Supergravity

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May 16, 2010

Abstract

After a short review of general relativity, the supergravity lagrangian is derived and discussed. When applying iteratively the Noether method on the supersymmetric Wess-Zumino model, we will find the need to introduce spin 3/2 and spin 2 fields, interpreted as gravitino and graviton, to obtain a lagrangian invariant under local supersymmetric transformation. We discuss the pure gravitational kinematic lagrangian, its gauge algebra and coupling to the Wess-Zumino model.

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1 Introduction

In this semester, FS10, a proseminar on supersymmetry is being organised by Babis Anastasiou. The objective of this proseminar is to discuss theoretical and phenomenological implications of supersymmetric theories. Previous talks derived rigid (or global) supersymmetry. In this framework it was shown how to incorporate gauge theories, which are invariant under global supersymmetry transformations.

In this report we wish to discuss the implications of making the supersymmetry transformation parameter local, $\varepsilon \to \varepsilon(x)$. We will find that local supersymmetry implies gravitation in a natural way, thus deserving the name "supergravity". Another motivation for including gravity in a supersymmetric theory is that, as can be observed, gravity is a force existing in our everyday world.

This report is organized as follows. After a short review of General Relativity in a field-theoretic approach using the Einstein-Hilbert-Lagrangian and a discussion of the spin statistics of gravitational interactions, it is reformulated in the vierbein formalism enabling us to include spinors which arise naturally in the context of quantum theories describing fermions.

In section 3 supergravity is introduced as a locally supersymmetric gauge theory. The gravitino, described by the Rarita-Schwinger field, is a hypothetical spin $\frac{3}{2}$ particle, introduced to gauge locally supersymmetric transformations, which then forces us to introduce interactions involving the metric $g_{\mu\nu}$ or vierbein as we apply the Noether method to gauge the theory. When discussing commutators acting on the supergravity multiplet, formed by the gravitino and vierbein, we are forced to add auxiliary fields to ensure the closure of the algebra.

Afterwards, the Wess-Zumino model of supersymmtry is coupled to gravity, which forces us to add additional terms to the lagrangian and modify the corresponding transformation properties.

2 Review of General Relativity

In this section we present a short review of Einstein's general relativity in a field-theoretic approach in four space-time dimensions. Here we use the convention that the signature of the space-time metric is $(+, -, -, -)$. We follow [MO] and [WA].

The gravitational action is $S = S_E + S_M$, where S_M is the matter part of the action which we will leave unspecified for the moment while S_E is the Einstein-Hilbert-Action given by

$$S_E = \frac{1}{16\pi G} \int \mathrm{d}^4x \sqrt{-g} R. \tag{1}$$

Here G is Newton's gravitational constant, $g = \det g_{\mu\nu}$, and R denotes the Ricci Scalar $R = R^{\mu}_{\mu}$ as contractions of the Ricci tensor, which is the contraction of the Riemann tensor, $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$. The Riemann tensor is defined by

$$R^{\mu}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} \tag{2}$$

where $\Gamma^{\rho}_{\mu\nu}$ are the Christoffel symbols(the torsion-free affine connection on the tangent bundle) are given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}(\partial_{\mu}g_{\sigma\nu} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}). \tag{3}$$

The energy momentum tensor $T^{\mu\nu}$ describing the matter by the action S_M is defined from the variation of S_M under a variation of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, according to

$$\delta S_M = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}. \tag{4}$$

Taking the variation of the total action with respect to $g_{\mu\nu}$, one finds the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{5}$$

As one observes, general relativity is invariant under diffeomorphisms,

$$x^{\mu} \to x'^{\mu}(x). \tag{6}$$

Under which the metric transforms as

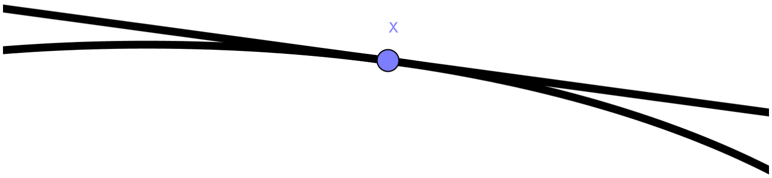
$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}g_{\rho\sigma}(x). \tag{7}$$

The linearized theory is defined by the expansion using flat-space metric $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \tag{8}$$

This approximation is valid due to the principle of equivalance. It states that we can choose for each space-time point a locally inertial system to describe physics. Geometrically, it means that for all points on our manifolds we can use the tangent space of a sufficiently large region around this point to describe physics.

