Representations of the N=1 Supersymmetry Algebra

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Abstract

In this proseminar report we review representations of the supersymmetry algebra, focusing on the N=1 case. After briefly motivating the extension of the Poincare algebra, we use it to demonstrate basic properties of supermultiplets before constructing them in the massive and massless case. Finally the chiral representation on fields is derived with some remarks about on-shell and off-shell representations illustrated by the Wess-Zumino model.

Contents

1 Introduction

The Poincare algebra

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, M_{\rho\sigma}] &= ig_{\nu\rho}M_{\mu\sigma} - ig_{\mu\rho}M_{\nu\sigma} - ig_{\nu\sigma}M_{\mu\rho} + ig_{\mu\sigma}M_{\nu\rho} \\
[M_{\mu\nu}, P_\rho] &= -ig_{\rho\mu}P_\nu + ig_{\rho\nu}P_\mu
\end{aligned} \tag{1.1}$$

comprises the generators of translations P_μ, rotations M_{ij}, and boosts M_{0i}. It summarizes the Poincare group of spacetime symmetries that forms part of the symmetries of the S-matrix. The other symmetries of the S-matrix are the discrete CPT transformations and global internal symmetries related to conserved quantities such as isospin or electric charge. The generators of internal symmetries obey a Lie algebra of the form

$$\begin{aligned}
[B_r, B_s] &= iC^t_{rs}B_t \\
[B_r, P_\mu] &= [B_r, M_{\mu\nu}] = 0
\end{aligned}$$

and the second line reflects the fact that they transform as Lorentz scalars. The Coleman-Mandula theorem [6] states that these are the *only* possible bosonic symmetries of the S-matrix. Thus the only way to extend this is to include anti-commuting spinorial generators, and if we further assume they form a Z_2 graded Lie algebra we are inevitably led to a supersymmetry relating bosons and fermions [1]. The enlarged supersymmetry algebra is found to be as follows:

$$\begin{aligned}
[P_\mu, Q^I_\alpha] &= 0 \\
[P_\mu, \bar{Q}^I_{\dot\alpha}] &= 0 \\
\{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta}Z^{IJ} \\
\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} &= \epsilon_{\dot\alpha\dot\beta}(Z^{IJ})^* \\
\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^{IJ} \\
[M_{\mu\nu}, Q^I_\alpha] &= i(\sigma_{\mu\nu})^\beta_\alpha Q^I_\beta \\
[M_{\mu\nu}, \bar{Q}^{I\dot\alpha}] &= i(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\dot\beta}Q^{I\dot\beta}
\end{aligned} \tag{1.2}$$

Q^I_α are the supersymmetry generators and the $\alpha = 1, 2$ indices denote two-component spinors (see appendix A.1). The extra $I = 1, ..., N$ index indicates the number of such generators. For the unextended $N = 1$ case the central charges Z^{IJ} that commutes with all other generators vanishes.

We can see from (1.2) that the supersymmetric generators raises or lowers the spin by one half:

$$[J_3, Q_\alpha] = [M_{12}, Q_\alpha] = i(\sigma_{12})^\beta_\alpha Q_\beta = \frac{1}{2}(\sigma_3)^\beta_\alpha Q_\beta = \pm\frac{1}{2}Q_\alpha \tag{1.3}$$

and similarly for $\bar{Q}_{\dot\alpha}$. From the spin-statistics theorem this changes bosons into fermions and vice versa. Our next task is to consider the supermultiplets that form a representation of this supersymmetry algebra.

2 Basic Properties

We start with some basic properties that all supermultiplets must obey.

Since P_μ and Q_α commute, $P^2 = m^2$ is still a Casimir operator and all particles within the same multiplet share the same mass.

