
Exercise 9.1 The Ward-Takahashi identities

In the lectures you saw how to exploit symmetries of the action (and of the integration measure) to derive the Slavnov-Taylor identities.
In the first part of this exercise, we will derive them explicitly for QED, in which case they are called *Ward-Takahashi identities.*
In part (b), we'll see how the W-T identities constrain the form of the photon propagator, and in part (c), we will derive a relation between the fermion-fermion-photon vertex and the fermion propagators.

a) Require the QED generating functional

$$Z[J,\rho,\bar\rho] = \int \mathcal{D}A_\mu \mathcal{D}\bar\psi \mathcal{D}\psi \exp\left\{ i\int d^4x \left[\mathcal{L}_{\text{inv}} \underbrace{-\frac{1}{2\lambda}(\partial_\mu A^\mu)^2 + J_\mu A^\mu + \bar\rho\psi + \bar\psi\rho}_{=:\Delta\mathcal{L}} \right] \right\} \quad ,$$

to be invariant under the infinitesimal gauge transformations

$$A'_\mu = A_\mu - \partial_\mu\theta, \qquad \psi' = (1-ie\theta)\psi, \qquad \bar\psi' = (1+ie\theta)\bar\psi,$$

i.e. impose

$$\int d^4x \langle \delta\Delta\mathcal{L}\rangle = 0 \quad .$$

Substituting

$$\langle A_\mu\rangle = \frac{\delta W}{\delta J_\mu} \quad , \quad \langle\psi\rangle = \frac{\delta W}{\delta\bar\rho} \quad \text{and} \quad \langle\bar\psi\rangle = -\frac{\delta W}{\delta\rho}$$

(do you know where the "−" sign in the last expression comes from?), derive the Ward-Takahashi identities in Lorentz gauge:

$$\frac{1}{\lambda}\partial^2\partial_\mu \frac{\delta W}{\delta J_\mu} - ie\bar\rho\frac{\delta W}{\delta\bar\rho} - ie\frac{\delta W}{\delta\rho}\rho + \partial_\mu J^\mu = 0 \qquad (1)$$

b) Differentiate equation (1) with respect to sources in such a way that the first term can be written as

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle \quad .$$

Then, go to momentum space i.e. write

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = (-i)\int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)}\tilde D_{\mu\nu}(k) \quad .$$

Using the Ward-Takahashi identity, you should arrive at the following constraint for the photon propagator:

$$\tilde D_{\mu\nu} = A(k^2)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{\lambda}{k^4}k_\mu k_\nu \quad .$$

Remark: The Ward-Takahashi identity thus constrains the (renormalised) propagator to all orders in perturbation theory. Comparing with the free propagator (eq. (259) in the script, we see that the gauge-fixing term is fully accounted for, i.e. it doesn't get affected by the interacting theory. Therefore, the gauge-fixing term will not need to be renormalised.

c)* Now, differentiate equation (1) such that you get

$$\langle 0|TA_\mu(x)\psi(y)\bar{\psi}(z)|0\rangle \quad .$$

Also in this case, you should be careful with the differentiation with respect to the fermionic sources to make sure you din't miss minus signs. Write all the terms as expectation values of time-ordered products. Then go to Fourier space; to make your life simpler, exploit translational invariance (i.e. for example $\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \langle 0|T\psi(x-y)\bar{\psi}(0)|0\rangle$). If necessary, work on the momenta you are integrating over so as to reduce all the exponentials to the same form, so that you can collect them. Your final result should be

$$\frac{1}{\lambda}p^2 p_\mu \tilde{V}^\mu(p,q) + e\tilde{S}_F(p+q) - e\tilde{S}_F(q) = 0 \quad ,$$

where $\tilde{V}^\mu(p,q)$ is the Fourier transform of the vertex and $(-i)\tilde{S}_F(q)$ is the Fourier transform of the fermionic propagator.