Figure 1: The tangent space



2.1 Why a spin-2 field?

From a field-theoretical point of view, all forces are mediated by exchange particles with integer spin, bosons, e.g. the photon for electromagnetism. In this subsection we assume the existence of such a particle mediating gravitation, called graviton and want to answer the question whether there are any physical constraints on its spin.

To be more precise, we try to write matter interactions for trial gravitons with spin 0, 1 and 2 and sketch their implications. To this end one can study tree-level scattering processes of matter fields with the trial gravitons and extract the gravitational potential in the non-relativistic limit. For more details see e.g. [MM].

The simplest trial graviton with even integer spin is a scalar particle ϕ (spin 0, hence scalar and without Lorentz index) described by the Klein Gordon equation. The only way to couple the field ϕ linearly to the energy-momentum tensor $T_{\mu\nu}$ is by coupling to its the trace, $T = T^{\mu}_{\mu}$. Thus we have the Klein-Gordon lagrangian with coupling constant g,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + g\phi T.$$

Note that the energy-momentum tensor is quadratic in matter fields, thus the vertex $g\phi T$ describes two matter field lines and one scalar ϕ. Calculating the potential yields with the identification $\frac{g^2}{4\pi} = G$,

$$V(r) = -\frac{Gm_1m_2}{r}, \tag{9}$$

the correct non-relativistic description of gravity by Newton. However, we know that the energy-momentum tensor of the electromagnetic field is traceless, thus no coupling between this scalar gravity and electromagnetism is possible, which is in contradiction to experiments. Thus scalar gravity has to be ruled out.

The next step would be trying to construct a theory of gravity mediated by spin 1 particles, A_μ, similar to electromagnetism. Unfortunately, a coupling given by

$$A_\mu A_\nu T^{\mu\nu} \tag{10}$$

is not gauge invariant and would also yield a potential proportional to $\frac{1}{r^3}$ because of the exchange of two particles. A derivative coupling,

$$\partial_\mu A_\nu T^{\mu\nu} \tag{11}$$

has to be excluded as well because after integration by parts, energy-momentum conservation would be violated. Another reason against a vector particle is a possible repulsive potential between two particles with positive mass, as described by the Coulomb force in electromagnetism (see [MM]).

Values of spin > 2 must be ruled out as well because of further inconsistencies.

Thus, the only possibility is a spin 2 field.

2.2 Tetrad formalism

An alternative formulation of general relativity, more suitable for local supersymmetry, is the vierbein (or tetrad) formalism, where we define the vierbein $e_\mu^m = e_\mu^m(x)$ where m is a tangent space-time index while μ is a curved space-time index. This alternative formulation is possible because of the principle of equivalence, with the same reasoning as the weak-field approximation discussed. Of course, the vierbein will differ from point to point, as the tangent space will differ point to point.

The motivation for introducing tetrads, or vierbeins, is the necessity to describe spinors in space-time. From quantum field theory we know that spinors describe fermionic particles which model fundamental matter particles, and if we wish to have a theory of gravity coupling to matter, we need to be able to handle these spinors in space-time. In ordinary gravity using the metric $g_{\mu\nu}$ one can only describe couplings to integer spin particles. When coupling to spinors, one is forced to introduce a spin connection ω_μ^{ab}. For a more detailed discussion on tetrads and spinors we refer to [WA], ch. 13.

We can rewrite the metric as

$$g_{\mu\nu} = e_\mu^m(x)e_\nu^n(x)\eta_{mn} \tag{12}$$

where η is the metric in flat space-time and under transformations we have

$$e_\mu^m \to e_\nu'^m = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu^m. \tag{13}$$

We can use the vierbein to express any contravariant vector A^μ as a vector in the locally inertial coordinate system,

$$A^m = e_\mu^m A^\mu. \tag{14}$$

We can raise and lower indices on e_μ^m by using $g^{\mu\nu}$ and η_{mn} to obtain the dual vielbein e_m^μ with the following properties,

$$e_m^\mu e_\mu^n = \delta_m^n, \quad e_m^\mu e_\nu^m = \delta_\nu^\mu. \tag{15}$$

The principle of equivalence requires that special relativity should apply in locally inertial frames; therefore, the index m will transform as a flat space vector under Lorentz transformation Λ_n^m,

$$e_\mu^m \to \Lambda_n^m e_\mu^n. \tag{16}$$

To write down the lagrangian in this formalism we will have to express the curvature tensor in terms of the vierbeins.