Another important property is that supermultiplets must always contain an equal number of bosonic and fermionic number of degrees of freedom. To show this we first define the operator $(-1)^{2s}$ to be $+1$ when acting on a bosonic state and -1 on a fermionic state. It then follows that

$$\{(-1)^{2s}, Q\} = 0 \tag{2.1}$$

Next consider a subspace of states $|i\rangle$ in a supermultiplet such that a supersymmetric generator acting on it gives another state $|i'\rangle$ within the supermultiplet. We can then use the completeness relation $\sum_i |i\rangle \langle i| = 1$ and (2.1) in taking the following trace:

$$\begin{aligned}
p^\mu Tr\{(-1)^{2s}\} &= \sum_i \langle i| (-1)^{2s} P^\mu |i\rangle \\
&= \sum_i \langle i| (-1)^{2s} Q Q^\dagger |i\rangle + \sum_i \sum_j \langle i| (-1)^{2s} Q^\dagger |j\rangle \langle j| Q |i\rangle \\
&= \sum_i \langle i| (-1)^{2s} Q Q^\dagger |i\rangle + \sum_j \langle j| Q (-1)^{2s} Q^\dagger |j\rangle \\
&= \sum_i \langle i| (-1)^{2s} Q Q^\dagger |i\rangle - \sum_j \langle j| (-1)^{2s} Q Q^\dagger |j\rangle \\
&= 0
\end{aligned}$$

Since $p^\mu \neq 0$ and the trace just counts the number of bosonic states minus fermionic states within the supermultiplet, the result being zero we conclude that they must be equal.

Finally the requirement of positivity on the Hilbert space means the energy P_0 must always be positive:

$$\begin{aligned}
\langle\phi| P_0 |\phi\rangle &= Tr\sigma^\mu_{\alpha\dot\alpha} \langle\phi| P_\mu |\phi\rangle \\
&= Tr \langle\phi| \{Q_\alpha, \bar{Q}_{\dot\alpha}\} |\phi\rangle \\
&= \|Q_\alpha |\phi\rangle \|^2 + \|\bar{Q}_{\dot\alpha} |\phi\rangle \|^2 \geq 0
\end{aligned}$$

In the first line we used the fact that $Tr\sigma^\mu = 2\delta^{\mu 0}$ and the second line is obtained by substituting from (1.2).

3 Massless Supermultiplets

The states of a massless supermultiplet are characterised by their helicity. Since $P^2 = 0$, for simplicity we can choose a reference frame where $P_\mu = (E, 0, 0, E)$ to construct our supermultiplet. In this case

$$\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu = 2(\sigma^0 P_0 - \sigma^3 P_3) = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}$$

Therefore $\{Q_1, \bar{Q}_{\dot{1}}\} = 0$ and from the positivity of the Hilbert space this implies that $Q_1 = 0$ and $\bar{Q}_{\dot{1}} = 0$, leaving us with 2N of the 4N supersymmetry generators. In the $N = 1$ case this is just Q_2 and $\bar{Q}_{\dot{2}}$.

To find the spectrum of states we define the ladder operators

$$a = \frac{1}{\sqrt{4E}} Q_2 \qquad a^\dagger = \frac{1}{\sqrt{4E}} \bar{Q}_{\dot{2}}$$

and a "ground state" $|\lambda_0\rangle$ with helicity λ_0 such that $a\,|\lambda_0\rangle = 0$. From (1.3) the ladder operators raise and lower helicity by one half which gives us N=1 massless supermultiplet states

$$|\lambda_0\rangle$$
$$a^\dagger\,|\lambda_0\rangle = \left|\lambda_0 + \frac{1}{2}\right\rangle$$

This supermultiplet will not be invariant under CPT since the operation flips the sign of the helicity, and in general the helicities will not be symmetrically distributed around zero. We therefore need to add the CPT conjugates with opposite helicity and quantum numbers to the $(\lambda_0, \lambda_0 + \frac{1}{2})$ supermultiplet.

From the requirement of an equal number of bosonic and fermionic degrees of freedom and different values of λ_0 we arrive at the following massless N=1 supermultiplets: The chiral multiplet $(0, \frac{1}{2})$ with CPT conjugate $(-\frac{1}{2}, 0)$ consists of a Weyl fermion and two real scalars, usually arranged into a single complex scalar, and the vector multiplet $(1, \frac{1}{2})$ plus $(-1, -\frac{1}{2})$ contains a massless vector gauge boson and a Weyl fermion.

