Path integral formalism
(script sections 2 and 3)

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Script section numbers refer to lecture notes of Prof. Babis Anastasiou.

A Motivation

Advantages of the path integral formalism wrt. “canonical quantization”:

- compact notation
- covers non-perturbative features
- natural interface to lattice gauge theory
- encodes $n$-point functions for all $n$ (effect. potential)
- explicite conservation symmetries (incl. Lorentz symmetry)
- easy derivation of Feynman rules
- easy quantization of non-abelian QFT (Fadeev-Popov method)

Literature (formal level): any modern QFT text book, e.g.:
- Peskin, Schroeder: An introduction to quantum field theory
- Weinberg: Quantum theory of fields I
- Ryder: Quantum field theory

Literature (in-depth, related fields):
- Roepstorff: Pfadintegrale in der Quantenphysik
- Zinn-Justin: Quantum field theory and critical phenomena
B  Path integral in QM

(script section 2.1)

Assume
\[ \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \]  

Show:
\[ \langle x_f | e^{-i\hat{H}(t_f - t_i)} | x_i \rangle = \frac{1}{\mathcal{N}} \int \mathcal{D}x \exp \left( \frac{i}{\hbar} \int_{t_i}^{t_f} dt \mathcal{L}(x(t)) \right) \]  

R.h.s. is the Feynman path integral of quantum mechanics. “Integration over all paths”.

Key points:
- transition from operators to functions (distrib.)
- limit $\delta t \to 0$ gives linearization in $\delta t$

![Diagram](image)

Figure 1: Supporting points for some path with $x(t_i) = x_i$ and $x(t_f) = x_f$.  

Note: it is straight-forward to generalize this to $N$ dimensions:

\[ \langle \vec{x}_f | e^{-i\hat{H}(t_f - t_i)} | \vec{x}_i \rangle = \frac{1}{\mathcal{N}} \int \mathcal{D}\vec{x} \exp \left( \frac{i}{\hbar} \int_{t_i}^{t_f} dt \mathcal{L}(\vec{x}(t)) \right) \]

where
\[ \vec{x} \equiv (x_1, \ldots, x_N), \quad \mathcal{D}\vec{x} \equiv \prod_{j=1}^{N} \mathcal{D}x_j. \]

Note: generalization to Hamiltonians with more complicated $\hat{p}_j$ dependence

\[ \langle \vec{q}_f | e^{-i\hat{H}(t_f - t_i)} | \vec{q}_i \rangle = \int \prod_{i=1}^{N} \mathcal{D}q_i \prod_{j=1}^{N} \mathcal{D}p_j \exp \left( \frac{i}{\hbar} \int_{t_1}^{t_2} dt \left( \sum_{k=1}^{N} p_k \dot{q}_k - H(p,q) \right) \right) \]

with integration over all paths $\vec{p}(t)$ and all paths $\vec{q}(t)$ which match boundary conditions $\vec{q}(t_i) = \vec{q}_i$ and $\vec{q}(t_f) = \vec{q}_f$ at the endpoints. Must take care of $\hat{p}_j$ and $\hat{q}_j$ ordering.
C Path integral for scalar QFT

(script section 3)

Consider scalar QFT as quantum mechanical system with “infinitely many oscillators” (and use high energy units with \( \hbar = 1 \)):

\[
[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \quad (6)
\]

\[
\rightarrow [\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \quad (7)
\]

Postulate: in analogy to the QM case assume that in QFT holds

\[
\langle \phi | e^{-i\hat{H}(t_f-t_i)} | \phi' \rangle = N \int D\phi \exp \left( i \int_{t_i}^{t_f} dt L \right) \quad (8)
\]

where the r.h.s. is the QFT path integral, \( L = \int d^3\vec{x}L \) and integration is over all field configurations with \( \phi(\vec{x}, t_i) = \phi_i(\vec{x}) \) and \( \phi(\vec{x}, t_f) = \phi_f(\vec{x}) \). Show in the following this makes sense.

D Greens functions

(script section 2.2 and 2.4 but for fields)

Show:

\[
\langle \phi | e^{-i\hat{H}t} T(\hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_n)) | \phi' \rangle = \int D\phi \phi(x_1) \cdots \phi(x_n) \exp \left( i \int_{t_i}^{t_f} dt L \right) \quad (9)
\]

Show:

\[
\langle 0 | T(\hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_n)) | 0 \rangle = \frac{\int D\phi \phi(x_1) \cdots \phi(x_n) \exp \left( i \int d^4\vec{x}L \right)}{\int D\phi \exp \left( i \int d^4\vec{x}L \right)} \quad (10)
\]

using a “damping term” \((1/2)i\epsilon\phi^2\) in \( L \).

E Generating functional \( Z[J] \)

(script sections 2.2 and 2.3 but for fields)

Show:

\[
\langle 0 | T(\hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_n)) | 0 \rangle = \frac{1}{i^n Z[0]} \frac{\delta^{(n)}}{\delta J(x_1) \cdots \delta J(x_n)} \bigg|_{J=0} Z[J] \quad (11)
\]

F Free propagator from \( Z[J] \)

(script section 3.1)

Assume: free massive scalar theory

Show:

\[
Z[J] = Z[0] \exp \left( -\frac{1}{2} \int d^4x d^4y J(x)D_F(x-y)J(y) \right) \quad (12)
\]

Then we get immediately:

\[
\langle 0 | T(\hat{\phi}_H(x_1) \hat{\phi}_H(x_n)) | 0 \rangle = D_F(x_1 - x_2) \quad (13)
\]

\[
= \int \frac{d^4x}{(2\pi)^4} e^{-ik(x_1-x_2)} \frac{i}{k^2 - m^2 + i\epsilon} \quad (14)
\]
G  Euclidean formulation and statistical mechanics

Perform Wick rotation clockwise in complex t-plane, with

\[ t \equiv -ix_4, \quad x_E = (x_1, x_2, x_3, x_4), \quad x^2 = -x_E^2, \quad (15) \]

With

\[ \mathcal{L}_E = \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 \quad (16) \]

the path integral is:

\[ Z[J] = \int D\phi \exp\left( -\frac{1}{\hbar} \int d^4x_E (\mathcal{L}_E - J\phi) \right) \quad (17) \]

Similar to partition function in statistical mechanics:

\[ Z = \sum_n \exp\left( -\frac{1}{kT} E_n \right). \quad (18) \]