We also need to redefine the covariant derivative such that it transforms appropriately under coordinate and Lorentz transformations. To this end we introduce the spin connection (also called Lorentz connection):

$$\omega_\mu^{ab} = e_\nu^a \partial_\mu e^{\nu b} + e_\nu^a e^{\sigma b}\Gamma_{\sigma\mu}^\nu. \tag{17}$$

Since the covariant derivative of the metric is zero, the same must be true for the vierbein,

$$D_\rho e_\sigma^m = \partial_\rho e_\sigma^m + \omega_\rho^{mn} e_{n\sigma} - \Gamma_{\rho\sigma}^\lambda e_\lambda^m = 0. \tag{18}$$

Using this we can express the spin connection in terms of the vierbein,

$$\omega_\mu^{mn} = \frac{1}{2} e^{m\nu}(\partial_\mu e_\nu^n - \partial_\nu e_\mu^n) + \frac{1}{4} e^{m\rho} e^{n\sigma}(\partial_\sigma e_\rho^l - \partial_\rho e_\sigma^l) e_{l\mu} - (a \leftrightarrow b). \tag{19}$$

The curvature tensor can then be rewritten as

$$R_{\mu\nu}^{mn} = \partial_\mu \omega_\nu^{mn} - \partial_\nu \omega_\mu^{mn} + \omega_\mu^{ml} \omega_{\nu l}^{n} - \omega_\nu^{ml} \omega_{\mu l}^{n}. \tag{20}$$

Note that $R_{\mu\nu}^{mn}$ is an equivalent formulation of the previously introduced Riemann curvature tensor $R_{\mu\nu\rho\sigma}$,

$$R_{\mu\nu\rho\sigma} e_m^\rho e_n^\sigma = R_{\mu\nu mn}. \tag{21}$$

The Einstein-Hilbert lagrangian is expressed using tetrads and $e = \det e_\mu^m$ by

$$\mathcal{L}_2 = -\frac{1}{2\kappa^2} e R(e, \omega). \tag{22}$$

The infinitesimal transformation under $x_\mu \to x'_\mu = x_\mu + \xi_\mu$ implies for the metric

$$g^{\mu\nu} \to g'^{\mu\nu} = g^{\mu\nu} + g^{\mu\sigma} \partial_\sigma \xi^\nu + g^{\rho\nu} \partial_\rho \xi^\mu + \xi^\rho \partial_\rho g_{\mu\nu} + \mathcal{O}(\xi^2) \tag{23}$$

which then implies

$$e_\mu^m \to e'^m_\mu = e_\mu^m + e_\lambda^m \partial^\mu \xi_\lambda + \xi_\rho \partial^\rho e_\mu^m + \mathcal{O}(\xi^2). \tag{24}$$

3 Supergravity

After a short review of local gauge transformations, the Noether method is applied to the Wess-Zumino model. Gauging locally supersymmetric transformation introduces spin $\frac{3}{2}$ fields identified with the Rarita-Schwinger field of the gravitino whose properties are quickly reviewed. Continuing applying Noether method yields terms proportional to the energy-momentum tensor of the Wess-Zumino fields, thus forcing us to introduce a coupling to a spin 2 particle, the graviton, to cancel additional terms. Then the free Lagrangian for supergravity is stated.

After a discussion of the gauge algebra properties, which indicates the need for auxiliary fields because of the non-closure of commutators of locally supersymmetric transformations applied on the gravitino, we discuss the coupling to the simple Wess-Zumino model consisting of two scalar particles and one fermion. We observe a gravitational coupling with mass dimension, as in the case of classical General Relativity in the field-theoretic approach, which is an indicator for non-renormalizability by superficial powercounting arguments.

For motivational purposes, we give a small hint for including gravity in a local supersymmetric theory: When applying two rigid supersymmetric transformations on the pseudoscalar B in the Wess-Zumino model, we find

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]B \sim \frac{1}{2}(\bar{\varepsilon}_2 \gamma^\mu \varepsilon_1) \partial_\mu B. \tag{25}$$

From the discussion of rigid supersymmetry we know that the application of two supersymmetric transformations yields a translation ∂_μ by the distance $d^\mu = \frac{1}{2}\bar{\varepsilon}_2\gamma^\mu\varepsilon_1$. Naively writing $\varepsilon \to \varepsilon(x)$ in both sides of the equation, we find

$$[\delta(\varepsilon_1(x)), \delta(\varepsilon_2(x))]B \sim \frac{1}{2}(\bar{\varepsilon}_2(x)\gamma^\mu\varepsilon_1(x))\partial_\mu B \tag{26}$$

and can interpret the distance as a function of a space-time point x, $d^\mu(x) = \frac{1}{2}\bar{\varepsilon}_2(x)\gamma^\mu\varepsilon_1(x)$. Now we have a situation where the amount of translation depends on the point discussed, which hints at the fact that space-time has become curved. As discussed before, this is precisely what happens when gravity is taken into account.

