Proseminar on supersymmetry: 3. Superspace and Superfields

Philipp GRUBER
Supervisor: Marco Panero

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Abstract

In this report we want to consider the superfield formalism. We start by reviewing space-time translations and then switch to supersymmetric transformations. We obtain the change in the superfield by inducing shifts in the spinorial coordinates. This will lead to a differential operator representation on the SuSy algebra. We then make the transition to chiral superfields and explore the terms that will appear in the expansion of the chiral superfield. We will take a look at other (more physical) representations of superfields, which unfortunately correspond to reducible representations. We will see how to fix this problem and finally obtain a real valued chiral superfield which has an irreducible representation.

Contents

1 Introduction

In this report we are going to review the superfield formalism. It was first introduced by Salam and Strathdee [6] for the N=1 case. There exist other approaches for N> 1 which include more coordinates than in the N=1 case, but we are not going to review these here because this would be beyond our scope. Salam and Strathdee introduced new fermionic coordinates. We will pay special attention on how to parametrize this manifold[1]. The superfield formalism allowes one to us to gather supermultiplets into superfields. In previous lectures we have seen the direct construction of field supermultiplets by applying the Jacobi identity over and over again on fields until we found an irreducible multiplet. This tedious work will be avoided by introducing a superfield and its expansion in the fermionic coordinates. We will see that an expansion will lead to irreducible multiplets only in the special case of chiral superfields which do not depend on one of its fermionic coordinates. The second part of this report will therefore concentrate on finding constraints which will yield irreducible multiplets even for expansions of general superfields to obtain, again, chiral multiplets. The case of vector, gravitino or graviton multiplets will not be discussed, but will be postponed to lectures about gauge transformations of supersymmetric lagrangians.

2 Transformations on fields

2.1 Review on space-time transformations

When we perform a translation in space-time [2], we change the coordinate by a constant, infinitesimal 4-vector ϵ^μ :

$$x'^\mu = x^\mu + \epsilon^\mu \tag{1}$$

Now we consider a scalar field $\phi(x)$ which we want to evaluate at a space-time coordinate x' and act upon the states $|\alpha\rangle$ which transform as $|\alpha\rangle' = U|\alpha\rangle$, with the infinitesimal translation

$$U = 1 + i\epsilon_\mu P^\mu \tag{2}$$

where the P^μ are the generators of this transformation. As the matrix elements must remain invariant we get:

$$\langle\beta|U^{-1}\phi(x')U|\alpha\rangle = \langle\beta|\phi(x)|\alpha\rangle \tag{3}$$

This should be true for all states, such that we infer

$$U^{-1}\phi(x')U = \phi(x) \tag{4}$$

or

$$U\phi(x)U^{-1} = \phi(x') = \phi(x+\epsilon) \tag{5}$$

We insert the explicit form of U , expand in ϵ and get

$$(1 + i\epsilon_\mu P^\mu)\phi(x)(1 - i\epsilon_\mu P^\mu) = \phi(x) + \epsilon^\mu \frac{\partial\phi}{\partial x^\mu} \tag{6}$$

[1] Group theory for this procedure and the inclusion of the Lorentz subgroup can be found in [7]. For the superfield formalism it is sufficient to work on the coset space super-Poincaré/Lorentz which contains the generators of space-time translations and the group elements of the supersymmetry algebra.

and deduce

$$\delta\phi(x) = i\epsilon_\mu[P^\mu, \phi(x)] = \epsilon_\mu\partial^\mu\phi(x) \tag{7}$$

where we can identify the differential operator $\hat{P}^\mu$:

$$\delta\phi(x) = \epsilon_\mu\partial^\mu\phi(x) = -i\epsilon_\mu\hat{P}^\mu\phi(x) \tag{8}$$

This is the known result from our "simple" translation. We are now going to use this derivation as an analogy to our derivation for the generators in superspace. But what do we actually mean by superspace? We are going to enlarge the space of coordinates, which used to be only x^μ up to now, by fermionic degrees of freedom, the spinorial coordinates θ and θ^*. We call the fields, which act upon this newly introduced superspace, superfields. We have just seen that the operator $\hat{P}^\mu$ generates a shift in the space-time components of ϕ, so we might want to find operators Q and $Q^\dagger$ that induce a shift in the spinorial coordinates. But these operators should also obey the supersymmetry algebra, defined in (19), which we have seen in previous lectures (and indeed they will, as we are going to see).