For renormalisable theories we must stop before spin $\frac{3}{2}$. If we include gravity we can also obtain the gravitino and graviton multiplet, but at helicities larger than two we can no longer consistently couple to gravity [8]. Note that since an N-extended supersymmetry will necessarily contain spin $\frac{N}{4}$ this also places limits at N=4 and N=8 respectively.

4 Massive Supermultiplets

The spacetime properties of massive one-particle states are described by the mass m, total spin s, and the spin projection along the z-axis s_z. We denote such a state by $|m, s, s_z\rangle$. Since $P^2 = m^2$ we can go to a rest frame $P^\mu = (m, 0, 0, 0)$ for convenience, in which

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2m\mathbb{1}$$

Define again ladder operators for N=1

$$a_{1,2} = \frac{1}{\sqrt{2m}} Q_{1,2} \qquad a^\dagger_{1,2} = \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{1},\dot{2}} \tag{4.1}$$

and "ground states" $|\Omega\rangle$ annihilated by the $a_{1,2}$. Note that since $|\Omega\rangle$ is an eigenstate of spin there are actually $2s_0 + 1$ ground states:

$$|\Omega\rangle = |m, s_0, s_3\rangle \qquad s_3 = -s_0, ..., s_0$$

Acting on them with the ladder operators (4.1) leaves us with $4(2s_0 + 1)$ states in the massive supermultiplet:

$$
\begin{aligned}
&|\Omega\rangle \\
a_1^\dagger &|\Omega\rangle \\
a_2^\dagger &|\Omega\rangle \\
\frac{1}{\sqrt{2}} a_1^\dagger a_2^\dagger &|\Omega\rangle = -\frac{1}{\sqrt{2}} a_2^\dagger a_1^\dagger |\Omega\rangle
\end{aligned}
\tag{4.2}
$$

For $s_0 = 0$ we have $s_3 = 0^+, \frac{1}{2}, -\frac{1}{2}, 0^-$ which corresponds to one massive Weyl fermion, one real scalar and one pseudoscalar. The pseudoscalar arises from the action of parity on the last state in (4.2) which has the effect of interchanging $a_1^\dagger$ and $a_2^\dagger$. We can also obtain a $(1, \frac{1}{2}, \frac{1}{2}, 0)$ supermultiplet plus its CPT conjugate $(-1, -\frac{1}{2}, -\frac{1}{2}, 0)$ corresponding to a Dirac fermion with 4 degrees of freedom, a vector boson with 3 degrees of freedom and a real scalar with 1 degree of freedom. This is the same as a massless chiral multiplet plus a massless vector mutiplet, and are related by a Higgs mechanism [9].

5 Representation on Fields

The most useful representation we are interested in is in terms of quantum fields. This is usually presented in the superspace formalism (see e.g. [11]) though we will avoid using superfields here.

The elements of a Hilbert space of a quantum field theory can be generated by the action of a field-valued operator $\phi(x)$ on a translationally invariant vacuum:

$$\phi(x)|0\rangle = |x\rangle$$

Noether's theorem guarantees us an energy-momentum operator P^μ that generates translations in spacetime:

$$e^{iy\cdot P}|x\rangle = |x+y\rangle$$

So the displacement of a field can be written as

$$\phi(x+y) = e^{iy\cdot P}\phi(x)e^{-iy\cdot P}$$

and for infinitesimal transformations we get

$$[\phi, P_\mu] = i\partial_\mu \phi \tag{5.1}$$

Since the action of P^μ on fields is known, we can then construct an N=1 supersymmetry representation using the algebra (1.2). In particular:

$$
\begin{aligned}
\{Q, \bar{Q}\} &= 2\sigma^\mu P^\mu \\
\{Q, Q\} &= \{\bar{Q}, \bar{Q}\} = 0
\end{aligned}
\tag{5.2}
$$

5.1 Chiral Field Multiplet

The basic idea is to define fields from the action of the supersymmetry generators Q, $\bar{Q}$ on a "ground state" field, then use (5.2) along with the Jacobi identities to constrain the results. We follow the seven step process outlined in Sohnius [8].