In this section we closely follow the arguments of [VN] and [MO]. For an introduction to supergravity using superspace formalism, the reader is referred to [WB]. [BL, BT, WE] present pedagogical introductions as well.

3.1 Gauge Theories

3.1.1 General Strategy and an Example

From the perspective of a field theory, supergravity is a gauge theory of three different symmetries and thus requires three different connections. The tetrad e_μ^m gauges space-time transformation, the spin-connection ω_μ^{ab} Lorentz transformation and the gravitino ψ_μ supersymmetric transformations. The algebra formed by these irreducible algebras is called the super Poincare algebra.

The method of gauging a theory is called Noether method. One starts by making the gauge parameter local, and will find as variation of the lagrangian a term proportional to the derivative of the gauge parameter called Noether current j^μ,

$$\delta\mathcal{L} = \partial_\mu\varepsilon j^\mu. \tag{27}$$

As first step one introduces a field with the same spin statistics as the derivative of the gauge parameter,A_μ, requiring its variation to be the $\delta A_\mu = \partial_\mu\varepsilon(x)$ and adds a coupling of this new field to the lagrangian with coupling constant g and the opposite sign to cancel the previously derived invariance,

$$\mathcal{L}' = -gA_\mu j^\mu. \tag{28}$$

One now checks the full Lagrangian whether new variations appear. If so, one continues with this iterative process by adding new terms to the lagrangian and changing the transformation properties until the the lagrangian is invariant.

As an analogy of deriving a local theory, it is shortly reviewed how the Noether method is applied on quantum electrodynamics following [AN]. In the next section, it is applied to supersymmetry transformation.

Given a Lagrangian

$$\mathcal{L} = i\bar{\chi}\partial\!\!\!/\chi, \tag{29}$$

with χ spinor field and $A\!\!\!/ = \gamma^\mu A_\mu$, which is invariant under a global gauge transformation $\chi \to \chi' = e^{i\varepsilon}\chi$ with scalar gauge paramter ε. To switch to local gauge transformations, we make the gauge parameter local, $\varepsilon = \varepsilon(x)$. Then the lagrangian is no longer invariant but changes by

$$\delta\mathcal{L} = -\bar{\chi}\gamma^\mu\chi\partial_\mu\varepsilon \quad \text{and} \quad j^\mu = -\bar{\chi}\gamma^\mu\chi. \tag{30}$$

Invariance can be restored by introducing a gauge field term, a vector field, to the Lagrangian,

$$\mathcal{L}' = -g\bar{\chi}\gamma^\mu A_\mu\chi \tag{31}$$

with the transformation property of

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{g}\partial_\mu\varepsilon \tag{32}$$

to restore invariance. In addition, one has to add kinetic terms for the new field A_μ,

$$\mathcal{L}'' = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{33}$$

with the invariant field strength tensor defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. After rewriting the partial derivative as covariant derivative, $\partial_\mu \to D_\mu = \partial_\mu + igA_\mu$, one can compactly write the quantum electrodynamics lagrangian as

$$\mathcal{L} = i\bar{\chi}D\!\!\!/\chi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{34}$$

This coupling of the Dirac field χ with the photon field A_μ is also known as minimal coupling.

Note that the scalar parameter ε of the gauge transformation enters as derivative $\partial_\mu\varepsilon$ under the variation of $\delta\chi$. Loosely speaking, it gets a Lorentz index and transforms as vector. To have an invariant Lagrangian, one is thus forced to introduce a vector field A_μ.

Applying the same logic to supersymmetry whose gauge parameter is a spinor, we are naturally lead to introduce a vectorial spinor field (or vector-spinor-field) ψ_μ called gravitino.

We will now make this intuition more precise by studying the Wess-Zumino model.

3.1.2 Local Supersymmetry Transformations: Example WZ-model

Let us for example study the Wess-Zumino model which we will use as guiding example for more general theories.