To perform a finite transformation, we do now exponentiate linear combinations of the generators and obtain for a simple space-time translation from $0 \to x$:

$$U = e^{ix\cdot P} \Rightarrow e^{ix\cdot P}\phi(0)e^{-ix\cdot P} = \phi(x) \tag{9}$$

2.2 Supersymmetry transformations

We now want to introduce the superfield formalism. The N=1 case the superspace consists of 4 (bosonic) space-time coordinates and 4 fermionic coordinates. These fermionic coordinates will be represented by set of anticommuting Grassman variables. Their properties can be found in the appendix.

To obtain a kind of transformation like (9) in superspace we define

$$U(x,\theta,\theta^*) = e^{ix\cdot P}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}} \tag{10}$$

which is, indeed, a bit arbitrary, since we could have also taken the exponential of $\bar{\theta}\cdot\bar{Q}$ before the exponential of $\theta\cdot Q$ which will gives us a different result, as they do not commute. We will cover this problem in the later sections. To remind us of the notation, we write the dot product as

$$\theta\cdot Q = \theta^T(-i\sigma_2)Q \tag{11}$$

$$\bar{\theta}\cdot\bar{Q} = \theta^\dagger(i\sigma_2)Q^{\dagger T} \tag{12}$$

Just as with the evolution from $0 \to x$ in standard space-time, we are going to do this transformation in superspace:

$$U(x,\theta,\theta^*)\Phi(0)U^{-1}(x,\theta,\theta^*) = \Phi(x,\theta,\theta^*) \tag{13}$$

where $\Phi(x,\theta,\theta^*)$ denotes our superfield. To calculate a more complex translation than that from $0 \to x$, we have to calculate the product $U(a,\xi,\xi^*)U(x,\theta,\theta^*)$ and see what happens[2]. What can we expect? Will it just be a normal translation from $x \to a$, $\theta \to \xi$, $\theta^* \to \xi^*$? In hindsight to the supersymmetry algebra, we might expect this not to happen since we need a nontrivial connection

[2]Following the procedure proposed in [1]

between the differential operators of the spinorial coordinates Q, $Q^\dagger$ and the generator of space-time translations P^μ. So let us try to find an answer. Explicitly, the product

$$U(a,\xi,\xi^*)U(x,\theta,\theta^*) = e^{ia\cdot P}e^{i\xi\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}}e^{ix\cdot P}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}} \tag{14}$$

is not as easy to obtain as before, since Q and $\bar{Q}$ obey the supersymmetry algebra (19). Therefore we cannot just use a simple algebraic approach, but use the Baker-Campbell-Hausdorff (B-C-H) identity:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots} \tag{15}$$

On the other hand, we know that P^μ and Q (respectively $\bar{Q}$) commute, so we can write the nontrivial part:

$$e^{i\xi\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}} \tag{16}$$

The first part is

$$e^{i\xi\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}} = e^{i\xi\cdot Q+i\bar{\xi}\cdot\bar{Q}-\frac{1}{2}[\xi\cdot Q,\bar{\xi}\cdot\bar{Q}]+\ldots} \tag{17}$$

with

$$\begin{aligned}
[\xi\cdot Q,\bar{\xi}\cdot\bar{Q}] &= [\xi^a Q_a, -\xi^{b*}Q_b^\dagger] \\
&= -\xi^a Q_a \xi^{b*}Q_b^\dagger + \xi^{b*}Q_b^\dagger \xi^a Q_a \\
&= \xi^a\xi^{b*}(Q_a Q_b^\dagger + Q_b^\dagger Q_a) \\
&= \xi^a\xi^{b*}(\sigma^\mu)_{ab}P_\mu
\end{aligned} \tag{18}$$

In the last line we have used the anticommutator

$$\{Q_a, Q_b^\dagger\} = (\sigma^\mu)_{ab}P_\mu \tag{19}$$

which was derived in the previous section about supersymmetry algebra.