Step 1) Choose a complex scalar field $A(x)$ as the ground state of our field multiplet.

Step 2) Impose the constraint

$$[A, \bar{Q}_{\dot{\alpha}}] = 0 \tag{5.3}$$

This defines the chiral multiplet we are constructing. It is possible to choose $[A, Q] = 0$ instead which would form an anti-chiral multiplet related to the former by Hermitian conjugation, or to drop this requirement completely to obtain a general supermultiplet, though this would be more laborious to construct.

Step 3) Define the fields $\Psi_\alpha(x)$, $F_{\alpha\beta}(x)$ and $X_{\alpha\dot{\beta}}(x)$ from successive actions of the supersymmetry generators:

$$\begin{aligned} [A, Q_\alpha] &\equiv 2i\Psi_\alpha \\ \{\Psi_\alpha, Q_\beta\} &\equiv -iF_{\alpha\beta} \\ \{\Psi_\alpha, \bar{Q}_{\dot{\beta}}\} &\equiv X_{\alpha\dot{\beta}} \end{aligned} \tag{5.4}$$

The factors are for later convenience. Note that Ψ_α must be an odd anti-commuting element since it is the result of an anticommutation relation between an even and odd generator. Similarly $F_{\alpha\beta}$ and $X_{\alpha\dot{\beta}}$ must be even.

Step 4) Enforce the algebra on $A(x)$ using the graded Jacobi identity:

$$\begin{aligned} [A, \{Q, \bar{Q}\}] &= \{[A, Q], \bar{Q}\} + \{[A, \bar{Q}], Q\} \\ 2i\sigma^\mu \partial_\mu A &= \{[A, Q], \bar{Q}\} \end{aligned}$$

We substituted (5.2) along with the action of P_μ on a field (5.1) to get the left-hand side, and used our chirality constraint (5.3) on the right-hand side. We see that if $A(x)$ were real the right-hand side would have to disappear entirely and A would just be constant, hence it is necessarily a complex scalar field. Using the definition of our fields (5.4) we get an equation for X:

$$\begin{aligned} 2i(\sigma^\mu)_{\alpha\dot{\beta}} \partial_\mu A &= 2i\{\Psi_\alpha, \bar{Q}_{\dot{\beta}}\} \\ (\sigma^\mu)_{\alpha\dot{\beta}} \partial_\mu A &= X_{\alpha\dot{\beta}} \end{aligned}$$

Similarly we get an equation for F from another Jacobi identity:

$$\begin{aligned} 0 &= [A, \{Q_\alpha, Q_\beta\}] = \{[A, Q_\alpha], Q_\beta\} + \{[A, Q_\beta], Q_\alpha\} \\ 0 &= F_{\alpha\beta} + F_{\beta\alpha} \\ &\Rightarrow F_{\alpha\beta} = \epsilon_{\alpha\beta} F \end{aligned}$$

Where $F(x)$ is a complex scalar field and $\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is an antisymmetric tensor (see appendix A.1).

Step 5) From (5.4) there remains the action of Q on F to explore. Define fields λ_α and $\bar{\chi}_{\dot{\alpha}}$ as

$$[F, Q_\alpha] \equiv \lambda_\alpha \qquad [F, \bar{Q}_{\dot{\alpha}}] \equiv \bar{\chi}_{\dot{\alpha}}$$

Step 6) Enforce the algebra on Ψ the same way we did for A:

$$\begin{aligned}
[\Psi_\alpha, \{Q_\beta, \bar{Q}_{\dot{\beta}}\}] &= [\{\Psi_\alpha, Q_\beta\}, \bar{Q}_{\dot{\beta}}] + [\{\Psi_\alpha, \bar{Q}_{\dot{\beta}}, Q_\beta] \\
2i(\sigma^\mu)_{\beta\dot{\beta}}\partial_\mu \Psi_\alpha &= -i\epsilon_{\alpha\beta}[F, \bar{Q}_{\dot{\beta}}] + (\sigma^\mu)_{\alpha\dot{\beta}}[\partial_\mu A, Q_\beta] \\
2i(\sigma^\mu)_{\beta\dot{\beta}}\partial_\mu \Psi_\alpha &= -i\epsilon_{\alpha\beta}\bar{\chi}_{\dot{\beta}} + 2i(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu \Psi_\beta \\
&\Rightarrow \bar{\chi}_{\dot{\alpha}} = 2\partial_\mu \Psi^\beta (\sigma^\mu)_{\beta\dot{\alpha}}
\end{aligned}$$