The action of the rigid Wess-Zumino model of supersymmetry is given by the sum of the Klein-Gordon action and the Dirac action,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{i}{2}\bar{\chi}\partial\!\!\!/\chi, \tag{35}$$

with scalar A, pseudoscalar B and Dirac field χ. This action is invariant (up to total derivatives) under rigid supersymmetry transformation defined by

$$\delta A = \bar{\varepsilon}\chi, \tag{36}$$

$$\delta B = +i\bar{\varepsilon}\gamma_5\chi, \tag{37}$$

$$\delta\chi = -i\partial\!\!\!/(A + iB\gamma_5)\varepsilon \tag{38}$$

where γ_5 is given by $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4$. Proceeding with the Noether method, changing the constant spinorial Majorana parameter to a local one, $\varepsilon \to \varepsilon(x)$, one finds up to total derivative

$$\delta\mathcal{L} = \partial_\mu\bar{\varepsilon}(\partial\!\!\!/(A - i\gamma_5 B))\gamma^\mu\chi \tag{39}$$

$$\equiv \partial_\mu\bar{\varepsilon}j^\mu \tag{40}$$

where we have introduced the Noether current j^μ. Note that the spinorial paramter ε appears under a derivative, $\partial_\mu\bar{\varepsilon}$, indicating that we need to introduce a spin $\frac{3}{2}$ particle. Thus, we quickly review properties of such spinorvectors described by the Rarita-Schwinger field.

3.1.3 Spin $\frac{3}{2}$ Rarita-Schwinger field

The massless Rarita-Schwinger field is described by a Majorana spinor with a Lorentz index, a vector-spinor, ψ_μ. Each of the Majorana spinor components satisfy the Dirac equations,

$$\partial\!\!\!/\psi_\lambda = 0,$$

and is subject to the subsidiary condition,

$$\gamma^\lambda\psi_\lambda = 0.$$

By contracting the Dirac equations with γ^λ one finds

$$\partial^\lambda\psi_\lambda = 0.$$

The Lagrangian for the massless Rarita-Schwinger field is given by

$$\mathcal{L}_{3/2} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma.$$

Using the Euler-Lagrange equation one immediately finds the equation of motion given by

$$-\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\partial_\rho\psi_\sigma = 0,$$

which is invariant under the transformation $\psi_\sigma \to \psi'_\sigma = \psi_\sigma + \frac{2}{\kappa}\partial_\rho\varepsilon$.

Counting the number of on-shell degrees of freedom for the massless gravitino gives 2, because the four degrees of freedom, given by the degrees of freedom of a massless vector field (2) times those of a Majorana spinor (2), are substracted by 2 by the contraints for the fermionic part due to the gauge condition $\gamma^\lambda\psi_\lambda$.

As the name "vector-spinor" or "bispinor-vector" suggests, it corresponds initially to the representation

$$[(\frac{1}{2},0)\oplus(0,\frac{1}{2})]\otimes(\frac{1}{2},\frac{1}{2})] \tag{41}$$

or

$$[4]\otimes\{[2_L]\oplus[2_R]\} \equiv \psi_\mu^{(16)}. \tag{42}$$

The before mentioned contraction corresponds to

$$[2_L]\oplus[2_R] \equiv (\gamma^\mu\psi_\mu)^{(4)}. \tag{43}$$

Decomposing we find

$$[4]\otimes\{[2_L]\oplus[2_R]\} = \{[6_R]\oplus[6_L]\}\oplus\{[2_R]\oplus[2_R]\}, \tag{44}$$

and after applying the constraints we are left with

$$[6_L]\oplus[6_R] \equiv \psi_\mu^{(12)}, \tag{45}$$

which corresponds to the irreducible representation of

$$(\frac{1}{2},1)\oplus(1,\frac{1}{2}). \tag{46}$$

For a more detailed discussion please see [WE, BT, MO, BI], or the original paper [RS].

3.1.4 Derivation of the kinetic Supergravity Lagrangian

Following the previously introduced procedure of gauging a theory (see also [VN]), we continue with the Noether method by adding as first term to the lagrangian (35) the coupling of the gauge field to the Noether current with opposite sign,

$$\mathcal{L}_1 = -\frac{\kappa}{2}\bar{\psi}_\mu j^\mu \tag{47}$$

$$= -\frac{\kappa}{2}\bar{\psi}_\mu(\partial\!\!\!/(A - i\gamma_5 B))\gamma^\mu\chi \tag{48}$$

and require that $\delta\psi_\mu \sim \frac{2}{\kappa}\partial_\mu\varepsilon(x)$. Note that since fermions have mass dimension 3/2 and ε has $-1/2$, a dimensional coupling κ appears.