What happened: We induced a shift in the spinorial coordinates and obtained not only a change in the spinorial coordinates, but also in the space-time coordinates!

$$e^{i\xi\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}} = e^{iA\cdot P}e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q})} \tag{20}$$

with

$$A^\mu = \frac{1}{2}i\xi^a(\sigma^\mu)_{ab}\xi^{b*} \tag{21}$$

The next factor will be:

$$\begin{aligned}
e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q})}e^{i\theta\cdot Q} &= e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q)-\frac{1}{2}[\xi\cdot Q+\bar{\xi}\cdot\bar{Q},\theta\cdot Q]+\ldots} \\
&= e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q)+\frac{1}{2}i\theta^a(\sigma^\mu)_{ab}\xi^{b*}P_\mu+\ldots}
\end{aligned} \tag{22}$$

To exclude the $\bar{Q}$ term from the exponential we use B-C-H again, but this time in the opposite direction:

$$e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q)} = e^{i(\xi+\theta)\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}}e^{\frac{1}{2}[(\xi+\theta)\cdot Q,\bar{\xi}\cdot\bar{Q}]} \tag{23}$$

Putting all the prefactors together leads to our final result:

$$e^{i\xi\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}} = e^{i[-i\theta^a(\sigma^\mu)_{ab}\xi^{b*}P_\mu]}e^{i(\xi+\theta)\cdot Q}e^{i(\bar{\xi}+\bar{\theta})\cdot\bar{Q}} \tag{24}$$

where all spinorial and space-time shifts are included. We read the transformations

$$
\begin{aligned}
0 &\to \theta \to \theta + \xi \\
0 &\to \theta^* \to \theta^* + \xi^* && (25) \\
0 &\to x^\mu \to x^\mu + a^\mu - i\theta^a (\sigma^\mu)_{ab} \xi^{b*} && (26)
\end{aligned}
$$

As expected, we can see that a change in the spinorial coordinates was induced, as well as a change in space-time through a^μ but also through θ^a (Why was no change induced by θ^*? The reason for this was the choice of $U(x, \theta, \theta^*) = e^{ix\cdot P} e^{i\theta\cdot Q} e^{i\bar{\theta}\cdot\bar{Q}}$. As mentioned we will discuss this issue later. For now it is ok to work with this result.)

3 Differential operator representation

If we were to perform a shift purely by spinorial arguments ξ^a, we would see that the change in the field is given by[3]

$$
\delta\Phi = -i\theta^a (\sigma^\mu)_{ab} \xi^{b*} \partial_\mu \Phi + \xi^a \frac{\partial \Phi}{\partial \theta^a} + \xi^*_a \frac{\partial \Phi}{\partial \theta^*_a} \qquad (27)
$$

or, in terms of differential operators $\hat{Q}_a$,$\hat{Q}^\dagger_a$,in analogy to (8)

$$
\begin{aligned}
\delta\Phi &= (-i\xi \cdot \hat{Q} - i\bar{\xi} \cdot \bar{\hat{Q}})\Phi \\
&= (-i\xi^a \hat{Q}_a - i\xi^*_a \hat{Q}^{\dagger a})\Phi && (28)
\end{aligned}
$$

Therefore we can identify the operators as

$$
\begin{aligned}
\hat{Q}_a &= i\frac{\partial}{\partial \theta^a} && (29) \\
\hat{Q}^\dagger_a &= -i\frac{\partial}{\partial \theta^{a*}} + \theta^b (\sigma^\mu)_{ba} \partial_\mu && (30)
\end{aligned}
$$

using $\xi^*_a \hat{Q}^{\dagger a} = -\xi^{a*} \hat{Q}^\dagger_a$ and anticommutation of Grassman variables. And, just as expected, they fulfill the supersymmetry algebra.

