi_{iL}),\\ & -\frac{1}{2}M_{ij}\left(\bar{\psi}_{iL}\psi_{jL}\right) - \frac{1}{2}M_{ij}^*\left(\bar{\psi}_{iL}\psi_{jL}\right)^* - (M^\dagger M)_{ij}\xi_i^*\xi_j\end{aligned}$$

with the symmetric matrix (M_{ij}) with components given by,

$$M_{ij} = \left.\frac{\partial^2 W}{\partial\phi_i \partial\phi_j}\right|_{\phi_0}.$$

By a redefinition of the fields, this matrix can be brought in the diagonal form $M_{ij} = m_i \delta_{ij}$, showing that the particle of the same supermultiplet represented by the superfield Φ_i have all the same mass m_i.

The Wess-Zumino model with superfields It is straightforward to show that the Wess-Zumino model can be obtained as a special case of the general renormalizable Lagrangian by choosing the parameters,

$$n = 1, \qquad c_1 = 0, \qquad m_{11} = \frac{m}{2}, \qquad \lambda_{111} = \frac{g}{3}.$$

A Properties of spinor bilinears

The 16 Dirac matrices,

$$\mathbb{1}\,(1), \qquad \gamma_\mu\,(4), \qquad \sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]\,(6), \qquad \gamma_\mu\gamma_5\,(4), \qquad \gamma_5\,(1),$$

form a basis of the space of 4×4-matrices, i.e. every 4×4-matrix can be written as a linear combination of them. For two Majorana 4-spinors ζ, ψ consisting of Grassmann numbers, we have the identities,

$$\bar{\zeta}\psi = \bar{\psi}\zeta, \quad \bar{\zeta}\gamma^\mu\psi = -\bar{\psi}\gamma^\mu\zeta, \quad \bar{\zeta}\sigma_{\mu\nu}\psi = \bar{\psi}\sigma_{\mu\nu}\zeta, \quad \bar{\zeta}\gamma^\mu\gamma_5\psi = \bar{\psi}\gamma^\mu\gamma_5\zeta, \quad \bar{\zeta}\gamma_5\psi = \bar{\psi}\gamma_5\zeta,$$

Using these properties, we can prove the Fierz rearrangement formula,

$$\psi\bar{\zeta} = -\frac{1}{4}(\bar{\zeta}\psi)\mathbb{1} - \frac{1}{4}(\bar{\zeta}\gamma^\mu\psi)\gamma_\mu - \frac{1}{8}(\bar{\zeta}\sigma^{\mu\nu}\psi)\sigma_{\mu\nu} + \frac{1}{4}(\bar{\zeta}\gamma^\mu\gamma_5\psi)\gamma_\mu\gamma_5 - \frac{1}{4}(\bar{\zeta}\gamma_5\psi)\gamma_5.$$

B On-shell algebra of the Wess-Zumino model : an explicit calculation

We show that the behaviour of the on-shell algebra (18) for each component field of section 3.5. In the case of the scalar component fields, it is not difficult to see that the algebra closes,

$$\begin{aligned}[\tilde{\delta}_1, \tilde{\delta}_2]A &= \tilde{\delta}_1(\bar{\zeta}_2\psi) - \tilde{\delta}_2(\bar{\zeta}_1\psi)\\ &= -\bar{\zeta}_2\left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right](A - i\gamma_5 B)\zeta_1 + \bar{\zeta}_1\left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right](A - i\gamma_5 B)\zeta_2\\ &= -i\bar{\zeta}_2\gamma^\mu\zeta_1\partial_\mu A - \bar{\zeta}_2\gamma^\mu\gamma_5\zeta_1\partial_\mu B + i\bar{\zeta}_1\gamma^\mu\zeta_2\partial_\mu A + \bar{\zeta}_1\gamma^\mu\gamma_5\zeta_2\partial_\mu B\\ &= 2i\bar{\zeta}_1\gamma^\mu\zeta_2\partial_\mu A,\\ [\tilde{\delta}_1, \tilde{\delta}_2]B &= \tilde{\delta}_1(-i\bar{\zeta}_2\gamma_5\psi) - \tilde{\delta}_2(-i\bar{\zeta}_1\gamma_5\psi)\\ &= i\bar{\zeta}_2\gamma_5\left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right](A - i\gamma_5 B)\zeta_1 - i\bar{\zeta}_1\gamma_5\left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right](A - i\gamma_5 B)\zeta_2\\ &= -\bar{\zeta}_2\gamma_5\gamma^\mu\zeta_1\partial_\mu A - i\bar{\zeta}_2\underbrace{\gamma_5\gamma^\mu\gamma_5}_{=-\gamma^\mu}\zeta_1\partial_\mu B + \bar{\zeta}_1\gamma_5\gamma^\mu\zeta_2\partial_\mu A + i\bar{\zeta}_1\underbrace{\gamma_5\gamma^\mu\gamma_5}_{=-\gamma^\mu}\zeta_2\partial_\mu B\\ &= 2i\bar{\zeta}_1\gamma^\mu\zeta_2\partial_\mu B,\end{aligned}$$