The last line is obtained by raising and contracting indices with $\epsilon^{\alpha\beta}$. Similarly to get λ_α:

$$\begin{aligned}
0 &= [\Psi_\alpha, \{Q_\beta, Q_\gamma\}] = [\{\Psi_\alpha, Q_\beta\}, Q_\gamma] + [\{\Psi_\alpha, Q_\gamma\}, Q_\beta] \\
0 &= \epsilon_{\alpha\beta}\lambda_\gamma + \epsilon_{\alpha\gamma}\lambda_\beta \\
&\Rightarrow \lambda_\alpha = 0
\end{aligned}$$

Step 7) This is everything we need, but we could also check that the following remaining conditions are satisfied:

$$\begin{aligned}
&[\Psi, \{Q, \bar{Q}\}] = [F, \{Q, Q\}] = [F, \{\bar{Q}, \bar{Q}\}] = 0 \\
&[F, \{Q, \bar{Q}\}] = 2i\sigma^\mu \partial_\mu F
\end{aligned}$$

Finally we have succeeded in constructing our N=1 chiral representation on fields in terms of a multiplet $\phi = (A; \Psi; F)$ containing two complex scalars A, F and one spinor Ψ. We will comment later on the degrees of freedom of this multiplet. The supersymmetry transformations summarised in terms of (anti)commutators are

$$\begin{aligned}
[A, Q_\alpha] &= 2i\Psi_\alpha \\
[A, \bar{Q}_{\dot{\alpha}}] &= 0 \\
\{\Psi_\alpha, Q_\beta\} &= -i\epsilon_{\alpha\beta}F \\
\{\Psi_\alpha, \bar{Q}_{\dot{\beta}}\} &= (\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu A \\
[F, Q_\alpha] &= 0 \\
[F, \bar{Q}_{\dot{\alpha}}] &= 2\partial_\mu \Psi^\beta (\sigma^\mu)_{\beta\dot{\alpha}}
\end{aligned}$$

A general infinitesimal variation of the supermultiplet Φ can be written by introducing an anticommuting spinor parameter η^α with the summation convention (A.2):

$$\delta\phi = -i[\phi, \eta Q + \bar{Q}\bar{\eta}]$$

$$\begin{aligned}
\delta A &= 2\eta\Psi \\
\delta\Psi &= -\eta F - i\partial_\mu A\sigma^\mu\bar{\eta} \\
\delta F &= -2i\partial_\mu\Psi\sigma^\mu\bar{\eta}
\end{aligned}$$

The algebra can be summarized by the commutator of two successive transformations:

$$[\delta_1, \delta_2]\phi = 2i(\eta_1\sigma^\mu\bar{\eta}_2 - \eta_2\sigma^\mu\bar{\eta}_1)\partial_\mu\phi$$

5.2 Majorana Form of the Chiral Multiplet

Since the four-component notation is also widely used we will rewrite our chiral multiplet in terms of Majorana spinors (see appendix A.2). The supersymmetric generator in Majorana form can be written in terms of our chiral ones as

$$Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix}$$

and the commutator (5.2) in the N=1 algebra becomes

$$\{Q, \bar{Q}\} = 2\gamma^\mu P_\mu$$

Similarly defining our anticommuting parameter η in four-component notation we can write the variation of the supermultiplet field:

$$\delta\phi = -i[\phi, \bar{\eta}Q] \qquad \bar{\eta}Q = \eta^\alpha Q_\alpha + \bar{Q}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}$$