Proceeding with the Noether method up to first order in κ and AA and BB terms, one finds

$$\delta(\mathcal{L} + \mathcal{L}_1) = i\kappa\bar{\psi}_\mu\gamma_\nu\varepsilon(\partial_\mu A\partial_\nu A + \partial_\mu B\partial_\nu B - \frac{1}{2}g_{\mu\nu}((\partial_\rho A)^2 + (\partial_\rho B)^2)) \tag{49}$$

$$= i\kappa\bar{\psi}_\mu\gamma_\nu\varepsilon T^{\mu\nu}. \tag{50}$$

This is nothing but the energy momentum tensor, strictly speaking, just of A and B field. More generally we therefore may write with $T^{\mu\nu}$ the energy-momentum tensor of the matter theory. This term can only be canceled by adding a second Noether coupling to the Noether current of translations $T_{\mu\nu}$. We introduce a new field $g_{\mu\nu}$ which we identify with the metric tensor. To find a non-zero transformation we require

$$\delta g_{\mu\nu} = -i\bar{\varepsilon}(\gamma_\mu\chi_\nu + \gamma_\nu\chi_\mu). \tag{51}$$

Thus, focusing on the gravitational sector, the kinetic terms of the locally supersymmetric lagrangian are then given by the kinetic terms for the fields ψ_μ and $g_{\mu\nu}$, or, using vierbeine, ψ_μ and e_μ^m. From earlier sections we know that the Rarita-Schwinger field and Riemann curvature tensor are to be used. We thus arrive at the following invariant lagrangian for the pure gravitational part

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{3/2} \tag{52}$$

with

$$\mathcal{L}_2 = -\frac{1}{2\kappa^2}eR(e,\omega), \tag{53}$$

$$\mathcal{L}_{3/2} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma, \tag{54}$$

$$D_\rho = \partial_\rho - \frac{1}{4}\sigma_{mn}\omega_\rho^{mn}, \tag{55}$$

where we have rewritten the curvature tensor by using $R(e,\omega) = e^{m\nu}e^{n\mu}R_{\mu\nu mn}(\omega)$. $\mathcal{L}$ is invariant under

$$\delta e_\mu^m = -i\kappa\bar{\varepsilon}\gamma^m\psi_\mu, \tag{56}$$

$$\delta\psi_\mu = \frac{2}{\kappa}(D_\mu\varepsilon). \tag{57}$$

It now remains to further add the matter part in a consistent manner. However, in order to make process on this front, we first have to sort out a conceptual issue.

3.2 Auxiliary Fields For The Gauge Algebra

As in the case of global superymmetry, we will need auxiliary fields to ensure that the commutators of the transformations close. In supergravity we have three types of local gauge transformations, two of which are bosonic and one fermionic, general coordinate transformations G with parameter ξ^μ and local Lorentz rotation L with parameter λ^{mn} and the fermionic local supersymmetry transformations Q with parameter ε.

Counting the degrees of freedom yields a discrepency of six bosonic degrees of freedom, because the tetrad has 6 degrees of freedom (16 a priori, reduced by 4 corresponding to general coordinate transformation and by 6 due to Lorentz transformations). Discussing the gravitino yields 12 degrees of freedom (16 a priori, reduced by 4 due to local supersymmetry transformations). Following the presentation of [VN], the minimal set of auxiliary fields is given by an axial vector A_m, an scalar S and an pseudoscalar P.

Another reason for the need of auxiliary fields is that we require transformation rules to be independent of matter fields. If the opposite were true, one could not sum the matter action of two different systems. With auxiliary fields, however, the sum is invariant as well and thus valid for any coupling system.

Here we merely state the results and the reader is referred to [VN] subsection 1.9 for more details.

When discussing the commutator of two supersymmetry transformations applied on the tetrad, one would find that it corresponds to the sum of a general coordinate, of a local Lorentz and of a supersymmetry transformation,

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_G(\xi^\mu) + \delta_Q(-\xi^\mu\psi_\mu) + \delta_L(\xi^\mu\omega_\mu^{mn}) \tag{58}$$

with $\xi^\mu = \frac{1}{2}\bar{\varepsilon}_2\gamma^\mu\varepsilon_1$.