As we can see, we have derived a representation of the generators purely out of the superspace formalism. In the next chapters we will recognize that this will be very useful, i.e. in obtaining the corresponding multiplets.

4 Chiral superfields

In this chapter we will expand superfields in the spinorial coordinates. A major advantage in doing so is that this expansion will always be exaxt because of the fermionic nature of the variables. Any quadratic term in $\theta_1$or θ_2 will vanish, thus giving a compact expansion of the superfield. To simplify

[3]The properties of the spinorial derivatives can be found in chapter 9.5 of [5].

this, we require our fields not to depend on the second spinorial coordinate θ^*. As it contains only the L-type spinor χ_a we are going to call it a left-chiral superfield. We write

$$\Phi(x,\theta) = \phi(x) + \theta \cdot \chi(x) + \frac{1}{2}\theta \cdot \theta F(x) \tag{31}$$

As one can see there are only three terms in this expansion:

- $\phi(x)$, a scalar field which is the part of Φ independent of θ
- $\chi_a(x)$, a spinor field, representing the coefficient of the part linear in θ
- $F(x)$, which will act as our auxilliary field.

Together they act as our chiral supermultiplet. We now want to compute the change in each field seperately. We proceed in calculating the space-time and spinorial derivatives and then compare each of the changes by sorting them in terms of θ:

$$\begin{aligned}\delta\Phi &= [-i\theta^a(\sigma^\mu)_{ab}\xi^{b*}\partial_\mu + \xi^a\frac{\partial}{\partial\theta^a}][\phi(x) + \theta\cdot\chi(x) + \frac{1}{2}\theta\cdot\theta F(x)] \\ &= \delta_\xi\phi + \theta^a\delta_\xi\chi_a + \frac{1}{2}\theta\cdot\theta\delta_\xi F(x)\end{aligned} \tag{32}$$

The space-time derivative can only act on $\phi(x)$ and $\chi_a(x)$ because any term cubic in θ will cancel out anyway. We obtain

$$-i\theta^b(\sigma^\mu)_{ba}\xi^{a*}\partial_\mu\phi + i\xi^{a*}\theta^b(\sigma^\mu)_{ba}\theta^c\partial_\mu\chi_c \tag{33}$$

The spinorial derivative gives

$$\frac{\partial}{\partial\theta^a}[\theta^c\chi_c + \frac{1}{2}\theta\cdot\theta F(x)] = \chi_a + \theta_a F(x) \tag{34}$$

Finally we read the changes induced by ξ (independent, linear or bilinear in θ):

$$\delta_\xi\phi = \xi^a\chi_a \tag{35}$$

$$\delta_\xi\chi_a = \xi_a F - i(\sigma^\mu)_{ab}\xi^{b*}\partial_\mu\phi \tag{36}$$

$$\delta_\xi F(x) = -i\xi^{a*}(\sigma^{\mu T})_{ab}\epsilon^{bc}\partial_\mu\chi_c \tag{37}$$

Observing these changes, we notice one of the most important results of this lecture: The change in the F-field is only induced by a total derivative $\partial_\mu\chi_c$ because all the other terms do not depend on x^μ ! Doing the same analysis for a right-chiral superfield we see that the F-component of a chiral superfield will always change by a total derivative. We will use this result again in the coming sections to analyze SuSy invariant actions.