To obtain the real form of the chiral supermultiplet we separate the real and imaginary parts of the complex fields A and F into A, B and F, G respectively, and Ψ becomes a Majorana spinor constructed from the two-component spinors Ψ_α and $\bar{\Psi}_{\dot{\alpha}}$. The chiral constraint (5.3) in step 2 then translates into the presence of a γ_5 on the transformations of B and G, which are otherwise identical to A and F. Note that this makes B and G pseudoscalars under parity. The real chiral supermultiplet $\phi = (A; B; \Psi; F; G)$ then transforms as

$$\begin{aligned}
\delta A &= \bar{\eta}\Psi \\
\delta B &= \bar{\eta}\gamma_5\Psi \\
\delta\Psi &= -(F + \gamma_5 G)\eta - i\not{\partial}(A + \gamma_5 B)\eta \\
\delta F &= i\bar{\eta}\not{\partial}\Psi \\
\delta G &= i\bar{\eta}\gamma_5\not{\partial}\eta
\end{aligned} \tag{5.5}$$

6 On-Shell and Off-Shell Representations

The chiral multiplet of section 5 contains 8 degrees of freedom: 4 fermionic and 4 bosonic ones. This is the minimum required since a spinor in four dimensions must have at least four real components (or two complex ones), hence the multiplet is irreducible. This result holds in all generality, but in quantum field theories we know that when we subject the fields to an equation of motion their dynamical degrees of freedom are reduced, for example the four components of a spinor are constrained by the Dirac equation to two degrees of freedom.

The field multiplet we derived is said to be an "off-shell" representation. If we use the equations of motion we are taking them "on-shell". But if the spinor is constrained what happens to the bosonic degrees of freedom? We proved in section 2 that the number of bosonic degrees of freedom must *always* equal the fermionic degrees of freedom!

6.1 The Wess-Zumino Model

A quick look at the Wess-Zumino model will make things clear. From dimensional arguments the Lagrangian in this model is the most general renormalisable Lagrangian one can write for a single chiral multiplet [1]:

$$L = \frac{1}{2}\phi T\phi - \frac{1}{2}m\phi\cdot\phi - \frac{g}{3}\phi\cdot\phi\cdot\phi$$

where $T\phi = (F; G; i\partial\!\!\!/\psi; -\partial^2 A; -\partial^2 B)$ is the kinetic multiplet. In components the Lagrangian can be written explicitly as

$$L = L_0 + L_m + L_g$$

$$\begin{aligned}
L_0 &= \frac{1}{2}(\partial_\mu A\partial^\mu A + \partial_\mu B\partial^\mu B + i\bar{\psi}\partial\!\!\!/\psi + F^2 + G^2) \\
L_m &= -m(AF + BG + \frac{1}{2}\bar{\psi}\psi) \\
L_g &= -g[(A^2 - B^2)F + 2ABG + \bar{\psi}(A - \gamma_5 B)\psi]
\end{aligned}$$

where we dropped a surface term in L_0 that has no effect on the action. A straight-forward calculation yields the Euler-Lagrange equations of motion:

$$\begin{aligned}
F &= mA + g(A^2 - B^2) \\
G &= mB + 2gAB \\
i\partial\!\!\!/\psi &= m\psi + 2g(A - \gamma_5 B)\psi \\
-\partial^2 A &= mF + 2g(AF + BG + \frac{1}{2}\psi\bar{\psi}) \\
-\partial^2 B &= mG + 2g(AG - BF - \frac{1}{2}\bar{\psi}\gamma_5\psi)
\end{aligned}$$

We see that while the equations of motion for ψ, A and B involve derivatives and describe propagation in spacetime, F and G have purely algebraic equations in terms of A and

B that can be eliminated. Such fields are called *auxiliary fields*, and their elimination explains what happens to the bosonic degrees of freedom when we take our Lagrangian on-shell. The result is

$$\begin{aligned} L =&\frac{1}{2}(\partial_\mu A\partial^\mu A - m^2A^2) + \frac{1}{2}(\partial_\mu B\partial^\mu B - m^2B^2)+ \\ &\frac{1}{2}\bar{\psi}(i\partial\!\!\!/ - m)\psi - mgA(A^2+B^2) - g\bar{\psi}(A-\gamma_5 B)\psi - \frac{g^2}{2}(A^2+B^2)^2 \end{aligned}$$