When discussing the same commutator applied on the gravitino, one will find that the algebra does not close exactly as in the Wess-Zumino model, see [VN]. [1]

[1] already discussed in the corresponding proseminar talk

Introducing auxiliary fields, using the same procedure as in the rigid Wess-Zumino model, one arrives at a result valid for all fields,

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_G(\xi^\mu) + \delta_Q(-\xi^\mu\psi_\mu) \tag{59}$$

$$+ \delta_L[\xi^\mu(\omega_{\mu mn} - \frac{i}{3}\varepsilon_{\mu mnl}A^l) + \frac{1}{3}\bar{\varepsilon}_2\sigma^{mn}(S - i\gamma_5 P)\varepsilon_1], \tag{60}$$

with $\xi^\mu = \frac{1}{2}\bar{\varepsilon}_2\gamma^\mu\varepsilon_1$, where the transformation properties of the all fields, including auxiliary fields, are given by

$$\delta e^m_\mu = -i\kappa\bar{\varepsilon}\gamma^m\psi_\mu \tag{61}$$

$$\delta\psi_\mu = \frac{2}{\kappa}(D_\mu + \frac{i\kappa}{2}A_\mu\gamma_5)\varepsilon - \frac{1}{2}\gamma_\mu\eta\varepsilon \tag{62}$$

$$\delta S = \frac{1}{4}\bar{\varepsilon}\gamma^\mu R^{cov}_\mu \tag{63}$$

$$\delta P = -\frac{i}{4}\bar{\varepsilon}\gamma_5\gamma^\mu R^{cov}_\mu \tag{64}$$

$$\delta A_m = \frac{3i}{4}\bar{\varepsilon}\gamma_5(R^{cov}_m - \frac{1}{3}\gamma_m\gamma^\mu R^{cov}_\mu \tag{65}$$

$$\eta = -\frac{1}{3}(S - i\gamma_5 P - i\not{A}\gamma_5) \tag{66}$$

$$R^{\mu,cov} = \varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu(D_\rho\psi_\sigma - \frac{i}{2}A_\sigma\gamma_5\psi_\rho + \frac{1}{2}\gamma_\rho\eta\psi_\sigma) \tag{67}$$

with the new Lagrangian given by

$$\mathcal{L} = -\frac{1}{2}eR(e,\omega) - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma - \frac{e}{3}(S^2 + P^2 - A^2_m). \tag{68}$$

The corresponding field equations are

$$S = P = A_m = 0 \tag{69}$$

$$R^\mu = \varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_\sigma = 0 \tag{70}$$

$$eG^{m\nu} = \frac{1}{4}\bar{\psi}_\mu\gamma_5\gamma^m\varepsilon^{\lambda\nu\rho\sigma}(D_\rho\psi_\sigma - D_\sigma\psi_\rho). \tag{71}$$

3.3 Coupling to matter

After having discussed the action of pure supergravity, the next step is to finish the discussion of the matter coupling.

We return to the WZ action with the appropriate auxiliary fields

$$\mathcal{L}_0 = \frac{1}{2}g^{\mu\nu}(\partial_\mu A\partial_\nu A + \partial_\mu B\partial_\nu B) + \frac{i}{2}e\bar{\chi}\gamma^\mu D_\mu\chi + \frac{1}{2}e(F^2 + G^2), \tag{72}$$

where F, G are auxiliary fields, with transformations

$$\delta A = \bar{\varepsilon}\chi, \tag{73}$$
$$\delta B = i\bar{\varepsilon}\gamma_5\chi, \tag{74}$$
$$\delta F = \bar{\varepsilon}\not{D}\chi, \tag{75}$$
$$\delta G = i\bar{\varepsilon}\gamma_5\not{D}\chi, \tag{76}$$
$$\delta\chi = \frac{1}{2}\not{\partial}(A - i\gamma_5 B)\varepsilon + \frac{1}{2}(F + i\gamma_5 G)\varepsilon. \tag{77}$$

Variation of the term $\mathcal{L}_1$ from (48) (now including AB-terms) yields

$$\delta\mathcal{L}_1 = -\,\partial_\mu\bar{\varepsilon}j^\mu + i\kappa\bar{\psi}^\mu\gamma^\nu\varepsilon T_{\mu\nu} \tag{78}$$
$$+\,\frac{i\kappa}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\partial_\rho\varepsilon A\overleftrightarrow{\partial_\rho}B \tag{79}$$
$$+\,\frac{i\kappa}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\rho\bar{\psi}_\mu\gamma_\nu\varepsilon A\overleftrightarrow{\partial_\sigma}B \tag{80}$$

The first term is cancelled by the variation of the gravitino, $\delta\psi_\mu$, and the second by the addition of the metric coupling to the energy-momentum tensor, as was shown in the previous subsection.