5 Other forms of chiral superfields

5.1 The 'real-type' superfield

As mentioned twice above, we encountered an ambiguity regarding the definition of $U(x,\theta,\theta^*)$. From now on we will refer to it as a 'type I' superfield. We postulated it to be

$$U(x,\theta,\theta^*) = U_I(x,\theta,\theta^*) = e^{ix\cdot P}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}} \tag{38}$$

And indeed, we could have also written one of the following terms:

$$\begin{aligned} U_{II}(x,\theta,\theta^*) &= e^{ix\cdot P}e^{i\bar{\theta}\cdot\bar{Q}}e^{i\theta\cdot Q} && (39)\\ U_{real}(x,\theta,\theta^*) &= e^{ix\cdot P}e^{i[\theta\cdot Q+\bar{\theta}\cdot\bar{Q}]} && (40)\end{aligned}$$

The transformation $U_{real}(x,\theta,\theta^*)$ will have a special physical meaning. Consider a superfield $\Phi_{real}(x,\theta,\theta^*)$ which is generated with this specific transformation:

$$\Phi_{real}(x,\theta,\theta^*) = e^{i[\theta\cdot Q+\bar{\theta}\cdot\bar{Q}]}\Phi(x,0,0)e^{-i[\theta\cdot Q+\bar{\theta}\cdot\bar{Q}]} \tag{41}$$

If we require $\Phi(x,0,0)$ to be real: $\Phi^\dagger(x,0,0) = \Phi(x,0,0)$ then we will infer that also $\Phi^\dagger_{real}(x,\theta,\theta^*) = \Phi_{real}(x,\theta,\theta^*)$ will be real. A quick check tells us that this is not true for 'type I' or 'type II' superfields. If we were to calculate the change in the 'real type' superfield, we would obtain:

$$\begin{aligned} &0 \to \theta \to \theta + \xi \\ &0 \to \theta^* \to \theta^* + \xi^* \\ &0 \to x^\mu \to x^\mu + a^\mu - \frac{1}{2}i\theta^a(\sigma^\mu)_{ab}\xi^{b*} + \frac{1}{2}i\xi^a(\sigma^\mu)_{ab}\theta^{b*} \end{aligned} \tag{42}$$

so to say a symmetric version of (26).

This time we would of course find a different representation of the differential operatiors. But, most importantly, they will also satisfy the same supersymmetry algebra. What can we do with these different representations? To begin, we want to know how they transform into each other. To do so, we write the 'real type' superfield (using B-C-H) as

$$\begin{aligned} \Phi_{real}(x,\theta,\theta^*) &= e^{i[\theta\cdot Q+\bar{\theta}\cdot\bar{Q}]}\Phi(x,0,0)e^{-i[\theta\cdot Q+\bar{\theta}\cdot\bar{Q}]} \\ &= e^{-iB\cdot P}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}}\Phi(x,0,0)e^{-i\bar{\theta}\cdot\bar{Q}}e^{-i\theta\cdot Q}e^{iB\cdot P} \\ &= e^{-iB\cdot P}\Phi_I(x,\theta,\theta^*)e^{iB\cdot P} \end{aligned} \tag{43}$$

with

$$B^\mu = \frac{1}{2}i\theta^a(\sigma)_{ab}\theta^{b*} \tag{44}$$

which yields a difference in the space-time coordinate:

$$\Phi_{real}(x,\theta,\theta^*) = \Phi_I(x^\mu - \frac{1}{2}i\theta^a(\sigma^\mu)_{ab}\theta^{b*},\theta,\theta^*) \tag{45}$$