We end by highlighting a subtlety involved in our on-shell formulation. By using the on-shell equations of motion

$$\begin{aligned} (\partial^2+m^2)A &= -mg(3A^2+B^2) - 2g^2A(A^2+B^2) - g\bar{\psi}\psi \\ (\partial^2+m^2)B &= -2mgAB - 2g^2B(A^2+B^2) + g\bar{\psi}\gamma_5\psi \\ (i\partial\!\!\!/ - m)\psi &= 2g(A-\gamma_5 B)\psi \end{aligned} \tag{6.1}$$

we can derive the on-shell supermultiplet transformations from (5.5):

$$\begin{aligned} \delta A &= \bar{\eta}\psi \\ \delta B &= \bar{\eta}\gamma_5\psi \\ \delta\psi &= -[i\partial\!\!\!/ + m + g(A+\gamma_5 B)](A+\gamma_5 B)\eta \end{aligned}$$

A problem appears if we take the commutator of two successive transformations on ψ:

$$[\delta_1,\delta_2]\psi = 2i\bar{\eta}_1\partial\!\!\!/\eta_2\psi - \gamma^\mu(i\partial\!\!\!/ - m - 2g(A-\gamma_5 B))\psi\bar{\eta}_1\gamma_\mu\eta_2$$

The on-shell algebra only closes if we use the equation of motion (6.1) to set the last term to zero! This has implications for calculations of quantum corrections where the fields have to be taken off-shell [8].

7 Conclusion

In the introduction we took the viewpoint that supersymmetry is the inevitable consequence of extending the symmetries of the S-matrix [3], but historically the search for a higher unifying symmetry that could relate particles of different spin was motivated by the early success of using symmetry principles to classify particles, most famously in Murray Gell-Mann's use of SU(3). Such a unification of force and matter particles would place fermions and bosons in the same multiplet, but since spin is related to behaviour under rotations the supersymmetry generator wouldn't commute with the Lorentz generators, in direct violation of the Coleman-Mandula no-go theorem that states all internal symmetry generators must transform as Lorentz scalars. An extension of the theorem by Haag, Lopuszanski and Sohnius [7] proved that supersymmetry is only realisable by spinorial generators carrying spin $\frac{1}{2}$.

In section 2 we showed that the states of a supermultiplet must all have the same mass. Since experiments rule out the existence of equal-mass superpartners, supersymmetry must be spontaneously broken in Nature i.e. the energy of the ground-state is non-zero and not invariant under supersymmetry [10]. This would preserve the structure of the supermultiplet and lift the mass degeneracy, with heavier superpartners outside the range of current experiments (though perhaps soon accessible by the LHC).

Nature, excluding gravity, is currently best described by the theoretical framework of quantum field theory. The realisation of supersymmetry in field theory is therefore of primary interest, and we considered the construction of field supermultiplets in section 5. The Wess-Zumino model was the first renormalisable supersymmetric model, and since then a large number of constructions, including the minimally supersymmetric standard model, have been written down and analysed. The behaviour of such models lead to remarkable cancellations of quantum divergences that has implications for the GUT hierarchy problem and supergravity [10].

Supersymmetry also provides the only known path to a unification of gravity with the other fundamental forces within the framework of quantum field theory [8]. If we assume that we can extrapolate general relativity to very small distances, we can argue that the gravitational field must be quantised in order not to violate Heisenberg's uncertainty principle by experiments that could, in principle, measure position and momentum with arbitrary accuracy through gravitational effects. In such a framework a purely attractive force must be carried by a field of spin 2, whereas the carrier of electromagnetism for example has spin 1. Since bosonic symmetry generators are forbidden the only way to relate particles of different integer spin is through intermediate fractional spin particles, thus if we wish to unify all the forces of Nature we require supersymmetry. The only alternative is to abandon unification or quantum field theory!