The third term can by cancelled by adding to the lagrangian the following term,

$$\mathcal{L}_3 = -\frac{i}{2}\kappa^2\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\psi_\rho A\overleftrightarrow{\partial_\rho}B. \tag{81}$$

The last term of $\delta\mathcal{L}_1$ can be cancelled by changing $\delta\psi_\mu$ to

$$\delta\psi_\mu = \frac{2}{\kappa}\partial_\mu\varepsilon + i\kappa\gamma_5\varepsilon A\overleftrightarrow{\partial_\mu}B. \tag{82}$$

However, this new variation changes $\delta\mathcal{L}_1$ to

$$\delta\mathcal{L}_1 = \frac{i\kappa^2}{2}\bar{\varepsilon}\gamma_5\not{\partial}(A - i\gamma_5 B)\gamma^\mu\chi A\overleftrightarrow{\partial_\mu}B + \ldots, \tag{83}$$

which forces us to add

$$\mathcal{L}_4 = -\frac{\kappa^2}{4}\bar{\chi}\gamma_5\gamma^\tau\chi A\overleftrightarrow{\partial_\tau}B \tag{84}$$

to our lagrangian excluding 4-fermion terms. As can be checked this lagrangian with the new definition of $\delta\psi_\mu$ is now invariant up to all orders.

Thus, the fully coupled supergravity Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{3/2} + \mathcal{L}_0 + \mathcal{L}_N + \mathcal{L}_I \tag{85}$$

with the $\mathcal{L}_2, \mathcal{L}_{3/2}$ the corresponding spin $2, 3/2$ Lagrangian given in the previous subsection, $\mathcal{L}_0, \mathcal{L}_N$ the coupled Wess-Zumino model and Noether current discussed in the previous paragraph, and the interaction Lagrangian $\mathcal{L}_I = \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_{4f}$ given by

$$\mathcal{L}_I = \kappa^2 \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \psi_\rho \left(\frac{1}{8}\bar{\chi}\gamma_5\gamma_\sigma\chi - \frac{i}{2} A \overleftrightarrow{\partial_\sigma} B\right) \tag{86}$$

$$+ e\kappa^2 \bar{\chi}\gamma_5\gamma^\tau\chi \left(-\frac{1}{8}\bar{\psi}_\pi\gamma_5\gamma_\tau\psi^\pi - \frac{i}{4} A \overleftrightarrow{\partial_\tau} B - \frac{1}{32}\bar{\chi}\gamma_5\gamma_\tau\chi\right), \tag{87}$$

with transformation rules

$$\delta A = \bar{\varepsilon}\chi, \tag{88}$$

$$\delta B = i\bar{\varepsilon}\gamma_5\chi, \tag{89}$$

$$\delta e^m_\mu = -i\kappa\bar{\varepsilon}\gamma^m\psi_\mu, \tag{90}$$

$$\delta\psi_\mu = \frac{2}{\kappa} D_\mu\varepsilon + \frac{i}{4}\kappa\gamma_5\varepsilon A \overleftrightarrow{D^{cov}_\nu} B + \frac{\kappa}{8}\sigma_{\mu\nu}\gamma_5\varepsilon(\bar{\chi}\gamma_5\gamma^\nu\chi), \tag{91}$$

$$\delta\chi = \frac{1}{2}(\not{D}^{cov}(A - i\gamma_5 B))\varepsilon + \frac{\kappa^2}{8}\gamma_5\lambda(A\bar{\varepsilon}\gamma_5\chi - iB\bar{\varepsilon}\chi), \tag{92}$$

$$D^{cov}_\mu A = \partial_\mu - \frac{\kappa}{2}\bar{\psi}_\mu\chi. \tag{93}$$

Please note that at low energies, the gravitational couplings are suppressed by inverse powers of M_p due to $\kappa^{-1} = \frac{M_P}{\sqrt{8\pi}} = 2.4 \times 10^{18}$GeV, which is derived by requiring that the low-energy limit of supergravity coincides with Newtonian gravity, where $\kappa^2 = 8\pi G/c^4$ and, in natural units, the before cited term, as is shown in [BT].

Coupling to other matter fields can be formulated analogously by using Noether method, or more sophisticated methods. This, however, will not be further discussed in this report and the reader is referred to [VN] or [BI].

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