5.2 General superfields

We recall the expansion of the superfield depending only on x^μ and θ, as given in (31). Of course, this will not be true for any superfield, we still have to deal with $\bar{\theta}$ in general. So we do now want to perform an expansion in both fermionic coordinates. Following standard conventions for general superfields we write[4]

$$\begin{aligned}\Phi(x,\theta,\theta^*) &= \phi(x)+\theta\chi(x)+\bar{\theta}\bar{\gamma}(x)+\theta\theta m(x)+\bar{\theta}\bar{\theta}n(x)\\ &\quad+\theta\sigma^\mu\bar{\theta}v_\mu(x)+\theta\theta\bar{\theta}\bar{\lambda}(x)+\bar{\theta}\bar{\theta}\theta\rho(x)+\theta\theta\bar{\theta}\bar{\theta}d(x)\end{aligned} \tag{46}$$

Where we have

- 4 complex scalar fields $\phi(x)$, $m(x)$, $n(x)$ and $d(x)$
- 1 complex vector field v_μ
- 2 L-type spinors χ and ρ
- 2 R-type spinors $\bar{\gamma}$ and $\bar{\lambda}$

(As a sidenote: if Φ carries extra vector indices, then so do the other component fields.)

Thus we have 16 bosonic and 16 fermionic field components. If we now consider a 'real type' superfield, we stress the fact that $[\Phi_{real}(x,\theta,\bar{\theta})]^\dagger = \Phi_{real}(x,\theta,\bar{\theta})$. This constraint halves the number of field components. We have now found the general multiplet which is formed by the field components of our general superfield. As we know, this multiplet is not an irreducible representation. We will now think about how to fix this problem.

5.3 Covariant spinor derivatives

If we want to reduce the number of field components, we have to impose so called supersymmetric conditions. Therefore we need to introduce covariant spinor derivatives[5]. We have previously calculated the transformation of the superfield by applying the change from the LHS:

$$U(a,\xi,\xi^*)U(x,\theta,\theta^*) = (1 - ia_\mu\hat{P}^\mu - i\xi\cdot\hat{Q} - i\bar{\xi}\cdot\hat{\bar{Q}})U(x,\theta,\theta^*)$$

This is called realization of the group. But group theory also states the associativity of group multiplication, implying anti-realization (inverted order):

$$U(x,\theta,\theta^*)U(a,\xi,\xi^*) = (1 + ia_\mu\hat{D}^\mu + i\xi\cdot\hat{D} + i\bar{\xi}\cdot\hat{\bar{D}})U(x,\theta,\theta^*)$$

Instead of the differential operators $\hat{Q}$ and $\hat{\bar{Q}}$ we will now get covariant spinor derivatives D_a and $\bar{D}_a$. We can use these to impose covariant conditions on the superfield. That is:

$$\bar{D}_a\Phi(x,\theta,\theta^*) = 0 \tag{47}$$

[4] Notation in analogy to [4]

[5] This part was inspired by [7] where additional information regarding the group- theoretical background can be found as well

and

$$D_a\bar{\Phi}(x,\theta,\theta^*) = 0 \tag{48}$$

In the case of 'type I' and 'type II' superfields this will be especially useful. For 'type I' we have already obtained

$$\begin{aligned}\hat{Q}_a &= i\frac{\partial}{\partial\theta^a} &(49)\\ \hat{Q}_a^\dagger &= -i\frac{\partial}{\partial\theta^{a*}} + \theta^b(\sigma^\mu)_{ba}\partial_\mu &(50)\end{aligned}$$

another quick calculation gives us the covariant derivatives

$$\begin{aligned}D_a &= \frac{\partial}{\partial\theta^a} - i\bar{\theta}^b(\sigma^\mu)_{ba}\partial_\mu &(51)\\ D_a^\dagger &= -\frac{\partial}{\partial\theta^{a*}} &(52)\end{aligned}$$

5.4 Chirality

Lets recall our left-chiral superfield $\Phi_I^L(x,\theta,\theta^*) = \Phi_I(x,\theta)$. In the first place we called it left-chiral because the only fields that were involved were L-type. That this led to the omission of θ^* was just a nice bonus. We will now see that the technical way to construct a left-chiral superfield actually takes the opposite way. We apply the constraint

$$D_a^\dagger\Phi_I(x,\theta,\theta^*) = -\frac{\partial}{\partial\theta^{a*}}\Phi_I(x,\theta,\theta^*) = 0$$