To summarize, we see that despite the complete lack of experimental evidence the attractiveness of supersymmetry lies in the fact that it hits several birds with one stone. It arises naturally out of strong physical constraints on combining internal and spacetime symmetries, unifies bosons and fermions, leads to "miraculous" cancellations in field theories, and provides the only path to incorporating gravity in a quantum framework.

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A Appendix

A.1 Two-Spinors

The algebra of the Lorentz group can be written in terms of six Hermitian generators K_i, J_i of boosts and rotations respectively, arranged into an antisymmetric 2nd rank tensor

$$M_{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

In order to find the representations we arrange the algebra in terms of six non-Hermitian operators $\vec{J}^{\pm}$

$$J_1^{\pm} \equiv \frac{1}{2}(M_{23} \pm iM_{01}) \tag{A.1}$$

and all cyclic combinations. We chose these operators in such a way that their algebra resembles that of two commuting SU(2) algebras, albeit with non-Hermitian generators:

$$\begin{aligned} [J_i^{\pm}, J_j^{\pm}] &= i\epsilon_{ijk}J_k^{\pm} \\ [\vec{J}^{\pm}, \vec{J}^{\mp}] &= 0 \end{aligned}$$

The Lorentz group is therefore a complexified SU(2)xSU(2), or Sl(2,C). Since we know the representations of SU(2) we can classify the representations of the Lorentz group by pairs of half-integer numbers (j_+, j_-) taken from the $j_{\pm}(j_{\pm}+1)$ eigenvalues of the $(\vec{J}_{\pm})^2$ operator.

For example inverting (A.1) we have

$$\begin{aligned} M_{23} &= J_1^+ + J_1^- \\ M_{01} &= -i(J_1^+ - J_1^-) \end{aligned}$$

and the representation $(\frac{1}{2}, 0)$ with $r(\vec{J}^+) = \frac{1}{2}\vec{\sigma}$ and $r(\vec{J}^-) = 0$ is then generated by

$$r(M_{23}) = \frac{1}{2}\sigma_1 \qquad r(M_{0i}) = -\frac{1}{2}i\sigma_i$$

where the σ pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The generators for the $(\frac{1}{2}, 0)$ representation can be written in general as $\frac{1}{2}\sigma_{\mu\nu}$ where

$$\begin{aligned} \sigma_{\mu\nu} &= \frac{1}{2}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu) \\ \sigma_\mu &= (\mathbb{1}, \sigma_i) \quad \bar{\sigma}_\mu = (\mathbb{1}, -\sigma_i) \end{aligned}$$

and similarly $\frac{1}{2}\bar{\sigma}_{\mu\nu} = \frac{1}{2}(\sigma_{\mu\nu})^\dagger$ generates the $(0, \frac{1}{2})$ representation.

We define the two-component spinors as objects that transform under these generators of the Sl(2,C) group. Note that if $r(g)$ is a homomorphism of the group into a matrix space, then so are $r^*(g)$, $r^{-1T}(g)$, and $r^{-1\dagger}(g)$. The spinors on which these act are denoted as u_α, $\bar{u}_{\dot{\alpha}}$, u^α and $\bar{u}^{\dot{\alpha}}$ respectively.

Two-spinor indices can be raised and lowered using the Levi-Civita tensors

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Our convention for summation over unwritten indices is

$$\eta\Psi \equiv \eta^\alpha\psi_\alpha = -\eta_\alpha\psi^\alpha \qquad \bar{\psi}\bar{\eta} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}} \tag{A.2}$$

A.2 Dirac and Majorana Spinors

A four-component Dirac spinor is constructed from a dotted and un-dotted two-component spinor:

$$\psi = \begin{pmatrix} \phi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

It transforms as a reducible $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz group with generators

$$S^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu}$$
$$\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{1}{2}\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}$$

and the Dirac matrices in the Weyl representation is defined as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

We can see that $\begin{pmatrix} \phi_\alpha \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$ will be chiral Dirac spinors, also called Weyl spinors. A Majorana spinor $\begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$ is a Dirac spinor with half as many independent components.