where as for the spinor field,

$$\begin{aligned}[\tilde{\delta}_1, \tilde{\delta}_2]\psi &= \tilde{\delta}_1\left(-\left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right](A - i\gamma_5 B)\zeta_2\right) - \tilde{\delta}_2\left(-\left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right](A - i\gamma_5 B)\zeta_1\right)\\ &= -g(\tilde{\delta}_1 A - i\gamma_5\tilde{\delta}_1 B)(A - i\gamma_5 B)\zeta_2 - \left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right](\tilde{\delta}_1 A - i\gamma_5\tilde{\delta}_1 B)\zeta_2\\ &\quad + g(\tilde{\delta}_2 A - i\gamma_5\tilde{\delta}_2 B)(A - i\gamma_5 B)\zeta_1 + \left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right](\tilde{\delta}_2 A - i\gamma_5\tilde{\delta}_2 B)\zeta_1\\ &= -g\left[(\bar{\zeta}_1\psi)\mathbb{1} - (\bar{\zeta}_1\gamma_5\psi)\gamma_5\right](A - i\gamma_5 B)\zeta_2 - \left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right]\left[(\bar{\zeta}_1\psi)\mathbb{1} - (\bar{\zeta}_1\gamma_5\psi)\gamma_5\right]\zeta_2\\ &\quad + g\left[(\bar{\zeta}_2\psi)\mathbb{1} - (\bar{\zeta}_2\gamma_5\psi)\gamma_5\right](A - i\gamma_5 B)\zeta_1 + \left[i\partial\!\!\!/ + m + g(A - i\gamma_5 B)\right]\left[(\bar{\zeta}_2\psi)\mathbb{1} - (\bar{\zeta}_2\gamma_5\psi)\gamma_5\right]\zeta_1\\ &= 2i\bar{\zeta}_1\gamma^\mu\zeta_2\partial_\mu\psi - \gamma^\mu\left[i\partial\!\!\!/ - m - 2g(A + i\gamma_5 B)\right]\psi\bar{\zeta}_1\gamma_\mu\zeta_2,\end{aligned}$$

where the last line is obtained by using the Fierz rearrangement formula (appendix A)for the square brackets containing fields bilinear and looking at the surviving terms. For the algebra to close, we need the second term to vanish, which is true, provided that ψ satisfies its equation of motion,

$$(i\partial\!\!\!/ - m)\psi = 2g(A + i\gamma_5 B)\psi,$$

the Euler-Lagrange equation for the on-shell Lagrangian (24).

References

[1] Ch. Anastasiou, *Quantum field theory I*, lecture at ETH Zürich

[2] Ch. Anastasiou, *Quantum field theory II*, lecture at ETH Zürich

[3] S. Coleman and J. Mandula, *All possible symmetries of the S matrix*, Physical Review **159**, 1251 (1967)

[4] Yu. A. Gol'fand and E. P. Likhtman, *Extension of the algebra of the Poincaré group generators and violations of P invariance*, JETP Letters **13**, 323 (1971)

[5] J. M. Lee, *Introduction to smooth manifolds*, Springer (2006)

[6] S. P. Martin, *A supersymmetry primer*, arXiv:hep-ph/9709356

[7] A. Salam and J. Strathdee, *Superfields and Fermi-Bose symmetry*, Physical Review D **11**, 1521 (1975)

[8] M. F. Sohnius, *Introducing supersymmetry*, Physics Reports **128**, 39 (1985)

[9] S. Weinberg, *The quantum theory of fields, Vol. III : Supersymmetry*, Cambridge University Press (2005)

[10] J. Wess and B. Zumino, *Supergauge transformations in four dimensions*, Nuclear Physics B **70**, 39 (1974)

[11] J. Wess and B. Zumino, *A Lagrangian model invariant under supergauge transformations*, Physics Letters **49B**, 52 (1974)

[12] P. West, *Introduction to supersymmetry and supergravity*, World scientific (1990)