Informally speaking, we could have also argued (in this special case) that $\frac{\partial}{\partial\theta^{a*}}(\delta\Phi) = \delta(\frac{\partial}{\partial\theta^{a*}}\Phi)$. To obtain a left chiral 'real type' superfield we perform the shift determined in (45) :

$$\Phi_{real}^L(x,\theta,\theta^*) = \Phi_I(x^\mu - \frac{1}{2}i\theta^a(\sigma^\mu)_{ab}\theta^{b*},\theta) \tag{53}$$

again, we can use the expansion from (31) to obtain

$$\begin{aligned}\Phi_{real}^L(x,\theta,\theta^*) &= \phi(x^\mu - \frac{1}{2}i\theta^a(\sigma^\mu)_{ab}\theta^{b*}) + \theta\cdot\chi(x^\mu - \frac{1}{2}i\theta^a(\sigma^\mu)_{ab}\theta^{b*})\\ &\quad +\frac{1}{2}\theta\cdot\theta F(x^\mu - \frac{1}{2}i\theta^a(\sigma^\mu)_{ab}\theta^{b*}) &(54)\end{aligned}$$

Taylor expansion in terms of x yields:

$$\begin{aligned}\Phi_{real}^L(x,\theta,\theta^*) &= \phi(x) + \theta\cdot\chi(x) + \frac{1}{2}\theta\cdot\theta F(x)\\ &\quad -\frac{1}{2}i\theta^a(\sigma^\mu)_{ab}\theta^{b*}\partial_\mu\phi - \frac{1}{2}i\theta\cdot\partial_\mu\chi\theta^a(\sigma^\mu)_{ab}\theta^{b*}\\ &\quad -\frac{1}{8}\theta^a(\sigma^\mu)_{ab}\theta^{b*}\theta^c(\sigma^\mu)_{cd}\theta^{d*}\partial_\mu\partial_\nu\phi &(55)\end{aligned}$$

further simplification gives us

$$\begin{aligned}\Phi^L_{real}(x,\theta,\theta^*) &= \phi(x)+\theta\cdot\chi(x)+\frac{1}{2}\theta\cdot\theta F(x)-\frac{1}{2}i\theta^a(\sigma^\mu)_{ab}\theta^{b*}\partial_\mu\phi \\ &\quad +\frac{1}{4}i\theta\cdot\theta\partial_\mu\chi^a(\sigma^\mu)_{ab}\theta^{b*}-\frac{1}{16}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}\partial^2\phi \end{aligned} \tag{56}$$

We have seen how to calculate irreducible multiplets using the superfield formalism. We can now go on to use these superfields to construct various Lagrangians. For example in the free Lagrangian the term $(\Phi^L_{real}(x,\theta,\theta^*))^\dagger\Phi^L_{real}(x,\theta,\theta^*)$ appears. Basic properties of products of superfields are given in the Appendix.

A Products of chiral superfields

In hindsight to the Wess-Zumino model, or in fact even more complex models like the MSSM we are now going to lay a basis for the non-gauge interactions of the fields. Because we haven't discussed the possible Lagrangians for supersymmetric theories yet, we are just going to postulate simple 'superpotentials'.

