Path integral formalism

(script sections 2 and 3)

Contents

Script section numbers refer to lecture notes of Prof. Babis Anastasiou.

A Motivation

Advantages of the path integral formalism wrt. "canonical quantization":

- compact notation
- covers non-perturbative features
- natural interface to lattice gauge theory
- encodes n-point functions for all n (effect. potential)
- explicite conservation symmetries (incl. Lorentz symmetry)
- easy derivation of Feynman rules
- easy quantization of non-abelian QFT (Fadeev-Popov method)

Literature (formal level): any modern QFT text book, e.g.:

- Peskin, Schroeder: An introduction to quantum field theory
- Weinberg: Quantum theory of fields I
- Ryder: Quantum field theory

Literature (in-depth, related fields):

- Roepstorff: Pfadintegrale in der Quantenphysik
- Zinn-Justin: Quantum field theory and critical phenomena

B Path integral in QM

(script section 2.1)

Assume
$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1}$$
Show:
$$\left\langle x_f | e^{-i\hat{H}(t_f - t_i)} | x_i \right\rangle = \frac{1}{\mathcal{N}} \int \mathcal{D}x \, \exp\left(\frac{i}{\hbar} \int_{t_i}^{t_f} \mathrm{d}t \, L(x(t)) \right) \tag{2}$$
R.h.s. is the Feynman path integral of quantum mechanics. "Integration over all paths".
Key points:

- transition from operators to functions (distrib.)
- limit $\delta t \to 0$ gives linearization in δt

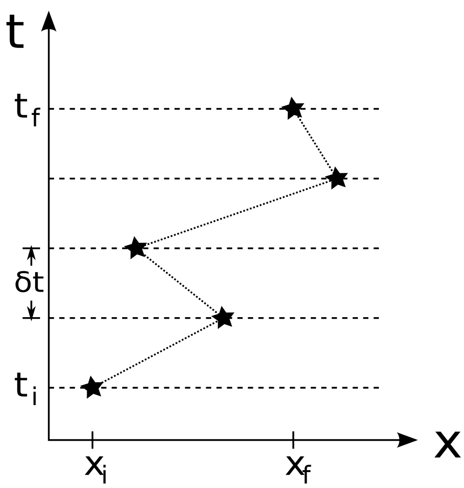

Figure 1: Supporting points for some path with $x(t_i) = x_i$ and $x(t_f) = x_f$.

Note: it is straight-forward to generalize this to N dimensions:
$$\left\langle \vec{x}_f | e^{-i\hat{H}(t_f - t_i)} | \vec{x}_i \right\rangle = \frac{1}{\mathcal{N}} \int \mathcal{D}\vec{x} \exp\left(\frac{i}{\hbar} \int_{t_i}^{t_f} \mathrm{d}t \, L(\vec{x}(t)) \right) \tag{3}$$
where
$$\vec{x} \equiv (x_1, \ldots, x_N), \qquad \mathcal{D}\vec{x} \equiv \prod_{j=1}^{N} \mathcal{D}x_j. \tag{4}$$
Note: generalization to Hamiltonians with more complicated $\hat{p}_j$ dependence
$$\left\langle \vec{q}_f | e^{-i\hat{H}(t_f - t_i)} | \vec{q}_i \right\rangle = \int \prod_{i=1}^{N} \mathcal{D}q_i \prod_{j=1}^{N} \mathcal{D}p_j \, \exp\left(\frac{i}{\hbar} \int_{t_1}^{t_2} \mathrm{d}t \left(\sum_{k=1}^{N} p_k \dot{q}_k - H(p,q) \right) \right) \tag{5}$$
with integration over all paths $\vec{p}(t)$ and all paths $\vec{q}(t)$ which match boundary conditions $\vec{q}(t_i) = \vec{q}_i$ and $\vec{q}(t_f) = \vec{q}_f$ at the endpoints. Must take care of $\hat{p}_j$ and $\hat{q}_j$ ordering.

C Path integral for scalar QFT

(script section 3)

Consider scalar QFT as quantum mechanical system with "infinitely many oscillators" (and use high energy units with $\hbar = 1$):

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \tag{6}$$

$$\rightarrow [\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \tag{7}$$

Postulate: in analogy to the QM case assume that in QFT holds

$$\left\langle \phi | e^{-i\hat{H}(t_f - t_i)} | \phi' \right\rangle = \mathcal{N} \int \mathcal{D}\phi \, \exp\left(i \int_{t_i}^{t_f} \mathrm{d}t \, L \right) \tag{8}$$

where the r.h.s. is the QFT path integral, $L = \int \mathrm{d}^3\vec{x}\mathcal{L}$ and integration is over all field configurations with $\phi(\vec{x}, t_i) = \phi_i(\vec{x})$ and $\phi(\vec{x}, t_f) = \phi_f(\vec{x})$. Show in the following this makes sense.

D Greens functions

(script section 2.2 and 2.4 but for fields)

Show:

$$\left\langle \phi | e^{-i\hat{H}t_f} T(\hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_n)) e^{i\hat{H}t_i} | \phi' \right\rangle = \int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) \exp\left(i \int_{t_i}^{t_f} \mathrm{d}t \, L \right) \tag{9}$$

Show:

$$\left\langle 0 | T(\hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_n)) | 0 \right\rangle = \frac{\int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) \exp\left(i \int \mathrm{d}^4\vec{x}\mathcal{L} \right)}{\int \mathcal{D}\phi \, \exp\left(i \int \mathrm{d}^4\vec{x}\mathcal{L} \right)} \tag{10}$$

using a "damping term" $(1/2)i\epsilon\phi^2$ in $\mathcal{L}$.

E Generating functional $Z[J]$

(script sections 2.2 and 2.3 but for fields)

Show:

$$\left\langle 0 | T(\hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_n)) | 0 \right\rangle = \frac{1}{i^n Z[0]} \frac{\delta^{(n)}}{\delta J(x_1) \cdots \delta J(x_n)} Z[J] \bigg|_{J=0} \tag{11}$$

F Free propagator from $Z[J]$

(script section 3.1)

Assume: free massive scalar theory
Show:

$$Z[J] = Z[0] \exp\left(-\frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y J(x) D_F(x-y) J(y) \right) \tag{12}$$

Then we get immediately:

$$\left\langle 0 | T(\hat{\phi}_H(x_1) \hat{\phi}_H(x_n)) | 0 \right\rangle = D_F(x_1 - x_2) \tag{13}$$

$$= \int \frac{\mathrm{d}^4x}{(2\pi)^4} e^{-ik\cdot(x_1 - x_2)} \frac{i}{k^2 - m^2 + i\epsilon} \tag{14}$$

G Euclidean formulation and statistical mechanics

Perform Wick rotation clockwise in complex t-plane, with

$$t \equiv -ix_4, \qquad x_E = (x_1, x_2, x_3, x_4), \qquad x^2 = -x_E^2, \tag{15}$$

With

$$\mathcal{L}_E = \frac{1}{2}\partial_\mu^E \phi \partial_\mu^E \phi + \frac{1}{2} m^2 \phi^2 \tag{16}$$

the path integral is:

$$Z[J] = \int \mathcal{D}\phi \, \exp\left(-\frac{1}{\hbar}\int d^4 x_E (\mathcal{L}_E - J\phi)\right) \tag{17}$$

Similar to partition function in statistical mechanics:

$$Z = \sum_n \exp\left(-\frac{1}{kT} E_n\right). \tag{18}$$