Consider a product of two chiral superfields:

$$\Phi_i \Phi_j = (\phi_i(x) + \theta \cdot \chi_i(x) + \frac{1}{2}\theta \cdot \theta F_i(x))(\phi_j(x) + \theta \cdot \chi_j(x) + \frac{1}{2}\theta \cdot \theta F_j(x)) \tag{57}$$

We identify the components of the product field:

- independent of θ: $\phi_i \phi_j$
- linear in θ: $\theta \cdot (\chi_i \phi_j + \chi_j \phi_i)$
- bilinear in θ: $\frac{1}{2}\theta \cdot \theta(\phi_i F_j + \phi_j F_i) + \theta \cdot \chi_i \theta \cdot \chi_j$

A similar calculation can be done for a product of three superfields. We denote the F-component:

$$\Phi_i \Phi_j \Phi_k|_F = \phi_i \phi_j F_k + \phi_j \phi_k F_i + \phi_k \phi_i F_j - \chi_i \cdot \chi_j \phi_k - \chi_j \cdot \chi_k \phi_i - \chi_k \cdot \chi_i \phi_j \tag{58}$$

We are now going to define the superpotentials. These will be just the F-component of the product of said superfields, because we always want to refer to a SuSy invariant action as they will only transform by a total derivative. M_{ij} , y_{ijk} are going to be taken symmetric in i, j ; respectively i, j, k.

$$\begin{aligned} W_{quad} &= \frac{1}{2} M_{ij} \Phi_i \Phi_j|_F \\ &= M_{ij} \phi_i F_j - \frac{1}{2} M_{ij} \chi_i \cdot \chi_j \end{aligned} \tag{59}$$

$$\begin{aligned} W_{cubic} &= \frac{1}{6} y_{ijk} \Phi_i \Phi_j \Phi_k|_F \\ &= \frac{1}{2} y_{ijk} \phi_i \phi_j F_k - \frac{1}{2} y_{ijk} \chi_i \cdot \chi_j \phi_k \end{aligned} \tag{60}$$

Another reason for the importance of the F-component of the products is following: when we integrate over the fermionic coordinates θ_1 , θ_2 we get rid of all but the terms proportional to $\theta \cdot \theta$ because of the definition of the integration over anticommutating variables.

B Properties of the Grassmann numbers

Grassmann numbers obey the anticommutation relation [3]

$$\{\theta_i, \theta_j\} = \theta_i \theta_j + \theta_j \theta_i = 0$$

This means any quadratic term will vanish,

$$\theta_i^2 = 0$$

We do now build a set of two Grassmann numbers: $\theta_a = (\theta_1, \theta_2)$
Using $\theta^a = \epsilon^{ab}\theta_b$, $\epsilon^{12} = 1$, $\epsilon^{21} = -1$, $\epsilon^{11} = \epsilon^{22} = 0$, a quadratic term $\theta \cdot \theta$ gives

$$\begin{aligned} \theta \cdot \theta &= \theta^a \theta_a \\ &= \theta^1 \theta_1 + \theta^2 \theta_2 \\ &= -2\theta^1 \theta^2 \end{aligned}$$

The derivative $\frac{\partial}{\partial \theta^a}(\theta^b \theta_b)$can be obatined by using

$$\begin{aligned} \frac{\partial}{\partial \theta^1}(\theta^b \theta_b) &= -2\theta^2 = 2\theta_1 \\ \frac{\partial}{\partial \theta^1}(\theta^b \theta_b) &= 2\theta_2 \end{aligned}$$

and therefore

$$\frac{\partial}{\partial \theta^a}(\theta \cdot \theta) = 2\theta_a$$

C Integration on superspace

We define the Berezin integrals

$$\int d\theta_1\, 1 = 0 \tag{61}$$

$$\int d\theta_1\, \theta_1 = 1 \tag{62}$$

$$\int d\theta_1 \int d\theta_2\, \frac{1}{2}\theta \cdot \theta = \int d\theta_1 \int d\theta_2\, \theta_2 \theta_1 = 1 \tag{63}$$

Keeping this in mind we can see that all terms linear or independent of θ vanish when integrating over $\int d\theta_1 d\theta_2$. Formally, integration and differentiation are the same.
As said, integrating the variation of a superfield over the whole superspace will be invariant:

$$\delta \int d^4x d^2\theta d^2\bar{\theta}\ \Phi(x, \theta, \bar{\theta}) = 0$$

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